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A CHARACTERIZATION OF THE CIRCLE AND INTERVAL

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Consider a connected T_1-space X. Take the Cartesian product of X with itself n times ($n \geqq 2$) and then remove the generalized diagonal $GD_n = \{(x_1, \cdots, x_n) \in X^n \mid x_i = x_j$ for some $i \neq j\}$ thus obtaining the deleted product $Z = X^n - GD_n$. If Z should be disconnected then a great deal can be said about X. For example, if X is compact and metrizable, then X is homeomorphic to the closed interval $[0, 1]$ or to the circle $C = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$. On the other hand, if it is only assumed (beyond X being T_1 and connected and Z disconnected) that X is Hausdorff, locally connected and separable, then X must be homeomorphic to either $(0, 1)$, $(0, 1]$, $[0,1]$ or C. In general, without any assumptions beyond X being T_1 and connected and Z disconnected it is possible to define an order on X which is a total order when restricted to X − *a certain finite set*, and such that the order topology is coarser (weaker, smaller) then the original topology on X. Furthermore, all connected subsets of X and the components of $X^m - GD_m$ for all $m \geqq 2$ (m not necessarily equal to n) are determined. In particular the number of components of $X^m - GD_m$ is either $(m-1)!$ or $m!/N!M!$ where $0 \leqq N, M < n$, $N + M \leqq m$ and each of these numbers is taken on for some X satisfying our hypothesis. The "generalized" cut point behavior of X is completely determined and an interesting result is that either there are no cut points or all but at most n points are cut points.

The analysis presupposes nothing but elementary concepts from general topology. In order to facilitate references to preceding lemmas and definitions a table of contents is included at the end.

1. Let X be a topological space. Set $X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$. We will denote by GD_n (the generalized diagonal) the subset of X^n consisting of all n-tuplets $(x_1, \cdots, x_n)$ such that $x_i = x_j$ for some $i \neq j$, $1 \leqq i, j \leqq n$.

CONVENTION. *Throughout this paper we will assume X is a connected T_1-space such that X^n-GD_n is not connected where n is a fixed integer greater than one.*

DEFINITION 1.1. Let Y be a topological space. Two points $x, y \in Y$ can be separated in Y if there exists disjoint open sets U,

V such that $x \in U$, $y \in V$ and $U \cup V = Y$. The open sets U and V are said to separate x and y in Y.

DEFINITION 1.2. If R is a relation on a set S then the negative relation R' is defined by: for $x, y \in S$, $xR'y$ if and only if xRy fails to hold.

In the following definition we introduce the central concept in our analysis of X.

DEFINITION 1.3. Let $\mathscr{F}$ be the collection of all finite subsets of X. For $\alpha \in \mathscr{F}$ we define the relation R_α on α by $xR_\alpha y$ if and only if $x, y \in \alpha$, $x \neq y$, and x cannot be separated from y in $(X - \alpha) \cup \{x, y\}$.

Clearly R_α is symmetric and $xR'_\alpha x$ for all $x \in \alpha$.

We associate with α and R_α a network (which we also call R_α) having the points of α as vertices and an edge between an x and $y \in \alpha$ if and only if $xR_\alpha y$ (or equivalently $yR_\alpha x$).

Our analysis consists of bringing into sharper and sharper focus our picture of the R_α's. First we see how the connectedness of X implies that each R_α is connected. Next, the disconnectedness of $X^n - GD_n$ is used to show that each vertex $x \in \alpha \in \mathscr{F}$ can have no more than n edges (in R_α) connected to it. The preceding two facts combine to show that there must be at least one long simple chain in α provided α has sufficiently many elements. Then bringing in the disconnectedness of $X^n - GD_n$ we see that except near the ends of such a long chain, each vertex x in the chain has exactly two edges (of R_α) connected to it which are of course the edges connecting x to the preceding are following vertex in the chain. This in turn enables us to distinguish (provided cardinality of α is sufficiently large) a unique long chain $C(\alpha)$ such that each vertex in it has the above property. The R_α's are sufficiently coherent for various α's to make it possible to use the $C(\alpha)$'s to define a simple order $<$ on most of X. Those points left out we will refer to here as exceptional points. The exceptional points are shown to be small in number (at most n) and clustered in two groups located at the ends of $C(\alpha)$. The simple order $<$ is then extended to a partial order on all of X by putting one group of exceptional points $>$ all other points and the other group of exceptional points $<$ all other points.

The relations R_α can now be determined quite easily from $<$. In fact if $x, y \in \alpha$, $xR_\alpha y$ if and only if there does not exist a $z \in \alpha$ such that $x < z < y$ or $y < z < x$. Meanwhile the topology is related to $<$ and all the connected subsets of X are determined (roughly just intervals). The exceptional points are shown to be basically (there may be 1 or 2 exceptions) the noncut points in the case that there are some cut points. The way in which $X^m - GD_m$, $m \geqq 1$, (m not necessarily equal to n) is disconnected is analyzed in terms of the numbers N and M of points in the two groups of exceptional points. We conclude that $X^m - GD_m$ has $m!/(N!M!)$ components.

The topology about the exceptional points (with possibly 1 or 2 exceptions) is shown to be necessarily not nice where nice means either locally compact or locally connected. The topology about the other points may or may not be nice but we prove that if it is nice then the order topology induced by $<$ agrees with the given topology at the points in question (same neighborhood system). Furthermore if separability is assumed (locally or globally) one can set up a homeomorphism (locally or globally) with a connected subset of the real line R. Combining these observations we prove our characterization of $I = \{x \in R \mid 0 \leqq x \leqq 1\}$ and $C = \{(x, y) \in R \times R \mid x^2 + y^2 = 1\}$.

(Actually, there are two cases for the general shape of $C(\alpha)$. The first is the one described above which leads to the final conclusion $X \cong I$. In the other case $C(\alpha)$ is a closed chain (circular chain) and in this case we finally conclude $X \cong C$.)

2. To simplify notation we will write x for the singleton $\{x\}$ when no confusion can arise.

LEMMA 2.1. *Let* $\alpha, \beta \in \mathscr{F}$, $\beta \subset \alpha$, $\alpha - \beta = \{z\}$, $x, y \in \beta$ *and* $x \neq y$. *If* $xR_\beta y$ *and* $xR'_\alpha y$ *then* $xR_\alpha z$ *and* $zR_\alpha y$.

Proof. Assume $xR_\beta y$ and $xR'_\alpha y$. We will show that $xR_\alpha z$. Assume the contrary, $xR'_\alpha z$. Set $X' = (X - \beta) \cup \{x, y\}$. The relation $xR'_\alpha y$ means that x can be separated from y in $(X - \alpha) \cup \{x, y\} = X' - z$ and consequently there exist sets X_1 and X_2 open in $X' - z$ such that $x \in X_1$, $y \in X_2$, $X_1 \cap X_2 = \phi$, and $X_1 \cup X_2 = X' - z$. Since $X' - z$ is open in X, X_1 and X_2 are open in X. Similarly $xR'_\alpha z$ implies the existence of open subsets of X, Y_1 and Y_2 such that $x \in Y_1$, $z \in Y_2$, $Y_1 \cap Y_2 = \phi$, and $Y_1 \cup Y_2 = X' - y$. Setting $X_3 = z$ and $Y_3 = y$ we have two partitions of X', $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$. Their product partition $\{X_i \cap Y_j\}$ is displayed below along with some relevant facts.

	$x \in Y_1$	$z \in Y_2$	$y = Y_3$
$x \in X_1$	$x \in X_1 \cap Y_1$	$X_1 \cap Y_2$	ϕ
$y \in X_2$	$X_2 \cap Y_1$	$X_2 \cap Y_2$	y
$z = X_3$	ϕ	z	ϕ

It is now apparent that x is separated from y in $X' = (X - \beta) \cup \{x, y\}$ by the open sets $X_1 \cap Y_1$ and $X_2 \cup Y_2$, a contradiction. Therefore $xR_\alpha z$. It follows similarly that $zR_\alpha y$.

DEFINITION 2.2. If R is a relation on a set S such that $xR'x$ for all $x \in S$, and $T \subset S$ then R induces a relation R^T on T as follows: For $x, y \in T$, $xR^T y$ if and only if $x \neq y$ and there is a finite sequence $x_0, x_1, x_2, \cdots, x_m \in (S - T) \cup \{x, y\}$ such that $x = x_0$, $x_{i-1}Rx_i$ for $1 \leqq i \leqq m$ and $x_m = y$.

It is easy to verify that $R = R^S$ and if $P \subset T \subset S$ then $R^P = (R^T)^P$.

LEMMA 2.3. *If $\alpha, \beta \in \mathscr{F}$ and $\beta \subset \alpha$ then $R_\beta = R_\alpha^\beta$.*

Proof. In light of the above observation it is sufficient to prove the lemma under the added restriction that $\alpha - \beta$ is a singleton $\{z\}$. Assume first that $x, y \in \beta$ and $xR_\beta y$. Either $xR_\alpha y$ or $xR'_\alpha y$. In the first case $xR_\alpha^\beta y$ follows immediately from Definition 2.2. If $xR'_\alpha y$ then Lemma 2.1 implies $xR_\alpha z$ and $zR_\alpha y$. Thus again from Definition 2.2, $xR_\alpha^\beta y$.

Now assume $x, y \in \beta$ and $xR_\alpha^\beta y$. Then there is a finite sequence $x_0, x_1, \cdots, x_m \in (\alpha - \beta) \cup \{x, y\} = \{z, x, y\}$ such that $x = x_0$, $x_{i-1}R_\alpha x_i$ for $1 \leqq i \leqq m$ and $X_m = y$. It follows readily that either $xR_\alpha y$ holds or both $xR_\alpha z$ and $zR_\alpha y$ hold. In the first case, $xR_\alpha y$, x cannot be separated from y in $(X - \alpha) \cup \{x, y\}$ and consequently x cannot be separated from y in $(X - \alpha) \cup \{x, y\} \cup \{z\} = (X - \beta) \cup \{x, y\}$. Hence, if $xR_\alpha y$ then $xR_\beta y$. Now in the second case, $xR_\alpha z$ and $zR_\alpha y$, and we again cannot separate x from y in $(X - \beta) \cup \{x, y\}$ because if A and B do so separate x from y then z is in either A or B, say A, and then $A - x$ and B separate z from y in $(X - \alpha) \cup \{z, y\}$. But then $zR'_\alpha y$ a contradiction. Thus $xR_\beta y$ in all cases.

DEFINITION 2.4. If R is a relation on a set A then a related relation $\bar{R}$ on A is defined by $x\bar{R}y$ if and only if there is a finite sequence $x_0, x_1, \cdots, x_m \in A$ such that $x = x_0$, $x_{i-1}Rx_i$ for $1 \leqq i \leqq m$ and $x_n = y$. Such a sequence, $x_0, \cdots, x_m$, is called an *R-chain* from x to y. The relations $x_{i-1}Rx_i$ are referred to as *links* and m as the *length* of the R-chain.

NOTATION 2.5. We will let ${}^\#s$ denote the cardinality of the set s.

LEMMA 2.6. *$x\bar{R}_\alpha y$ for all $x, y \in \alpha$ provided ${}^\#\alpha > 1$.*

Proof. We will use induction on the number N of elements of

α. If $N = 2$ the lemma follows from the fact that if $x, y \in \alpha$ and $x \neq y$ then $(X - \alpha) \cup \{x, y\} = X$ and X is connected. Thus $xR_\alpha y$ and $yR_\alpha x$ which implies $a\bar{R}_\alpha b$ for all $a, b \in \alpha$. Now assume the lemma holds for $N = m$. Let $\alpha = \{x_1, \cdots, x_{m+1}\}$ where $x_i \neq x_j$ for $i \neq j$. Take any two distinct elements of α, say x_1 and x_2. Set $\beta = \{x_1, x_2, \cdots, x_m\}$. By the induction hypothesis $x_1\bar{R}_\beta x_2$, i.e., there is a sequence $y_0, \cdots, y_p \in \beta$ such that $x_1 = y_0$, $y_{i-1}R_\beta y_i$ for $1 \leqq i \leqq p$ and $y_p = x_2$. This R_β chain from x_1 to x_2 can be converted into an R_α chain from x_1 to x_2 as follows. For each i, $1 \leqq i \leqq p$, if $y_{i-1}R_\alpha y_i$ then we replace the link $y_{i-1}R_\beta y_i$ by $y_{i-1}R_\alpha y_i$; if $y_{i-1}R'_\alpha y_i$ then by Lemma 2.1 $y_{i-1}R_\alpha x_{m+1}$ and $x_{m+1}R_\alpha y_i$, and we replace $y_{i-1}R_\beta y_i$ by $y_{i-1}R_\alpha x_{m+1}$ and $x_{m+1}R_\alpha y_i$. Thus $x_1\bar{R}_\alpha x_2$. The relations $x\bar{R}_\alpha x$ for $x \in \alpha$ follow from $x\bar{R}_\alpha y$ and $y\bar{R}_\alpha x$ where y is any element of α different from x. Therefore $x\bar{R}_\alpha y$ for all $x, y \in \alpha$. This completes the induction step and hence the lemma is proved.

3. For later reference we state the following trivial observations as lemmas.

LEMMA 3.1. *If $x, y \in B \subset A$ where A is a topological space, B is a subspace, and x is separated from y in A then x is separated from y in B.*

LEMMA 3.2. *If $\alpha, \beta \in \mathscr{F}$ and $x, y \in \beta \subset \alpha$ then $xR_\alpha y$ implies $xR_\beta y$.*

In the next lemma we start to investigate the relation between connectedness in X and connectedness in $X^n - GD_n$.

LEMMA 3.3. *Suppose $\alpha = \{x_1, \cdots, x_{n+1}\}$ and $x_l \neq x_j$ for $l \neq j$. If $x_iR_\alpha x_{n+1}$ then $\bar{x} = (x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ cannot be separated from $\bar{y} = (x_1, \cdots, x_{i-1}, x_{n+1}, x_{i+1}, \cdots, x_n)$ in $X^n - GD_n$.*

Proof. Consider the set $X' = x_1 \times x_2 \times \cdots \times x_{i-1} \times X \times x_{i+1} \times \cdots \times x_n \subset X^n$. X' is homeomorphic to X under the projection onto the i^{th} coordinate p_i and

$$\begin{aligned} p_i(X' \cap (X^n - GD_n)) &= X - \{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\} \\ &= (X - \alpha) \cup \{x_i, x_{n+1}\} \,. \end{aligned}$$

The conclusion now follows from $x_iR_\alpha x_{n+1}$ and Lemma 3.1.

COROLLARY 3.4. *Suppose $\alpha = \{x_1, \cdots, x_{n+1}\}$ and $x_i \neq x_j$ for $i \neq j$ and $1 \leqq i, j \leqq n + 1$. If $\alpha \subset \beta \in \mathscr{F}$ and $x_iR_\beta x_{n+1}$ then $\bar{x} = (x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ cannot be separated from*

$$\bar{y} = (x_1, \cdots, x_{i-1}, x_{n+1}, x_{i+1}, \cdots, x_n)$$

in $X^n - GD_n$.

Proof. By Lemma 3.2 $x_i R_\beta x_{n+1}$ implies $x_i R_\alpha x_{n+1}$ and so the conclusion follows from Lemma 3.3.

(3.5) Pictorial interpretation of Corollary 3.4. (see Figure 1.) Given an $\alpha \in \mathscr{F}$ let the vertices of the network R_α be represented by dots on a sheet of paper with lines between dots corresponding to the edges of R_α. That is two dots corresponding to points $x, y \in \alpha$ have a line between them if and only if $x R_\alpha y$. Suppose we have n

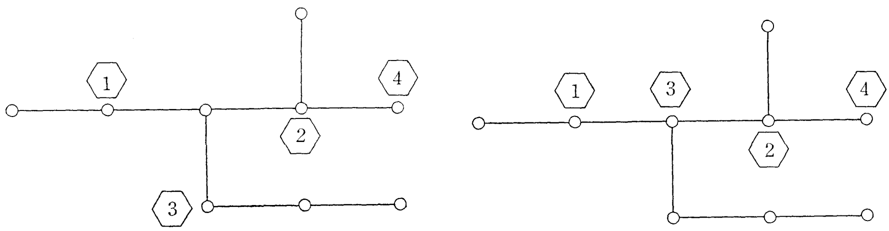

Figure 1

markers labled $1, 2, \cdots, n$. Then an n-tuplet $(y_1, \cdots, y_n) \in X^n - GD_n$ such that $y_i \in \alpha$ for $1 \leqq i \leqq n$, corresponds in a natural manner with an arrangement of the n markers on n distinct dots. Call such an arrangement of markers admissible. The above correspondence is one to one and onto from the set of all $(y_1, \cdots, y_n) \in X^n - GD_n$ such that $y_i \in \alpha$ for $1 \leqq i \leqq n$, to the set of all admissible arrangements of markers. Now the content of Corollary 3.4 is that if one admissible arrangement of markers is altered by moving one marker from the dot it is on to an unoccupied dot which is connected to the original dot by a line (such a change in the positions of the markers is called allowable) then the new and old arrangements correspond to connected (i.e., nonseparated) points of $X^n - GD_n$.

DEFINITION 3.6. The relation S is defined on $X^n - GD_n$ by: for $a, b \in X^n - GD_n$, aSb if and only if a can be separated from b in $X^n - GD_n$. Note that S' is transitive and symmetric and reflexive, i.e., S' is an equivalence relation.

DEFINITION 3.7. Let S_n be the permutation group on n objects. If $(x_1, \cdots, x_n) \in X^n$ then set $\sigma(x_1, \cdots, x_n) = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)})$. Note that $\sigma(X^n - GD_n) = X^n - GD_n$ for each $\sigma \in S_n$.

DEFINITION 3.8. Let R be a relation on a set S. An R-chain $x_0, \cdots, x_m$ is *simple* if and only if $x_i \neq x_j$ for $i \neq j$. If $x_0, \cdots, x_m$ is an R-chain from x_0 to $x_m, x_0 \neq x_m$, then one may obtain from $x_0, x_1, \cdots, x_m$ a simple R-chain from x_0 to x_m by removing "loops".

LEMMA 3.9. *If* $\bar{x} = (x_1, \cdots, x_n)$ *and* $\bar{y} = (y_1, \cdots, y_n)$ *are elements of* $X^n - GD_n$ *then* $\bar{x}S'\sigma\bar{y}$ *for some* $\sigma \in S_n$.

Proof. Since S' is transitive and $\{y_1, \cdots, y_n\} = B$ can be obtained from $\{x_1, \cdots, x_n\} = A$ by replacing elements of $A - B$ one by one with elements of $B - A$ it is sufficient to consider the case where $A - B = \{x_i\}$ and $B - A = \{y_j\}$. Then $\alpha = \{x_1, \cdots, x_n, y_1, \cdots, y_n\}$ is really a set of $n+1$ distinct elements of X. Because each $\tau \in S_n$ acts as a homeomorphism on $X^n - GD_n$ we have $\tau\bar{x}S'\sigma\bar{y}$ if and only if $\bar{x}S'\tau^{-1}\sigma\bar{y}$. Thus we need only show that $\tau\bar{x}S'\sigma\bar{y}$ for some $\tau, \sigma \in S_n$. According to Lemma 2.6 there is a R_α-chain from y_j to x_i. Let $z_0, \cdots, z_N$ be a simple R_α-chain from y_j to x_i. Let $z_{N+1}, \cdots, z_n$ be an enumeration of $\alpha - \{z_0, \cdots, z_N\}$. Then $\alpha = \{z_0, \cdots, z_n\}$, $z_k \neq z_l$ if $k \neq l$, $x_i = z_N$, $y_j = z_0$ and $z_{k-1}R_\alpha z_k$ for $1 \leqq k \leqq N$. It follows that $(\hat{z}_0, \cdots, z_n) = \tau\bar{x}$ and $(z_0, \cdots, \hat{z}_N, \cdots, z_n) = \sigma\bar{y}$ for some $\tau, \sigma \in S_n$ where the hat (^) signifies that the element is missing. From Lemma 3.3, (with z_{k-1} and z_k taking the parts of the x_{n+1} and x_i of Lemma 3.3 respectively) $(z_0, \cdots, \hat{z}_{k-1}, \cdots, z_n)\ S'\ (z_0, \cdots, \hat{z}_k, \cdots z_n)$ for $1 \leqq k \leqq N$. Thus $\tau\bar{x}S'\sigma\bar{y}$ as we wished to show.

LEMMA 3.10. *If* $a \in X^n - GD_n$ *then* $aS\sigma a$ *for some* $\sigma \in S_n$.

Proof. Let $a \in X^n - GD_n$ and assume $aS'\sigma a$ for all $\sigma \in S_n$. According to Lemma 3.9 for each b and $c \in X^n - GD_n$ there exist $\sigma, \tau \in S_n$ such that $bS'\sigma a$ and $cS'\tau a$. Since S' is transitive it follows from $bS'\sigma a$, $\sigma aS'a$, $aS'\tau a$, $\tau aS'c$ that $bS'c$ for all b and $c \in X^n - GD_n$. But this contradicts our fundamental assumption that $X^n - GD_n$ is not connected.

4. DEFINITION 4.1. If R is a relation on a set Y and $x \in Y$ then we will set $sp_R x = {}^{\#}\{y \in Y \mid xRy\}$.

LEMMA 4.2. *If* $x \in \alpha \in \mathscr{F}$ *then* $sp_{R_\alpha} x \leqq n$.

Proof. Assume the contrary, i.e., assume $x \in \alpha \in \mathscr{F}$ and $sp_{R_\alpha} x \geqq n + 1$. (see Figure 2.) Then there are $n + 1$ distinct points $x_1, x_2, \cdots, x_{n+1}$ of α such that $xR_\alpha x_i$ for $1 \leqq i \leqq n + 1$. From the definition of R_α we have x distinct from each x_i, $1 \leqq i \leqq n + 1$. Let $a = (x_1, \cdots, x_n) \in X^n - GD_n$. We will show that $aS'\sigma a$ for all $\sigma \in S_n$. Since S' is transitive it is sufficient to show $aS'\sigma a$ for all simple permutations σ. Also because each $\tau \in S_n$ acts as a homeomorphism on $X^n - GD_n$ we have $\tau a S' \tau \sigma a'$ if and only if $aS'\sigma a$ and consequently we may assume $\sigma(x_1, x_2, x_3, \cdots, x_n) = (x_2, x_1, x_3, \cdots x_n)$. Using Corollary 3.4 it follows from $x_1 R_\alpha x$, $x R_\alpha x_{n+1}$, $x_2 R_\alpha x$, $x R_\alpha x_1$, $x_{n+1} R_\alpha x$ and

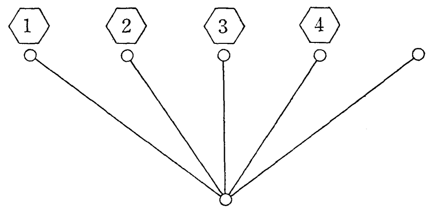

Figure 2

$x R_\alpha x_2$ that $(x_1, x_2, x_3, \cdots, x_n)$ S' $(x, x_2, x_3, \cdots, x_n)$ S' $(x_{n+1}, x_2, x_3, \cdots, x_n)$ S' $(x_{n+1}, x, x_3, \cdots, x_n)$ S' $(x_{n+1}, x_1, x_3, \cdots, x_n)$ S' $(x, x_1, x_3, \cdots, x_n)$ S' $(x_2, x_1, x_3, \cdots, x_n)$. This calculation is illustrated in Figure 2. Thus $(x_1, x_2, x_3, \cdots, x_n)$ S' $(x_2, x_1, x_3, \cdots, x_n)$ as we desired. So we have $aS'\sigma a$ for all $\sigma \in S_n$ which contradicts Lemma 3.10. Therefore the present lemma holds.

In the light of Lemmas 2.6 and 4.2 the following lemma tells us that for ${}^\#\alpha$ large, α must contain at least one long simple R_α-chain.

LEMMA 4.3. *If R is a relation on a nonempty set Y such that no simple R-chain has length more than N, $x\bar{R}y$ for all $x, y \in Y$, and $sp_R x \leqq M$ for all $x \in Y$ where $M \geqq 2$ then ${}^\#Y \leqq M^{N+1}$.*

Proof. Pick an $x_0 \in Y$. Set $Y_m = \{y \in Y \mid$ there is a simple R-chain of length m from x_0 to $y\}$ for $m \geqq 0$. Note that x_0 is an R-chain of length 0 and the only R-chain of length 0 starting at x_0. Thus $Y_0 = \{x_0\}$. Let $Z_m = \{$all simple R-chains of length m from x_0 to some point of $Y\}$ for $m \geqq 0$. Clearly ${}^\#Y_m \leqq {}^\#Z_m$ for all m. From the hypothesis we have $Z_m = \varnothing$ for $m > N$. Hence $Y_m = \varnothing$ for $m > N$. Also $x\bar{R}y$ for all $x, y \in Y$ implies $\bigcup_0^\infty Y_m = Y$. Thus $Y = \bigcup_0^\infty Y_m = \bigcup_0^N Y_m$ and consequently ${}^\#Y \leqq \sum_0^N {}^\#Y_m \leqq \sum_0^N {}^\#Z_m$. Now each simple R-chain of length $m+1$ starting at x_0 is obtained from a simple R-chain of length m starting at x_0 by adjoining a link. Since $sp_R x \leqq M$ for all $x \in Y$ we see that each simple R-chain of length m can give rise to no more than M simple R-chains of length $m+1$. It follows that ${}^\#Z_{m+1} \leqq M\,{}^\#Z_m$ for $m \geqq 0$. Since ${}^\#Z_0 = {}^\#\{x_0\} = 1$, an easy induction gives ${}^\#Z_m \leqq M^m$. Thus

$$ {}^\#Y \leqq \sum_0^N Z_m \leqq \sum_{m=0}^N M^m = \frac{M^{N+1} - 1}{M - 1} \leqq M^{N+1} . $$

LEMMA 4.4. *Let $\alpha \in \mathscr{F}$ and $x_1, \cdots, x_N$ be a simple R_α-chain* *in* α. *If* $2n+1 \leqq i \leqq N-2n$ *then* $x_i R_\alpha y$ *holds only for* $y = x_{i-1}$ *and* $y = x_{i+1}$. *Thus* $sp_{R_\alpha} x_i = 2$.

Proof. Assume the hypotheses and suppose $y \in \alpha$, $y \neq x_{i-1}$, $y \neq x_{i+1}$, and $x_i R_\alpha y$. (see Figure 3.) We will show that $(x_1, \cdots, x_n)$ S'

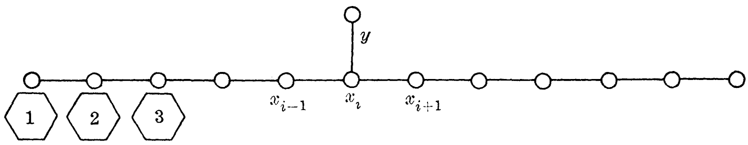

Figure 3

$\sigma(x_1, \cdots, x_n)$ for all $\sigma \in S_n$. It is sufficient to consider only simple interchanges of two adjacent objects. So assume $\sigma(x_1, \cdots, x_n) = (x_1, \cdots, x_{l-1}, x_{l+1}, x_l, x_{l+2}, \cdots, x_n)$. We will consider three cases. Case 1: $y \neq x_k$, $i - n \leqq k \leqq i + n$. Case 2: $y = x_k$ with $i - n \leqq k \leqq i - 2$. Case 3: $y = x_k$ with $i + 2 \leqq k \leqq i + n$.

Consider Case 1. The diagram of dots is illustrated in Figure 3. It is now easy to see in light of the discussion 3.5 that $(x_1, \cdots, x_n)$ S' $(x_{i-l}, x_{i-l+1}, \cdots, x_{i-1} x_i, x_{i+2}, \cdots, x_{i-l+n})$ S' $(x_{i-l}, x_{i-l+1}, \cdots, x_{i-1}, y, x_{i+2}, \cdots, x_{i-l+n})$ S' $(x_{i-l}, x_{i-l+1}, \cdots, x_{i-2}, x_{i+1}, y, x_{i+2}, \cdots, x_{i-l+n})$ S' $(x_{i-l}, x_{i-l+1}, \cdots, x_{i-2}, x_{i+1}, x_i, x_{i+2}, \cdots, x_{i-l+n})$ S' $(x_1, x_2, \cdots, x_{l-1}, x_{l+1}, x_l, x_{l+2}, \cdots, x_n)$. Thus $(x_1, \cdots, x_n)$ S' $\sigma(x_1, \cdots, x_n)$ as desired.

Next consider Case 2. It is easy to see that $(x_1, \cdots, x_n)$ S' $(x_{k-l+1}, \cdots, x_{k-1}, x_k, x_{k+1}, x_{i+1}, \cdots, x_{i+n-l-1})$ S' $(x_{k-l+1}, \cdots, x_{k-1}, x_i, x_{k+1}, x_{i+1}, \cdots, x_{i+n-l-1})$ S' $(x_1, \cdots, x_{l-1}, x_{l+1}, x_l, x_{l+2}, \cdots, x_n)$. Thus $(x_1, \cdots, x_n)$ S' $\sigma(x_1, \cdots, x_n)$ as we wished to show.

Case 3 is perfectly analogous to Case 2 and is left to the reader. This completes the proof.

5. LEMMA 5.1. *If* $^{\#}\alpha \geqq n^{8n+1} + 1$ *then there is a unique subset* $C(\alpha)$ *of* α *satisfying*

(i) $C(\alpha)$ *is a simple* R_α*-chain—i.e.,* $C(\alpha)$ *can be indexed so that* $C(\alpha) = \{x_1, \cdots, x_m\}$, $^{\#}C(\alpha) = m$, *and* $x_1, \cdots, x_m$ *is a simple* R_α *-chain.*

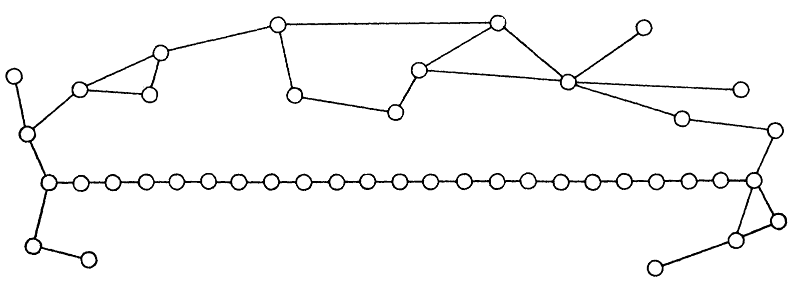

Figure 4

(ii) ${}^{\#}C(\alpha) \geqq 4n + 1$

(iii) $x \in C(\alpha) \Rightarrow sp_{R_\alpha} x = 2$

(iv) $C(\alpha)$ *is a maximal set satisfying* (i), (ii) *and* (iii). (see Figure 4.)

Proof. By Lemma 4.2, $sp_{R_\alpha} x \leqq n$ for all $x \in \alpha$. Thus from Lemma 4.3 there must be a simple chain $C_1 = x_1, \cdots, x_{8n+1}$ of length $8n+1$. Consider the simple chain $C_2 = x_{2n+1}, x_{2n+2}, \cdots, x_{6n+1}$. By Lemma 4.4 $sp_{R_\alpha} x_i = 2$ for $2n + 1 \leqq i \leqq 6n + 1$. Thus the set of points in the chain C_2 satisfies (i), (ii) and (iii). Since α is a finite set it is easy to see that there must be a maximal set C_3 satisfying (i), (ii), and (iii).

We will now prove that C_3 is the only subset of α satisfying (i), (ii), (iii) and (iv). Let $C_4 \neq C_3$ be another such set and let $C_3 = \{y_1, \cdots, y_m\}$ and $C_4 = \{z_1, \cdots, z_p\}$ where $y_i \neq y_j$ for $i \neq j$, $z_i \neq z_j$ for $i \neq j$, $y_{i-1} R_\alpha y_i$ for $2 \leqq i \leqq m$, $z_{i-1} R_\alpha z_i$ for $2 \leqq i \leqq p$, $m \geqq 4n + 1$ and $p \geqq 4n + 1$. Since $sp_{R_\alpha} y_1 = 2$ there must be a unique element y_0 of α such that $y_0 R_\alpha y_1$ and $y_0 \neq y_2$. Similarly there exists a unique y_{m+1} such that $y_m R_\alpha y_{m+1}$ and $y_{m-1} \neq y_{m+1}$. Analogously we have z_0 and z_{p+1} with corresponding properties.

We break up the proof into three cases. Case 1: $y_0 = y_m$ or $z_0 = z_p$; Case 2: $y_0 \neq y_m$, $z_0 \neq z_p$ and $C_3 \cap C_4 \neq \emptyset$; Case 3: $C_3 \cap C_4 = \emptyset$. We will reach a contradiction in each case. Consider Case 1 and for definiteness assume $y_0 = y_m$. Note that in this case $y_{m+1} = y_1$. Since $C_4 \neq C_3$ and C_4 is maximal we cannot have $C_4 \subset C_3$. Thus there is a $z \in \alpha - C_3$. By Lemma 2.6 we know that there is a simple R_α-chain C_5 from z to y_1. Let y_i be the first element of C_5 in C_3 and t the element of C_5 preceding y_i. Note that $t \notin C_3$. Since $y_0 = y_m$, $C_6 = y_{i-2n}, y_{i-2n+1}, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_{i+2n}$ is a simple R_α-chain where we temporarily have set $y_j = y_{m+j}$ if $j \leqq 0$ and $y_j = y_{j-m}$ if $j > m$. But, because $y_{i-1} R_\alpha y_i$, $y_i R_\alpha y_{i+1}$, $t R_\alpha y_i$ and $t \neq y_{i-1}, y_{i+1}$ and $y_{i-1} \neq y_{i+1}$ we must have $sp_{R_\alpha} y_i \geqq 3$. This contradicts Lemma 4.4.

Next consider Case 2. $y_0 \neq y_m$, $z_0 \neq z_p$ and $C_3 \cap C_4 \neq \emptyset$. In this case $y_{m+1} \neq y_1$. Also note that $y_0 \notin C_3$ for otherwise $y_j R_\alpha y_1$ for some j, $2 < j < m$ which would contradict $sp_{R_\alpha} y_j = 2$. Similarly $y_{m+1} \notin C_3$, $z_0, z_{p+1} \notin C_4$. Since $C_3 \cap C_4 \neq \emptyset$ we have $y_i = z_j$ for some i, j satisfying $1 \leqq i \leqq m$ and $1 \leqq j \leqq p$. Because $y_{i+1} R_\alpha y_i$, $sp_{R_\alpha} z_j = 2$, $z_{j-1} R_\alpha z_j$, and $z_j R_\alpha z_{j+1}$ we must have either $y_{i+1} = z_{j-1}$ or $y_{i+1} = z_{j+1}$. By renumbering if necessary we may assume $y_{i+1} = z_{j+1}$.

If $i + 1 = m + 1$ but $j + 1 < p + 1$ we can conclude that $sp_{R_\alpha} y_{m+1} = sp_{R_\alpha} z_{j+1} = 2$. This then implies $y_1, \cdots, y_m, y_{m+1}$ is a simple R_α-chain and in fact $C_7 = \{y_1, \cdots, y_m, y_{m+1}\}$ satisfies conditions (i), (ii) and (iii). This contradicts the maximality of C_3. Using the same argument with the roles of C_3 and C_4 reversed we can conclude that either $i+1=m+1$ and $j+1=p+1$ or $i+1<m+1$ and $j+1<p+1$. Now if the latter condition holds we may reason as above and use the additional facts that $y_{i+2} \neq y_i = z_j$ to conclude that $y_{i+2} = z_{j+2}$. And again either $i+2=m+1$ and $j+2=p+1$ or $i+2<m+1$ and $j+2<p+1$. The latter condition leads to another step in this process and since α is finite the process must stop. Hence $i+k=m+1$ and $j+k=p+1$ for some k.

Now start the above process going the other way. That is, consider y_{i-1}. It is easily seen that without renumbering again we must have $y_{i-1} = z_{j-1}$. Continuing as far as we can we discover that $i-l=0$ and $j-l=0$ for some l. Thus $i=l=j$ and from above $m+1= i+k=j+k=p+1$. Hence $y_q = z_q$ for $1 \leqq q \leqq m=p$ and so $C_3 = C_4$ a contradiction.

Finally consider Case 3—$C_3 \cap C_4 = \varnothing$. By Lemma 2.6 there is a simple R_α-chain C_8 from y_1 to z_1. Let y_r be the last element of C_8 in C_3 and z_s the first element of C_8 following y_r and in C_4. In order not to contradict condition (iii) for either C_3 or C_4 we must have $r=1$ or m and $s=1$ or p. By renumbering if necessary we may assume $r=m$ and $s=1$. Let $C_9 = t_0, t_1, \cdots, t_v$ be that portion of C_8 from y_r to z_s. Then $C_{10} = y_1, y_2, \cdots, y_m = y_r = t_0, t_1, \cdots, t_v = z_s = z_1, z_2, \cdots, z_p$ is a simple R_α-chain of length at least $4n+1$. It follows from Lemma 4.4, $m \geqq 4n+1$ and $p \geqq 4n+1$ that $sp_{R_\alpha} t_l = 2$ for $0 \leqq l \leqq v$. Thus $C_{11} = \{y_1, y_2, \cdots, y_m, t_1, \cdots, t_{v-1}, z_1, \cdots, z_p\}$ satisfies conditions (i), (ii), (iii) and contradicts the maximality of C_3.

Since all cases lead to contradictions we conclude that there is no $C_4 \neq C_3$ satisfying conditions (i), (ii), (iii) and (iv).

The following corollary follows readily from the proof of Lemma 5.1.

COROLLARY 5.2. *If* $\alpha \in \mathscr{F}$ *and* $\alpha \geqq n^{10n+1}+1$ *then* ${}^{\#}C(\alpha) \geqq 6n+1$.

LEMMA 5.3. *If* $\alpha, \beta \in F$ *and* ${}^{\#}\alpha$, ${}^{\#}\beta \geqq n^{8n+1}+1$ *then* $C(\alpha)=\alpha$ *if and only if* $C(\beta)=\beta$.

Proof. It is sufficient to prove the lemma in the special case $\alpha \subset \beta$, for the general case then follows by applying the special case to α, $\alpha \cup \beta$, and $\alpha \cup \beta$, β. Now since β can be obtained from α by adjoining the elements of $\beta - \alpha$ one at a time we may further assume that $\beta - \alpha = \{z\}$.

First assume $C(\beta)=\beta$. The conclusion $C(\alpha)=\alpha$ follows readily from the definitions and the fact that $R_\alpha = R_\beta^\alpha$.

Now assume $C(\alpha)=\alpha$. Let $\alpha = x_0, x_1, x_2, \cdots, x_m$ with $x_0 = x_m$,

$x_iR_\alpha x_{i+1}$ for $0 \leqq i \leqq m-1$ and $sp_{R_\alpha}x_i = 2$ for all i, i.e., $x_iR_\alpha x_{i+1}$, $0 \leqq i \leqq m-1$ are the only R_α relations to hold. (Such a representation of α is arrived at by writing $\alpha = C(\alpha) = \{x_1, x_2, \cdots, x_m\}$ and nothing that $sp_{R_\alpha}x_i = 2$ all i leads to $x_mR_\alpha x_1$.) Since β is R_β connected (see Lemma 2.6) we must have $zR_\beta x_j$ for some j, $1 \leqq j \leqq m$. Renumbering α if necessary we may assume $j = 2n+1$. If $x_iR_\beta x_{i+1}$ for all i, $0 \leqq i \leqq m-1$, then $sp_{R_\beta}x_j \geqq 3$ and Lemma 4.4 is contradicted since $j = 2n+1$. Thus $x_iR'_\beta x_{i+1}$ for some i, $0 \leqq i \leqq m-1$. Let i_0 be any such i, i.e., $x_{i_0}R'_\beta x_{i_0+1}$. Consequently, by Lemma 2.1 $x_{i_0}R_\beta z$ and $zR_\beta x_{i_0+1}$. Since $x_{i_0}R_\beta z$, $zR_\beta x_j$ and $R_\alpha = R_\beta^\alpha$ we can conclude that $i_0 \in \{j-1, j, j+1\}$. Similarly $i_0 + 1 \in \{j-1, j, j+1\}$ and therefore $i_0 = j-1$ or j. We have thus shown that $x_iR_\beta x_{i+1}$ for all $i \neq j-1$ or j and yet $x_iR'_\beta x_{i+1}$ for some i. We have two cases. Case 1. $x_{j-1}R'_\beta x_j$ and $x_jR'_\beta x_{j+1}$. In this case we can conclude as above that $x_{j-1}R_\beta z$ and $zR_\beta x_{j+1}$. Combining this with $zR_\beta x_j$ we have $sp_{R_\beta}z \geqq 3$ and thus Lemma 4.4 is contradicted. Thus we are left with Case 2: $x_{j-1}R'_\beta x_j$ or $x_jR'_\beta x_{j+1}$ holds but not both. For definiteness we will assume $x_{j-1}R'_\beta x_j$ and $x_jR_\beta x_{j+1}$. Again we can conclude that $zR_\beta x_{j-1}$. We cannot have $zR_\beta x_i$ for $i \notin \{j-1, j\}$ for if we did then $zR_\beta x_{j-1}$ and $zR_\beta x_j$ would imply $x_iR_\alpha x_{j-1}$ and $x_iR_\alpha x_j$ which is impossible. (Note that $m \geqq 4n+1 \geqq 9$ and thus $x_{j+1} = x_{2n+2} \neq x_{2n-1} = x_{j-2}$.) It follows from Lemma 3.2 that $x_iR'_\beta x_k$ for all i, k, $0 < i, k \leqq m$, $|i-k| \neq 1, m-1$. We have thus determined R_β completely and it is easy to see that $C(\beta) = \beta$. In fact β is the simple closed (circular) R_β-chain $x_0, x_1, \cdots, x_{j-1}, z, x_jx_{j+1} \cdots x_m$. This completes the proof.

6. Let $\mathscr{F}' := \{\alpha \in \mathscr{F} \mid {}^\#\alpha \geqq n^{10n+1} + 1\}$. Lemma 5.3 implies $C(\alpha) = \alpha$ for all $\alpha \in \mathscr{F}'$ or $C(\alpha) \neq \alpha$ for all $\alpha \in \mathscr{F}'$. We will call X *circular* or *noncircular* according to whether the first or second possibility holds. In §'s 6 through 11 we will consider the noncircular case exclusively. *Thus in* §'s 6 *through* 11 *we assume X is noncircular, i.e,. $C(\alpha) \neq \alpha$ for all $\alpha \in \mathscr{F}'$.*

DEFINITION 6.1. Let $\alpha \in \mathscr{F}'$. If $(x_1, \cdots, x_m)$ is an m-tuple such that $m = {}^\#C(\alpha)$, $C(\alpha) = \{x_1, \cdots, x_m\}$ and $x_iR_\alpha x_{i+1}$ for $1 \leqq i \leqq m-1$, then $(x_1, \cdots, x_m)$ is called a *presentation* of $C(\alpha)$ (in symbols $C(\alpha) \sim (x_1, \cdots, x_m)$). Condition (i) of Lemma 5.1 implies that $C(\alpha)$ always has at least one presentation for $\alpha \in \mathscr{F}'$. It also follows from Lemma 5.1 that if $(x_1, \cdots, x_m)$ is a presentation for $C(\alpha)$, $\alpha \in \mathscr{F}'$, then $sp_{R_\alpha}x_i = 2$ and consequently $x_iR'_\alpha x_j$ when $|i-j| \neq 1$ and $\{i, j\} \neq \{1, m\}$.

LEMMA 6.2. *If $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_p)$ are two presentations of $C(\alpha)$, $\alpha \in \mathscr{F}'$, then $m = p$ and either $x_i = y_i$ for $1 \leqq i \leqq m$ or $x_i = y_{m-i+1}$ for $1 \leqq i \leqq m$.*

Proof. First, $m = {}^{\#}C(\alpha) = p$. Next, we claim that $x_1 R'_\alpha x_m$. Assume the contrary, $x_1 R_\alpha x_m$. Since $C(\alpha) \neq \alpha$ there is a $z \in \alpha - C(\alpha)$. By Lemma 2.6 there must be an R_α-chain C_1 from z to x_1. Let x_i be the first element of $C(\alpha)$ on C_1 and t the preceding element of C_1. Then $t \in \alpha - C(\alpha)$ and $t R_\alpha x_i$. Consequently $sp_{R_\alpha} x_i \geqq 3$ which contradicts condition (iii) for $C(\alpha)$. Thus $x_1 R'_\alpha x_m$ and similarly $y_1 R'_\alpha y_m$.

Now note that $\{x_1, \cdots, x_m\} = C(\alpha) = \{y_1, \cdots, y_m\}$ and that x_1 and x_m are distinguished from all other elements of $C(\alpha)$ by the fact that $\{z \in C(\alpha) \mid z R_\alpha x_1\}$ and $\{z \in C(\alpha) \mid z R_\alpha x_m\}$ are singletons, ($\{x_2\}$ and $\{x_{m-1}\}$ respectively), whereas $\{z \in C(\alpha) \mid z R_\alpha x_i\} = \{x_{i-1}, x_{1+1}\}$ a set with two elements provided $x_i \neq x_1, x_m$. The same thing holds for y_1 and y_m and thus $\{x_1, x_m\} = \{y_1, y_m\}$. Stated briefly,

$$\{x_1, x_m\} = \{t \in C(\alpha) \mid {}^{\#}\{z \in C(\alpha) \mid z R_\alpha t\} = 1\} = \{y_1, y_2\} .$$

Case 1: $x_1 = y_1$ and $x_m = y_m$. Case 2: $x_1 = y_m$ and $x_m = y_1$. Consider Case 1: $x_1 = y_1$ and $x_m = y_m$. x_2 is the unique element z of $C(\alpha)$ such that $x_1 R_\alpha z$. But this is also true of y_2, and thus $x_2 = y_2$. x_3 is the unique element of $C(\alpha)$ different from $x_1 = y_1$ such that $x_2 R_\alpha x_3$. But this is also true of y_3, and thus $x_3 = y_3$. Proceeding in this way we arrive at the desired conclusion, $x_i = y_i$ for $1 \leqq i \leqq m$. Case 2 can be reduced to Case 1 by renumbering the y_i's in the reverse order ($y'_i = y_{m-i+1}$). We then conclude that $x_i = y'_i = y_{m-i+1}$ as we wished.

In the first paragraph of the proof of Lemma 6.2 we proved the following result.

LEMMA 6.3. *If $C(\alpha) \sim (x_1, \cdots, x_m)$, $\alpha \in \mathscr{F}'$ then $x_1 R'_\alpha x_m$.*

DEFINITION 6.4. Let $\alpha \in \mathscr{F}'$ and $C(\alpha) \sim (x_1, \cdots, x_m)$. Since $sp_{R\alpha} x_1 = 2$ and x_2 is the only element t_1 of $C(\alpha)$ such that $x_1 R_\alpha t_1$ there must be a unique $t \in \alpha - C(\alpha)$ such that $x_1 R_\alpha t$. Designate this t by x_0. Similarly, let x_{m+1} be the unique $t_2 \in \alpha - C(\alpha)$ such that $x_m R_\alpha t_2$. Note that the definitions of x_0 and x_{m+1} depend on α and the presentation $(x_1, \cdots, x_m)$ of $C(\alpha)$. Taking into account Lemma 6.2 we see that if the presentation of $C(\alpha)$ is changed from one of the two possibilities to the other, then x_0 and x_{m+1} simply interchange places. We will use the following notations:

$$\begin{aligned} \mathscr{L}(\alpha) &= \{x_0, x_{m+1}\} \\ \mathscr{E}(\alpha) &= \alpha - \{x_0, x_1, \cdots, x_m, x_{m+1}\} \\ &= \alpha - (C(\alpha) \cup \mathscr{L}(\alpha)) . \end{aligned}$$

Thus α is the disjoint union of $C(\alpha)$, $\mathscr{L}(\alpha)$ and $\mathscr{E}(\alpha)$.

LEMMA 6.5. *Let* $\alpha \in \mathscr{F}'$ *and* $C(\alpha) \sim (x_1, \cdots, x_m)$. *If* $x_i R_\alpha x_j$ *and* $0 \leqq i, j \leqq m+1$ *then* $|i - j| = 1$.

Proof. Since $x_i R_\alpha x_{i+1}$ for $i = 0, 1, \cdots, m$ and $sp_{R_\alpha} x_i = 2$ for $1 \leqq i \leqq m$ the conclusion clearly holds for all i and j such that $\{i,, j\} \cap \{0, m+1\} = \emptyset$. Next we shall show that $x_0 \neq x_{m+1}$. Suppose the contrary, $x_0 = x_{m+1}$. Using already familiar techniques and Lemmas 2.6 and 4.4 it is easily shown that $\alpha - C(\alpha) = \emptyset$, i.e., $\alpha = C(\alpha)$. But then X would be circular contrary to our assumption. Thus $x_0 \neq x_{m+1}$. The same line of reasoning shows that $x_0 R'_\alpha x_{m+1}$. From $sp_{R_\alpha} x_i = 2$ for $1 \leqq i \leqq m$ it now follows that $x_0 R_\alpha x_i$ and $1 \leqq i \leqq m$ implies that $i = 1$. Similarly $x_{m+1} R_\alpha x_i$ and $1 \leqq i \leqq m$ implies $i = m$. Thus we have established the conclusion in all but one case, $\{i, j\} = \{0, m+1\}$. But $x_0 R'_\alpha x_{m+1}$ (see above) and so in this case the lemma is vacuously true. The lemma is thus proved.

7. In the proof of Lemma 7.1 we go into a fairly complete analysis of the structure of R_β in terms of that of R_α when $\alpha \subset \beta$, $\beta - \alpha = \{z\}$, and $\alpha, \beta \in \mathscr{F}'$. We will have several occasions to refer back to this analysis.

LEMMA 7.1. *If* $\alpha, \beta \in \mathscr{F}'$ *and* $\alpha \subset \beta$ *then* $\mathscr{E}(\alpha) \subset \mathscr{E}(\beta)$.

Proof. It is sufficient to consider the case where $\beta - \alpha = \{z\}$. In the following discussions it will be important to remember that due to $R_\alpha = R_\beta^\alpha$ we have $x R_\alpha y$ if and only if either $x R_\beta y$ or $x R_\beta z$ and $z R_\beta y$. Let $C(\alpha) \sim (x_1, \cdots, x_m)$.

We will consider two cases.

Case 1: $z R'_\beta x_i$ for $i = 1, \cdots, m$. In this case we must have $x_i R_\beta x_{i+1}$ for $i = 0, 1, \cdots, m$, and $x_i R'_\beta t$ for $i = 1, \cdots, m$ and $t \notin \{x_{i-1}, x_{i+1}\}$. Thus $C(\alpha)$ is an R_β-chain at least $4n+1$ long such that $sp_{R_\beta} t = 2$ for each $t \in C(\alpha)$. Since $C(\alpha)$ can be extended to a maximal such set in β we can conclude that $C(\alpha) \subset C(\beta)$. Let $C(\beta) \sim (y_1, \cdots, y_p)$. It is clear that with the proper choice of presentation for $C(\beta)$ we may write $y_{j+i} = x_i$ for $i = 1, \cdots, m$ where $0 \leqq j \leqq p - m$. (see proof of Lemma 6.2) It follows that $x_0 = y_j$ and $x_{m+1} = y_{j+m+1}$.

We now claim that $j = 0$ or 1 and if $j = 1$ then $z = y_0$. In order not to contradict the maximality of $C(\alpha)$ we must have $sp_{R_\beta} x_0 \neq 2$. We distinguish two cases: Case 1a, $sp_{R_\alpha} x_0 = 1$; Case 1b, $sp_{R_\alpha} x_0 \geqq 3$.

Consider Case 1a, $sp_{R_\alpha} x_0 = 1$. Assume $j > 0$. Then $sp_{R_\beta} x_0 = sp_{R_\beta} y_j = 2$. Since $x_0 R_\beta x_i$ only for $i = 1$ we must have $x_0 R_\beta t$ for some $t \notin \{x_0, x_1, \cdots, x_m, x_{m+1}\}$. If $t \in \alpha$ then $x_0 R_\alpha t$ which is impossible and so $t = z$. We assert that $sp_{R_\beta} z = 1$. Suppose the contrary. Then $z R_\beta s$ for some $s \neq x_0$. Since we are considering Case 1 we have assumed $zR'_\beta x_i$ for $1 \leqq i \leqq m$. Thus we can conclude that $s \neq x_1$. From $x_0 R_\beta z$ and $zR_\beta s$ we have $x_0 R_\alpha s$ which contradicts $sp_{R_\alpha} x_0 = 1$. Since $zR_\beta y_j$, $x_0 = y_j$, and $z \neq y_{j+1} = x_1 \in \alpha$ we must have $z = y_{j-1}$. It follows that $j - 1 = 0$ and $z = y_0$ as we claimed. Thus the claim is established for Case 1a.

Case 1b, $sp_{R_\alpha} x_0 \geqq 3$. Assume $j > 0$. Now there must be distinct elements $t_1, t_2 \in \alpha - \{x_0, x_1, \cdots, x_m, x_{m+1}\}$ such that $t_1 R_\alpha x_0$ and $t_2 R_\alpha x_0$. Since $sp_{R_\beta} x_0 = sp_{R_\beta} y_j = 2$ and $x_0 R_\beta x_1$ we must have $t_i R'_\beta x_0$ for $i = 1$ or 2, say for $i = 1$. Then $t_1 R_\beta z$ and $zR_\beta x_0$. Hence $t_2 R'_\beta x_0$ $(sp_{R_\beta} x_0 = 2)$ and consequently $t_2 R_\beta z$ and $zR_\beta x_0$. Thus $sp_{R_\beta} z \geqq 3$, $(t_i \neq x_0$ since $t_i R_\alpha x_0)$. Since $zR_\beta y_j$, $(x_0 = y_j)$, z must be y_{j-1}. It then follows that $j - 1 = 0$ and so the claim has been established for Case 1b. This completes the proof of the claim.

Using the same arguments (or just renumbering the x_i and y_i backwards) one may show that $j + m + 1 = p$ or $p + 1$ and if $j + m + 1 = p$ then $z = y_{p+1}$. In all the above eventualities we never have an element of $\mathscr{E}(\alpha) = \alpha - \{x_0, x_1, \cdots, x_m, x_{m+1}\}$ become an element of $\{y_0, y_1, \cdots, y_p, y_{p+1}\}$ as we go from α to β. Thus $\mathscr{E}(\alpha) \subset \mathscr{E}(\beta) = \beta - \{y_0, y_1, \cdots, y_y, y_{p+1}\}$ as we wished to show. This proves Case 1.

Case 2. $zR_\beta x_i$ for some i, $1 \leqq i \leqq m$.

Case 2a. $zR'_\beta y$ for all $y \neq x_i$ $y \in \beta$. In this case $t, s \in \alpha$ and $tR_\alpha s$ implies $tR_\beta s$. Thus $C(\alpha) \sim (x_1, \cdots, x_m)$ is a simple R_β-chain. From Corollary 5.2 we have $m = C(\alpha) \geqq 6n + 1$. Lemma 4.4 implies $i \notin (2n + 1, 2n + 2, \cdots, m - 2n\}$. We may assume without loss of generality that $m - 2n < i \leqq m$. Then $C_1 = \{x_1, \cdots, x_{i-1}\}$ satisfies conditions (i), (ii) and (iii) of Lemma 5.1 with respect to β and from $sp_{R_\alpha} x_0 = sp_{R_\beta} x_0 \neq 2$, $x_0 R_\beta x_1$, $sp_{R_\beta} x_i = 3$ and $x_{i-1} R_\beta x_i$ it is easily seen that C_1 is maximal. Thus $C_1 = C(\beta)$, $C_1 \sim (y_1, \cdots, y_{i-1})$ with $x_j = y_j$ for $1 \leqq j < i$, $y_0 = x_0$, and $y_i = x_i$. It follows that

$$\mathscr{E}(\alpha) = \alpha - \{x_0, x_1, \cdots, x_m, x_{m+1}\} \subset \beta - \{y_0, y_1, \cdots, y_{i-1}, y_i\} = \mathscr{E}(\beta) .$$

This completes the proof of Case 2a.

Case 2b. $zR_\beta y$ for some $y \neq x_i$. Then $x_i R_\alpha y$ and so $y = x_{i-1}$ or $y = x_{i+1}$. If both relations $zR_\beta x_{i-1}$ and $zR_\beta x_{i+1}$ held then $x_{i-1} R_\alpha x_{i+1}$ would hold, which is impossible. Thus just one holds.

We distinguish Case 2b (i) $zR_\beta x_{i-1}$ and Case 2b (ii) $zR_\beta x_{i+1}$. Consider Case 2b (i) $zR_\beta x_{i-1}$. Since $zR'_\beta x$ for all $x \in \beta$ except $x = x_{i-1}$ or x_i it follows that, provided $x, y \in \alpha$ and $\{x, y\} \neq \{x_{i-1}, x_i\}$, we have $xR_\alpha y$ if and only if $xR_\beta y$. If $x_{i-1} R'_\beta x_i$ then clearly

$$C(\beta) \sim (x_1, \cdots, x_{i-1}, z, x_i, x_{i+1}, \cdots, x_m)$$

and $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$. Thus $\mathscr{E}(\alpha) = \mathscr{E}(\beta)$ and hence $\mathscr{E}(\alpha) \subset \mathscr{E}(\beta)$ as desired. On the other hand if $x_{i-1} R_\beta x_i$ then $x_1, \cdots, x_m$ is a simple

R_β-chain and from Lemma 4.4 we can conclude that

$$i \notin \{2n+1, 2n+2, \cdots, m-2n\} .$$

Assume $i \geqq m - 2n + 1$. Then as in Case 2a we can conclude that $C(\beta) \sim (x_1, \cdots, x_{i-2})$. It follows that $\mathscr{E}(\alpha) \subset \mathscr{E}(\beta)$ as we wished. This proves Case 2b (i). The proof of Case 2b (ii) is very similar and thus left to the reader. Thus we have shown that $\mathscr{E}(\alpha) \subset \mathscr{E}(\beta)$ is all cases and the lemma is established.

With the notation as in Lemma 7.1 and $\beta - \alpha = \{z\}$, $C(\alpha) \sim (x_1, \cdots, x_m)$ we list possible presentations $(y_1, \cdots, y_p)$ for $C(\beta)$ occuring in all the various cases.

Case 1. $C(\beta) \sim (x_0, x_1, \cdots, x_m)$
$C(\beta) \sim (x_1, \cdots, x_m, x_{m+1})$ and $z = y_{p+1}$
$C(\beta) \sim (x_0, x_1, \cdots, x_m)$ and $z = y_0$
$C(\beta) \sim (x_0, \cdots, x_m, x_{m+1})$ and $y_0 = z = y_{p+1}$.
(This case is impossible by Lemma 6.5.)

Case 2. $C(\beta) \sim (x_1, \cdots, x_j)$ some j, $m - 2n - 1 \leqq j < m$
$C(\beta) \sim (x_j, \cdots, x_m)$ some j, $1 \leqq j \leqq m$
$C(\beta) \sim (x_1, \cdots, x_i, z, x_{i+1}, \cdots, x_m)$ some

$$i, 0 \leqq i \leqq m .$$

8. Lemma 8.1. *X is infinite.*

Proof. Since $X^n - GD_n$ is disconnected, $X^n - GD_n \neq 0$. Because $n \geqq 2$, X has at least two distinct elements. Now using the fact that X is a T_1-space we see that if X were finite it would be disconnected. But X is assumed connected and hence X is infinite.

Remark. This lemma and its proof obviously hold in general, not just the noncircular case.

Lemma 8.2. *If $\alpha \in \mathscr{F}'$, $x \in \mathscr{E}(\alpha)$ and $C(\alpha) \sim (x_1, \cdots, x_m)$ then*

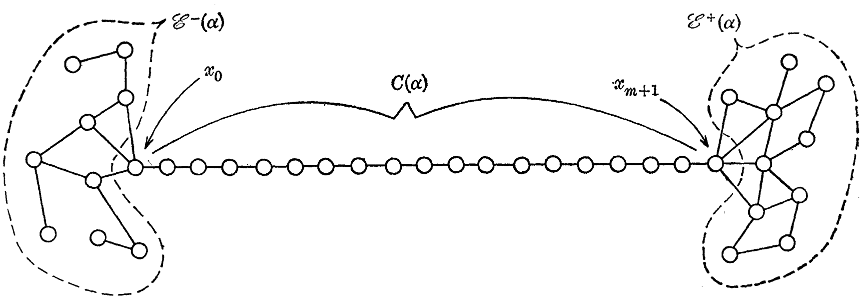

Figure 5

either there is a simple R_α-chain C not intersecting $C(\alpha)$, $C \cap C(\alpha) = \emptyset$, from x to x_0 or from x to x_{m+1} but not both. (see Figure 5.)

Proof. By Lemma 2.6 there is an R_α-chain C_1, which we may assume to be simple, from x to x_0. Let x_i be the first element of $\{x_0, x_1, \cdots, x_m, x_{m+1}\}$ on the R_α-chain C_1. In order not to contradict $sp_{R_\alpha} x_j = 2$ for $1 \leqq j \leqq m$, we must have $i = 0$ or $m+1$. Thus, that portion of C_1 from x to x_i is the desired simple R_α-chain.

Now if C_2 and C_3 are simple R_α chains from x to x_0 and x_{m+1} respectively each not intersecting $C(\alpha)$ then we may construct from them a simple R_α chain $C_4 = y_1, \cdots, y_p$ from x_{m+1} to x_0 not intersecting $C(\alpha)$. Now if we apply Lemma 4.4 to x_{m+1} and the simple R_α-chain $x_{m-2n}, x_{m-2n+1}, \cdots, x_m, x_{m+1}$ $(= y_1), y_2, \cdots, y_p$ $(= x_0), x_1, \cdots, x_{2n}$ we see that $sp_{R_\alpha} x_{m+1} = 2$. But this contradicts the maximality of $C(\alpha)$. Consequently both C_2 and C_3 cannot exist. This completes the proof.

DEFINITION 8.3. Set $\mathscr{E} = \bigcup_{\alpha \in \mathscr{F}'} \mathscr{E}(\alpha)$.

LEMMA 8.4. *$\mathscr{E}$ is a finite set.*

Proof. In light of the fact that $\mathscr{E}(\alpha) \subset \mathscr{E}(\beta)$ for $\alpha \subset \beta$, α, $\beta \in \mathscr{F}'$, it is sufficient to establish the inequality ${}^{\#}\mathscr{E}(\alpha) \leqq 2n^{4n+1}$ for all $\alpha \in \mathscr{F}'$. Assume the contrary, ${}^{\#}\mathscr{E}(\alpha) > 2n^{4n+1}$ for some $\alpha \in \mathscr{F}'$. Let $C(\alpha) \sim (x_1, \cdots, x_m)$ and consider the sets $A^{(\pm)} = \left\{x \in \mathscr{E}(\alpha) \mid \text{there is a simple } R_\alpha\text{-chain not intersecting } C(\alpha) \text{ from } x \text{ to } \begin{pmatrix} x_0 \\ x_{m+1} \end{pmatrix}\right\}$. According to Lemma 8.2 we have $A^+ \cup A^- = \mathscr{E}(\alpha)$ and $A^+ \cap A^- = \emptyset$. Consequently either ${}^{\#}A^+ > n^{4n+1}$ or ${}^{\#}A^- > n^{4n+1}$. For definiteness we will assume ${}^{\#}A^+ > n^{4n+1}$. Now consider the set $A^+ \cup \{x_0\}$ and the restriction R of the relation R_α to $A^+ \cup \{x_0\}$. It follows from the definition of A^+ that $x\bar{R}y$ for all $x, y \in A$. Thus by Lemma 4.3 $A^+ \cup \{x_0\}$ must contain a simple R-chain $C_1 = y_1, \cdots, y_p$ of length at least $4n+1$. By considering an R-chain C_2 from x_0 to y_1 we can obtain a simple R-chain C_3 (made up of parts of C_1 and C_2) of length at least $2n+1$ and starting from x_0. C_3 is actually an R_α-chain disjoint from $C(\alpha)$ and by combining C_3 and $C(\alpha)$ and using Lemma 4.4 we find that $sp_{R_\alpha} x_0 = 2$. But this contradicts the maximality of $C(\alpha)$. This completes the proof.

DEFINITION 8.5. Set $\mathscr{M} = X - \mathscr{E}$. Note that $C(\alpha) \subset \mathscr{M}$ all $\alpha \in \mathscr{F}'$. Since X is infinite and $\mathscr{E}$ is finite, $\mathscr{M}$ must be infinite. Pick two distinct elements u and v of $\mathscr{M}$ and let them be fixed in all discussions of the noncircular case. Set $\mathscr{G} = \{\alpha \in \mathscr{F}' \mid u, v \in \alpha\}$.

For each $\alpha \in \mathscr{G}$ set $\overline{\mathscr{C}}(\alpha) = C(\alpha) \cup \mathscr{L}(\alpha)$. By a presentation for $\overline{\mathscr{C}}(\alpha)$ (notation: $\overline{\mathscr{C}}(\alpha) \sim (x_0, x_1, \cdots, x_m, x_{m+1})$) we will mean an $m+2$ tuple $(x_0, x_1, \cdots, x_m, x_{m+1})$ such that $C(\alpha) \sim (x_1, \cdots, x_m)$, x_0 and x_{m+1} are as is Definition 6.4 and if $u = x_i$ and $v = x_j$ then $i < j$. It is clear from Lemma 6.2 that for each $\alpha \in \mathscr{G}$ there is exactly one presentation for $\overline{\mathscr{C}}(\alpha)$.

DEFINITION 8.6. If $\alpha \in \mathscr{G}$ and $x \in \overline{\mathscr{C}}(\alpha)$ then set $I_\alpha(x) =$ the unique i such that $\overline{\mathscr{C}}(\alpha) \sim (x_0, \cdots, x_{m+1})$ and $x = x_i$ with $0 \leqq i \leqq m+1$.

DEFINITION 8.7. We define a relation $<$ on $\mathscr{M}$ as follows. If $x, y \in \mathscr{M}$ pick any $\alpha \in \mathscr{G}$ such that $x, y \in \alpha$ and set $x < y$ if and only if $I_\alpha(x) < I_\alpha(y)$. We proceed with the obvious task of showing that $<$ is well defined.

LEMMA 8.8. *For $x, y \in \mathscr{M}$ $x < y$ is well defined.*

Proof. Let $\alpha, \beta \in \mathscr{G}$ and $x, y \in \alpha$, and $x, y \in \beta$. By considering the pairs $\alpha, \alpha \cup \beta$ and $\alpha \cup \beta$, β we can reduce the proof to the case where $\alpha \subset \beta$. Then using induction we can further reduce the proof to the case $\alpha \subset \beta$, $\beta - \alpha = \{z\}$. In the proof of Lemma 7.1 we worked out a presentation for $C(\beta)$ in terms of one for $C(\alpha)$ in each of various cases. The present lemma follows by a direct inspection of these related presentations.

Now that we have seen that the relation $x < y$ is independent of the α used in its definition it is easy to see that $<$ is a total (linear) ordering of $\mathscr{M}$. We now take up the problem of extending $<$ in a natural way to a partial ordering on X.

DEFINITION 8.9. Let $\alpha \in \mathscr{G}$ and $\overline{\mathscr{C}}(\alpha) \sim (x_0, \cdots, x_{m+1})$. Set $\mathscr{E}^{(\pm)}(\alpha) = \Big\{x \in \mathscr{E}(\alpha) \,|\,$ there is a simple R_α-chain not intersecting $C(\alpha)$ from x to $\begin{pmatrix} x_{m+1} \\ x_0 \end{pmatrix}\Big\}$. From Lemma 8.2 we have $\mathscr{E}(\alpha) = \mathscr{E}^+(\alpha) \cup \mathscr{E}^-(\alpha)$ and $\mathscr{E}^+(\alpha) \cap \mathscr{E}^-(\alpha) = \varnothing$. (see Figure 5.)

From the analysis in the proof of Lemma 7.1 and the idea in the proof of Lemma 2.6 we can draw the following conclusion.

LEMMA 8.10. *If $\alpha, \beta \in \mathscr{G}$ and $\alpha \subset \beta$ then $\mathscr{E}^+(\alpha) \subset \mathscr{E}^+(\beta)$ and $\mathscr{E}^-(\alpha) \subset \mathscr{E}^-(\beta)$.*

DEFINITION 8.11. Set $\mathscr{E}^+ = \bigcup_{\alpha \in \mathscr{G}} \mathscr{E}^+(\alpha)$ and $\mathscr{E}^- = \bigcup_{\alpha \in \mathscr{G}} \mathscr{E}^-(\alpha)$.

It follows from $\mathscr{E}(\alpha) = \mathscr{E}^+(\alpha) \cup \mathscr{E}^-(\alpha)$ for $\alpha \in \mathscr{G}$ that $\mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^-$. Also, it is not hard to show from $\mathscr{E}^+(\alpha) \cap \mathscr{E}^-(\alpha) = \varnothing$ for $\alpha \in \mathscr{G}$ and Lemma 8.10 that $\mathscr{E}^+ \cap \mathscr{E}^- = \varnothing$.

DEFINITION 8.12. Extend the definition of $<$ by setting $\mathscr{E}^- < \mathscr{M} < \mathscr{E}^+$, i.e., by setting $y < x$, $x < z$ and $y < z$ for all $y \in \mathscr{E}^-$, $x \in \mathscr{M}$ and $z \in \mathscr{E}^+$. The resulting relation is still antisymmetric and transitive (i.e., a partial ordering) and of course is a total ordering when restricted to $\mathscr{M}$.

9. In this section we relate the ordering $<$ to the topology of X. This leads to a complete determination of all the R_α's, $\alpha \in \mathscr{F}$, in terms of $<$, and sharp bounds on the size of the sets $\mathscr{E}^+$ and $\mathscr{E}^-$.

LEMMA 9.1. *If* $x \in \mathscr{M}$ *then* $\{y \mid y < x\}$ *and* $\{y \mid y > x\}$ *are open subsets of* X.

Proof. Consider first the case where there exists $s, t \in \mathscr{M}$ such that $s < x < t$. Let $\alpha \in \mathscr{G}$ be such that $s, x, t \in \alpha$. Set $\beta = \{s, x, t\}$. Since $R_\beta = R_\alpha^\beta$ it is easy to calculate R_β. The important relation is that $sR'_\beta t$. This means that there are disjoint open subsets U and V of X such that $U \cup V = X - \{x\}$, $s \in U$, and $t \in V$. We will show that $U = \{y \mid y < x\}$ and $V = \{y \mid y > x\}$. First suppose $y \in X$ and $y < x$. Let $\gamma \in \mathscr{G}$ be such that $y, s, x \in \gamma$. Since $y < x$ we must have either $y \in \overline{\mathscr{C}}(\gamma)$ and $I_\gamma(y) < I_\gamma(x)$ or $y \in \mathscr{E}^-(\gamma)$. Also because $s \in \mathscr{M} = \bigcap_{\delta \in \mathscr{G}} \overline{\mathscr{C}}(\delta)$ and $s < x$ we have $s \in \overline{\mathscr{C}}(\gamma)$ and $I_\gamma(s) < I_\gamma(x)$. We now have a clear enough picture of R_γ to partially calculate $R_\sigma = R_\gamma^\sigma$ where $\sigma = \{y, s, x\}$. We get $yR_\sigma s$. This means that y cannot be separated from s in $(X - \sigma) \cup \{y, s\} = X - \{x\}$. This implies $y \in U$. We have thus shown that $\{y \mid y < x\} \subset U$. Similarly $\{y \mid y > x\} \subset V$ and consequently $\{y \mid y < x\} = U$ and $\{y \mid y > x\} = V$ as we wished to show.

Now we will consider the case where x is either an initial or terminal element of $\mathscr{M}$. For definiteness we will assume that x is an initial element, i.e., $y \geqq x$ for all $y \in \mathscr{M}$. Since X is infinite and $\mathscr{E}$ is finite we must have a $t \in \mathscr{M}$ such that $x < t$. If $\{y \mid y < x\} = \varnothing$ then $\{y \mid y < x\}$ and $\{y \mid y > x\} = X - \{x\}$ are clearly open. So assume $\{y \mid y < x\} \neq \varnothing$. Next note that $\{y \mid y < x\} = \mathscr{E}^-$. Let $y < x$ and consider $\psi = \{y, x, t\}$. Let $\alpha \in \mathscr{G}$ be such that $\psi \subset \alpha$, and $\mathscr{E}^-(\alpha)$. Then since $x, t \in \mathscr{M}$ we have $x, t \in \overline{\mathscr{C}}(\alpha) \sim (x_0, \cdots, x_{m+1})$.

Next, from $R_\psi = R_\alpha^\psi$ it follows that $yR'_\psi t$. Thus there exist disjoint open sets U_y and V_y such that $U_y \cup V_y = X - \{x\}$, $y \in U_y$ and $t \in V_y$. As in the first case above we have $\{z \mid z > x\} \subset V_y$. Set $U = \bigcup_{y \in \mathscr{E}^-} U_y$ and $V = \bigcap_{y \in \mathscr{E}^-} V_y$. Then we have $\{z \mid z < x\} = \mathscr{E}^- \subset U$ and

$\{z \mid z > x\} \subset V$. Thus $\{z \mid z < x\} = U$ and $\{z \mid z > x\} = V$ and because $\mathscr{E}^-$ is finite U and V are open. This completes the proof.

LEMMA 9.2. *If $\mathscr{M}$ contains a minimum element then $\mathscr{E}^- = \varnothing$. Similarly, if $\mathscr{M}$ contains a maximum element then $\mathscr{E}^+ = \varnothing$.*

Proof. Suppose $x \in \mathscr{M}$ and $x \leqq y$ for all $y \in \mathscr{M}$. Then $\{y \mid y < x\} = \mathscr{E}^-$ is an open set. $\mathscr{E}^-$ is also finite and thus closed. (X is a T_1-space.) But this implies X is disconnected unless $\mathscr{E}^- = \varnothing$. Thus $\mathscr{E}^- = \varnothing$ as we wish to show. The second statement follows similarly.

LEMMA 9.3. *Let $\alpha \in \mathscr{G}$ and $\overline{\mathscr{C}}(\alpha) \sim (x_0, \cdots, x_{m+1})$. Then $x \in \mathscr{E}^-(\alpha)$ implies $xR_\alpha x_0$, and $x \in \mathscr{E}^+(\alpha)$ implies $x_{m+1}R_\alpha x$.* (see Figure 6.)

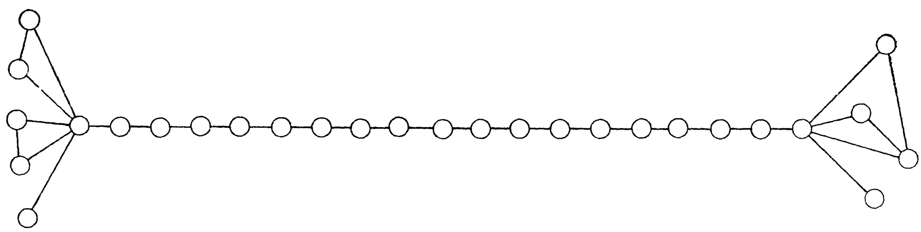

Figure 6

Proof. Suppose $x \in \mathscr{E}^-(\alpha)$ and $xR'_\alpha x_0$. Since in general $x \in \mathscr{E}^-(\alpha)$ implies $xR'_\alpha y$ for all $y \in (\overline{\mathscr{C}}(\alpha) \cup \mathscr{E}^+(\alpha)) - \{x_0\}$ there is no R_α-chain from x to x_0 which does not intersect $\mathscr{E}^-(\alpha) - \{x\}$. Thus $xR'_\varphi x_0$ where $\varphi = \mathscr{E}^-(\alpha) \cup \{x_0\}$. Therefore there are disjoint open sets U and V such that $U \cup V = X - (\mathscr{E}^-(\alpha) - \{x\})$, $x \in U$ and $x_0 \in V$.

We now claim that $\mathscr{M} \subset V$ or $\mathscr{M} \subset U$. Let $t, y \in M$. Set $\beta = \alpha \cup \{t, y\}$. We must have $t, y \in \mathscr{C}(\beta)$ and consequently there is an R_β chain from y to x_0 which does not intersect $\mathscr{E}^-(\beta)\, \mathscr{E}^-(\alpha)$. Set $\gamma = (\mathscr{E}^-(\alpha) - \{x\}) \cup \{t_0, y\}$ and observe that $t_0 R_\gamma y$. Thus x_0 cannot be separated from y in $X - (\mathscr{E}^-(\alpha) - \{x\})$ and so $t, y \in V$ or $t, y \in U$. This shows that $\mathscr{M} \cup \mathscr{E}^+ \subset V$ or $\mathscr{M} \subset U$ as claimed.

Now from $\mathscr{M} \subset V$ it follows that $U \subset \mathscr{E}$. Thus U would be a nonempty open finite set which clearly contradicts the connectedness of X. Similarly, $\mathscr{M} \subset U$ leads to a contradiction. Therefore $xR_\alpha x_0$ as we wished to show. The second statement of the lemma follows similarly.

LEMMA 9.4. *If $\alpha \in \mathscr{G}$ and $x, y \in \mathscr{E}^-(\alpha)$ or $x, y \in \mathscr{E}^+(\alpha)$ then $xR_\alpha y$.* (see Figure 7.)

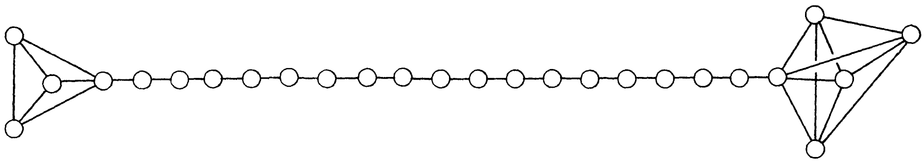

Figure 7

Proof. Let $\alpha \in \mathscr{G}$ and $x, y \in \mathscr{E}^-(\alpha)$, and $\overline{\mathscr{C}}(\alpha) \sim (x_0, \cdots, x_{m+1})$. According to Lemma 9.2 $\mathscr{M}$ cannot contain a minimum element. Thus there is a $z \in \mathscr{M}$ such that $z < x_0$. Set $\beta = \alpha \cup \{z\}$ and note that $z \in \overline{\mathscr{C}}(\beta)$, $\mathscr{E}^-(\alpha) \subset \mathscr{E}^-(\beta)$, and $\mathscr{E}^+(\alpha) \subset \mathscr{E}^+(\beta)$. It follows that $\overline{\mathscr{C}}(\beta) \subset \overline{\mathscr{C}}(\alpha) \cup \{z\}$ and since $z < x_0 \leqq t$ for all $t \in \overline{\mathscr{C}}(\alpha)$ we must have $I_\beta(z) \leqq I_\beta(t)$ for all $t \in \overline{\mathscr{C}}(\beta)$. Thus $z = y_0$ where $\overline{\mathscr{C}}(\beta) \sim (y_0, \cdots, y_{p+1})$. Now from Lemma 9.3 we have $xR_\beta z$ and $yR_\beta z$. It follows immediately that $xR_\alpha y$. The case where $x, y \in \mathscr{E}^+(\alpha)$ is completely analogous. This completes the proof.

LEMMA 9.5. *R_α for $\alpha \in \mathscr{F}$ is completely determined by $<$. In fact for $x, y \in \alpha$, $xR_\alpha y$ if and only if $x \neq y$ and there does not exist a $z \in \alpha$ such that either $x < z < y$ or $y < z < x$.* (see Figure 8.)

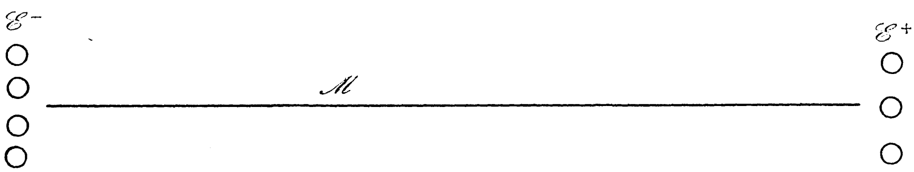

Figure 8

Proof. First consider an $\alpha \in \mathscr{G}$ such that $\mathscr{E}^- = \mathscr{E}^-(\alpha)$ and $\mathscr{E}^+ = \mathscr{E}^+(\alpha)$. We have a complete description of R_α. Indeed, α is the disjoint union of $\mathscr{E}^-(\alpha)$, $\overline{\mathscr{C}}(\alpha)$ and $\mathscr{E}^+(\alpha)$. For $x, y \in \alpha$ we have $xR_\alpha y$ if and only if $x \neq y$ and either (1) $x, y \in \mathscr{E}^-(\alpha)$, (2) $x, y \in \mathscr{E}^+(\alpha)$, (3) $x, y \in \overline{\mathscr{C}}(\alpha)$ and $|I_\alpha(x) - I_\alpha(y)| = 1$ (Lemma 6.5), (4) $x_0 \in \{x, y\}$ and $\{x, y\} \cap \mathscr{E}^-(\alpha) \neq \varnothing$ where $\overline{\mathscr{C}}(\alpha) \sim (x_0, \cdots, x_{m+1})$, or (5) $x_{m+1} \in \{x, y\}$ and $\{x, y\} \cap \mathscr{E}^+(\alpha) \neq \varnothing$.

It follows from $\mathscr{E}^- = \mathscr{E}^-(\alpha)$ and $\mathscr{E}^+ = \mathscr{E}^+(\alpha)$ that $\overline{\mathscr{C}}(\alpha) \subset \mathscr{M}$. Now recalling the definition of $<$ (Definitions 8.7 and 8.12) we see that the lemma holds for α.

Next consider an arbitrary $\beta \in \mathscr{F}$. Pick an $\alpha \in \mathscr{G}$ such that $\mathscr{E}^- = \mathscr{E}^-(\alpha)$, $\mathscr{E}^+ = \mathscr{E}^+(\alpha)$ and $\beta \subset \alpha$. The lemma now follows for β from the equation $R_\beta = R_\alpha^\beta$.

LEMMA 9.6. *${}^\#\mathscr{E} \leqq n$, ${}^\#\mathscr{E}^+ < n$ and ${}^\#\mathscr{E}^- < n$.*

Proof. We will rely heavily on the visual method introduced in (3.5). Consider the claim ${}^\#\mathscr{E} \leqq n$. Suppose ${}^\#\mathscr{E} \geqq n + 1$. Set

${}^{\#}\mathscr{E}^- = N$ and ${}^{\#}\mathscr{E}^+ = M$. Let $\alpha \in \mathscr{G}$ such that $\mathscr{E}^- = \mathscr{E}^-(\alpha)$ and $\mathscr{E}^+ = \mathscr{E}^+(\alpha)$. Let

$$\overline{\mathscr{C}}(\alpha) \sim (x_0, x_1, \cdots, x_{m+1})\ .$$

We know that $m \geqq 6n + 1$ (Corollary 5.2). Consider the n marker placed on the "dots" $x_1, x_2, \cdots, x_n$ in their natural order. We wish to rearrange them into an arbitrary permutation of this original placement through allowable changes. (see 3.5.) It is obviously sufficient to show how to interchange an arbitrary pair of markers. Let the markers be called m_1, m_2, etc. Then at the start m_1 is on x_1, m_2 is on x_2, etc. We wish to interchange the positions of m_i and m_j where $i < j$. We distinguish three cases. Case 1. $j \leqq N$, Case 2. $n - M < i$. Case 3. $i \leqq N$ and $n - M < j$. There are no other cases since $N + M > n$. In either Case 1 or Case 2 one simply uses the dots in $\mathscr{E}^-$ or $\mathscr{E}^+$ to perform the desired interchange. The moves are very similar to those in Lemma 4.2 and will be left to the reader.

Now for Case 3. Since $N + M = {}^{\#}\mathscr{E} \geqq n + 1$ there must be a $k \leqq N$ and $n - M < k$. Using Cases 1 and 2 on the pairs m_i, m_k and m_i, m_j and m_k, m_j in that order one can interchange m_i and m_j. Thus the markers can be rearranged into an arbitrary permutation of their original placement through allowable changes. But this contradicts Lemma 3.9. Thus ${}^{\#}\mathscr{E} \leqq n$ as we wished to show. The inequalities ${}^{\#}\mathscr{E}^+ < n$ and ${}^{\#}\mathscr{E}' < n$ follow from the same considerations as in Cases 1 and 2 above.

LEMMA 9.7. *${}^{\#}\mathscr{E}^- \neq 1$ and ${}^{\#}\mathscr{E}^+ \neq 1$.*

Proof. Suppose $\mathscr{E}^- = \{y\}$. Pick an $\alpha \in \mathscr{G}$ such that $\mathscr{E}^- = \mathscr{E}^-(\alpha)$ and $\mathscr{E}^+ = \mathscr{E}^+(\alpha)$. It then follows from Lemma 9.5 that $yR_\alpha x_0$ holds only for $x \in \mathscr{E}^- \cup \{x_1\}$ where $\overline{\mathscr{C}}(\alpha) \sim (x_0, \cdots, x_{m+1})$. Since then $sp_{R_\alpha}(x_0) = 2$ we must have $x_0 \in \mathscr{C}(\alpha)$ which contradicts the definition of x_0. Thus $\mathscr{E}^-$ cannot be a singleton, i.e., ${}^{\#}\mathscr{E}^- \neq 1$. ${}^{\#}\mathscr{E}^+ \neq 1$ follows analogously.

10. In this section we will determine all the connected subsets of X.

LEMMA 10.1. *If $x, y \in \mathscr{M}$ then the sets*

$$\left\{t \mid x \begin{Bmatrix} \leqq \\ \leqq \\ < \\ < \end{Bmatrix} t \begin{Bmatrix} \leqq \\ < \\ \leqq \\ < \end{Bmatrix} y \right\}, \qquad \mathscr{M} \cap \left\{t \mid t \begin{Bmatrix} \leqq \\ < \\ \geqq \\ > \end{Bmatrix} x \right\},$$

and $\mathscr{M}$ *are connected. Furthermore, any connected subset of* $\mathscr{M}$ *is of one of the above forms.*

Proof. We will show that $A = \{t \mid x \leqq t < y\}$ is connected. The other cases are very similar and so will be left to the reader. Suppose A is not connected, that is suppose U and V are open subsets of X such that $U \cup V \supset A$, $U \cap V \cap A = \emptyset$, $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. Let $a \in U \cap A$ and $b \in V \cap A$. We may assume without loss of generality that $a < b$. So we have $x \leqq a < b < y$. Set

$$U' = (U \cap \{t \mid t < b\}) \cup \{t \mid t < a\}$$

and $V' = (V \cap \{t \mid t > a\}) \cup \{t \mid t > b\}$. Then it is easy to see that U' and V' are disjoint nonempty open sets whose union is all of X. But this contradicts the connectivity of X. Thus A is connected.

Next we will show that if A is a connected subset of $\mathscr{M}$ then A is of one of the above forms. First we need to observe that Lemma 9.1 and the connectedness of $\mathscr{M}$ (proved in the above paragraph) imply that if S is a subset of $\mathscr{M}$ with an upper bound a in $\mathscr{M}$ then S has a least upper bound b (notation: $b = \operatorname{lub} S$). For if S had no least upper bound then S could not have a maximum element and so $U = \bigcup_{s \in S}\{t \in \mathscr{M} \mid t < s\}$ and

$$V = \bigcup_{c \text{ an upper bound for } s \text{ in } \mathscr{M}}\{t \in \mathscr{M} \mid c < t\}$$

would be two disjoint nonempty open sets such that $U \cup V = \mathscr{M}$. But this contradicts the connectedness of $\mathscr{M}$. Similarly each subset of $\mathscr{M}$ with a lower bound has a greatest lower bound (glb).

Now A may or may not have a lower or upper bound in $\mathscr{M}$ and should $\operatorname{glb} A$ or $\operatorname{lub} A$ exist, these points may or may not be elements of A. These various possibilities lead directly to the various forms given above. We will consider one typical case. Suppose A has an upper bound but no lower bound and that $x = \operatorname{lub} A \in A$. We claim that $A = \mathscr{M} \cap \{t \mid t \leqq x\}$. Clearly $A \subset \mathscr{M} \cap \{t \mid t \leqq x\}$. Next we will show that $\mathscr{M} \cap \{t \mid t < x\} = \{t \in \mathscr{M} \mid t < x\} \subset A$. Suppose $z \in \mathscr{M}$ and $z < x$. Suppose further that $z \notin A$. Because A has no lower bound $U = \{t \in A \mid t < z\} \neq \emptyset$. Because $x = \operatorname{lub} A$ and

$$z < x,\ V = \{t \in A \mid t > z\} \neq \emptyset\ .$$

Clearly U and V are disjoint open (in the relative topology of A) subsets of A such that $U \cup V = A$. Thus A is not connected contradicting our hypothesis. Therefore

$$\mathscr{M} \cap \{t \mid t < x\} \subset A \subset \mathscr{M} \cap \{t \mid t \leqq x\}\ .$$

Only the fate of x is left to be decided. But $x \in A$ by our hypothesis.

Thus $A = \mathscr{M} \cap \{t \mid t \leqq x\}$ as we wish to prove.

COROLLARY 10.2. *If* $x, y \in X$ *then* $\{t \mid x < t < y\}$, $\{t \mid x < t\}$, $\{t \mid t < x\}$ *are connected.* $\{t \mid x \leqq t\}$ *and* $\{t \mid t \geqq x\}$ *are also connected provided* $x \in \mathscr{M}$.

Proof. We will consider the set $\{t \mid t < x\}$ and leave the others to the reader. It is sufficient to consider the nontrivial case where $\mathscr{E}^- \cap \{t \mid t < x\} \neq \varnothing$. Then $x \notin \mathscr{E}^-$ and x is not a minimum for $\mathscr{M}$ (see Lemma 9.2). Thus $\{t \mid t < x\} \cap \mathscr{M} \neq \varnothing$. Now assume $\{t \mid t < x\}$ is not connected and let U and V be disjoint nonempty open subsets of $\{t \mid t < x\}$ such that $U \cup V = \{t \mid t < x\}$. Then neither U nor V can be completely contained in $\{t \mid t < x\} - (\{t \mid t < x\} \cap \mathscr{M}) = \mathscr{E}^-$ because $\mathscr{E}^-$ is a finite set and X is connected. Thus $U \cap (\{t \mid t < x\} \cap \mathscr{M}) \neq \varnothing$ and $V \cap (\{t \mid t < x\} \cap \mathscr{M}) \neq \varnothing$ and consequently $\{t \mid t < x\} \cap \mathscr{M}$ is not connected. But $\{t \mid t < x\} \cap \mathscr{M}$ is connected when $x \in \mathscr{M}$ by Lemma 10.1 and if $x \in \mathscr{E}^+$ then $\{t \mid t < x\} \cap \mathscr{M} = \mathscr{M}$ and again is connected by Lemma 10.1. Thus $\{t \mid t < x\}$ must be connected.

LEMMA 10.3. *A is a connected subset of X if and only if A is of the form $A = I - E$ where I is one of the sets listed in either Lemma* 10.1 *or Corollary* 10.2 *or is X or is a singleton and $E \subset \mathscr{E}$.*

Proof. First suppose A is a connected subset of X. If $A \cap \mathscr{E} = \varnothing$ the desired conclusion follows from Lemma 10.1. So assume $A \cap \mathscr{E} \neq \varnothing$. Consider the set $B = A \cap \mathscr{M}$. Consider the case where $B = \varnothing$. Then $A \subset \mathscr{E}$ and is thus a finite connected space. Since X is a T_1-space, A is also a T_1-space in its relative topology. Thus A has the discrete topology and because A is connected it must be a singleton. So the lemma holds in this case. Now assume $B \neq \varnothing$. If $B = \mathscr{M}$ then the conclusion is obvious. So assume $x \in \mathscr{M} - B$. Then either $\{t \mid t < x\} \cap A$ or $\{t \mid t > x\} \cap A$ is empty for otherwise A would be disconnected. For definiteness assume $\{t \mid t > x\} \cap A = \varnothing$. Since we assumed at the outset that $A \cap \mathscr{E} \neq \varnothing$ we must now have $A \cap \mathscr{E}^- \neq \varnothing$. Set $b = \operatorname{lub} B$. Clearly $B \supset \{t \mid t < b\} \cap \mathscr{M}$ for otherwise we could disconnect A. Therefore $A = I - E$ where I is either $\{t \mid t < b\}$ or $\{t \mid t \leqq b\}$ and $E = \mathscr{E}^- - A$. This proves the "only if" part of the lemma.

Now assume $A = I - E$ where I is one of the sets listed in either Lemma 10.1 or Corollary 10.2 or is X or is a singleton and $E \subset \mathscr{E}$. If I is a set listed in Lemma 10.1 then $A = I$ and thus connected by Lemma 10.1. If I is a singleton $I - \mathscr{E}$ is a singleton or the empty set and is thus connected. Consider a typical case where I is a set listed in Corollary 10.2. Say $I = \{t \mid t \leqq x\}$ for some

$x \in \mathscr{M}$. Suppose U and V are open sets such that $U \cap V \cap A = \varnothing$, $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, and $U \cup V \supset A$. We may assume without loss of generality that $x \notin U$. Replacing U by $U \cap \{t \mid t < x\}$ if necessary we may assume $U \subset \{t \mid t < x\}$. Set

$$B = A \cap \mathscr{M} = \mathscr{M} \cap \{t \mid t \leqq x\} .$$

We have $x \in V \cap B$ so $V \cap B \neq \varnothing$. We also must have $U \cap B \neq \varnothing$ for otherwise $U \subset \mathscr{E}^-$ and U would be a finite open subset of X which is impossible. But this shows that B is disconnected contradicting Lemma 10.1. Thus A is connected. The other cases can be handled similarly. This completes the proof.

Since the connected subsets of X are determined by a finite number of yes or no choices and at most two choices of points from X we have the following corollary.

COROLLARY 10.4. *The cardinality of the set of all connected subsets of X equals the cardinality of X.*

11. In this section we determine the number of components of $X^m - GD_m$ for all $m \geqq 2$.

DEFINITION 11.1. Let x be a point of a topological space Y. The quasicomponent of x is the set $[x] = \{y \in Y \mid y$ cannot be separated from $x\}$.

LEMMA 11.2. *Suppose $^{\#}\mathscr{E}^- = N$ and $^{\#}\mathscr{E}^+ = M$. Then $X^m - GD_m$ has exactly $m!/(N!M!)$ components provided $m \geqq 2$ and $N + M \leqq m$. If $N + M > m$ then $X^m - GD_m$ is connected.*

Proof. We will first investigate how the symmetric group S_m acts on the set Q of quasicomponents of $X^m - GD_m$. It turns out that Q is in one-to-one correspondence with the left cosets of a certain subgroup G of S_m. We then determine G completely and calculate $^{\#}Q$ by $^{\#}Q = {}^{\#}S_n/{}^{\#}G$. Finally, because the number of quasicomponents turns out to be finite, the quasicomponents of $X^m - GD_m$ are in fact the components of $X^m - GD_m$.

If $y \in X^m$ set $y_i =$ the i^{th} component of y, for $1 \leqq i \leqq m$. Then $y = (y_1, \cdots, y_m)$. For each $\sigma \in S_m$ (S_m considered as the permutation group of the set $\{1, \cdots, m\}$) and $y \in X^m$, define σy by setting $(\sigma y)_i = y_{\sigma^{-1}(i)}$. It follows that $\tau(\sigma y) = (\tau\sigma)y$ for all $\tau, \sigma \in S_m$ and $y \in X^m$. Clearly each $\sigma \in S_m$ considered as a function from X^m into itself takes $X^m - GD_m$ into $X^m - GD_m$ and is a homeomorphism of $X^m - GD_m$ with itself. Hence each σ takes quasicomponents of $X^m - GD_m$ into quasicomponents of $X^m - GD_m$ and thus induces a function $\pi(\sigma)$: $Q \to Q$. We have for each $y \in X^m - GD_m$ and $\sigma \in S_m$ the equation $\pi(\sigma)([y]) = [\sigma y]$.

Now let $y_1, y_2, \cdots, y_m$ be m points of $\mathscr{M}$ such that $y_1 < y_2 < \cdots < y_m$. Set $y = (y_1, \cdots, y_m) \in X^m - GD_m$. Consider the subgroup G of S_m consisting of all $\sigma \in S_m$ such that $\pi(\sigma)[y] = [y]$. It is easily seen that $\pi(\sigma)[y] = \pi(\tau)[y]$ for $\sigma, \tau \in S_m$ if and only if σ and τ are in the same left coset of G. Furthermore, according to Lemma 3.9 for each $q \in Q$ there is some $\sigma \in S_m$ such that $q = \pi(\sigma)[y]$. (Lemma 3.9 is stated for the case where $m = n$ but its proof does not use the assumption that $X^n - GD_n$ is disconnected and hold for any $m \geqq 2$ in place of n.) Thus there is a one-to-one correspondence between Q and the set of left cosets of G. Therefore ${}^{\#}Q = {}^{\#}S_m/{}^{\#}G$.

We will now find G explicitly. We claim that $G = G' \equiv \{\sigma \in S_n \mid \sigma(i) = i$ for $N < i \leqq m - M$, and $\sigma(i) \leqq N$ for $i \leqq N$, and $\sigma(i) > n - M$ for $i > n - M\}$. The argument used in Lemma 9.6, Cases 1 and 2, show that $G' \subset G$. We will now prove the reverse inclusion. To this end, suppose $\sigma \in S_m - G'$. We will show that $\pi(\sigma)[y] \neq [y]$, i.e., $\sigma y S y$ (σy is separated from y in $X^m - GD_m$). Consider the sets $U = \Big\{w \in X^m - GD_m \mid w_i < w_j$ for $i \leqq N$ and $j > m - M$, and $w_i \left\{\begin{matrix} < \\ > \end{matrix}\right\} w_j$ for $N < j \leqq m - M$ and $i \left\{\begin{matrix} < \\ > \end{matrix}\right\} j\Big\}$ and $V = \{w \in X^m - GD_m \mid$ either $w_i > w_j$ for *some* $i \leqq N$ and $j > m - M$ or $w_i > w_j$ for *some* i and j with $N < j \leqq m - M$ and $i < j$ or $w_i < w_j$ for *some* i and j with $N < j \leqq m - M$ and $i > j\}$. Using the fact that X is connected and that $\mathscr{M}$ has no initial (terminal) point if $\mathscr{E}^- \neq \emptyset (\mathscr{E}^+ \neq \emptyset)$ it is not hard to see that U and V are open. They are obviously disjoint and clearly $y \in U$ and $\sigma y \in V$. It is only necessary now to establish that $U \cup V = X^m - GD_m$ to show that $\sigma y S y$. So let $\omega \in X^m - GD_m$ and suppose $\omega \notin U \cup V$. Then from $\omega \notin U$ we can conclude that $\omega_i < \omega_j$ fails for an appropriate pair i, j. Consider a typical case: $i \leqq N$ and $N < j \leqq m - M$. Since $\omega \notin V$ we have that $\omega_j < \omega_i$ fails. Because $\mathscr{M}$ is totally ordered and $\mathscr{E}^- < \mathscr{E}^+$ we must have $\omega_i, \omega_j \in \mathscr{E}^-$ or $\omega_i, \omega_j \in \mathscr{E}^+$. We take the second case leaving the first to the reader. Now since $\omega \notin V$, $\omega_k < \omega_j$ cannot hold for any $k > m - M$. Thus $\omega_k \in \mathscr{E}^+$ for all $k = m - M + 1, m - M + 2 \cdots, m$. Combining this with $\omega_i, \omega_j \in \mathscr{E}^+$ we see that we have $M + 2$ distinct ($\omega \in X^m - GD_m$) elements in a set $\mathscr{E}^+$ of M elements. This is absurd. Thus $\omega \notin U \cup V$ is untenable and so $U \cup V = X^m - GD_m$ as we wished to show. Hence U and V separate y and σy and consequently $\sigma y S y$. Therefore $\sigma \notin G'$ implies $\sigma \notin G$ and this combined with $G' \subset G$ shows that $G = G'$ as claimed.

Now we can calculate ${}^{\#}Q = {}^{\#}S_m/{}^{\#}G$. ${}^{\#}S_m = m!$ and clearly ${}^{\#}G = N!M!$. Thus ${}^{\#}Q = m!/(N!M!)$ and because there are only a finite number of quasicomponents, the quasicomponents coincide with the components and thus the first statement of the lemma is proved. The proof of the second statement is completely analogous to the proof of Lemma 9.6.

REMARK. It is not hard to show that the set $U = [y]$.

12. The circular case. We will now consider the circular case. That is, in this section we assume that $\mathscr{C}(\alpha) = \alpha$ for all $\alpha \in \mathscr{F}'$. Since α is R_α connected (i.e., $x\bar{R}_\alpha y$ for all $x, y \in \alpha$) and $sp_{R_\alpha} x = 2$ for every $x \in \alpha$ it is clear that the network representing R_α (i.e., the network consisting of the points of α as vertices and having a line segment between two points $x, y \in \alpha$ if and only if $xR_\alpha y$) is one simple circular chain. (see Figure 9.) Choose a triplet (u, v, w) of

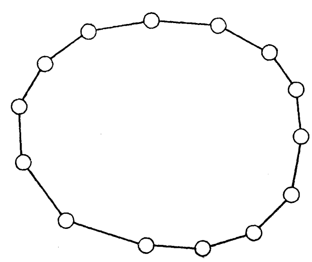

Figure 9

distinct points of X and let it be fixed from now on. (Recall that X is infinite—see Lemma 8.1 and the remark that follows it.) Set $\mathscr{G} = \{\alpha \in \mathscr{F}' \mid u, v, w \in \alpha\}$. It is clear that for each $\alpha \in \mathscr{G}$ there is a unique presentation $(x_1, x_2, \cdots, x_m)$ of $\mathscr{C}(\alpha) = \alpha$ such that $x_1 = u$ and if $v = x_i$ and $w = x_j$ then $i < j$. We change our notation slightly and now write $\alpha \sim (x_1, \cdots, x_m)$ only for the distinguished presentation $(x_1, \cdots, x_m)$ mentioned above. Next, let $\alpha \in \mathscr{G}$ and $x \in \alpha$, and $\alpha \sim (x_1, \cdots, x_m)$. Set $I_\alpha(x) =$ the unique i such that $x = x_i$.

DEFINITION 12.1. Let $x, y \in X$ and pick an $\alpha \in \mathscr{G}$ such that $x, y \in \alpha$. Set $x < y$ if and only if $I_\alpha(x) < I_\alpha(y)$. We need to show that $x < y$ is well defined, i.e., does not depend on choice of α.

LEMMA 12.2. *If $x, y \in X$, $\alpha, \beta \in \mathscr{G}$ and $x, y \in \alpha \cap \beta$ then $I_\alpha(x) < I_\alpha(y)$ if and only if $I_\beta(x) < I_\beta(y)$.*

Proof. As in Lemma 8.8 it is sufficient to consider the case where $\beta - \alpha = \{z\}$ and $\alpha \subset \beta$. From $R_\alpha = R_\beta^\alpha$ it follows that the R_β

network is obtained from the R_α network by removing the link between x_i and x_{i+1} and adding the vertex z along with a link from x_i to z and one from z to x_{i+1}. It is now clear that $I_\alpha(x) < I_\alpha(y)$ if and only if $I_\beta(x) < I_\beta(y)$. This completes the proof.

It is clear that $<$ is a total order on X.

LEMMA 12.3. *If $a, b \in X$ and $a < b$ then $A = \{x \mid a < x < b\}$ and $B = \{x \mid x < a$ or $b < x\}$ are open subsets of X.*

Proof. If either $A = \varnothing$ or $B = \varnothing$ then the other set equals $X - \{a, b\}$ and consequently both A and B would be open. So we may assume there is a $c \in A$ and $d \in B$. Let $\alpha \in \mathscr{G}$ be such that $a, b, c, d \in \alpha$. It is clear from the definition of $<$ that a and b separate c and d in the R_α network. That is in going around α we would come to a then c then b then d then a. It follows that $cR'_\gamma d$ where $\gamma = \{a, b, c, d\}$. Consequently, there exists disjoint open sets U and V such that $c \in U$, $d \in V$ and $U \cup V = X - \{a, b\}$. We claim that $U = A$ and $V = B$. Let $x \in A$. Choose a $\beta \in \mathscr{G}$ such that $\delta = \{a, b, c, x\} \subset \beta$. From $c, x \in A$ it follows that $I_\beta(a) < I_\beta(c)$, $I_\beta(x) < I_\beta(b)$. Consequently $cR_\delta x$ and so c cannot be separated from x in $X - \{a, b\}$. Thus $x \in U$. This shows that $A \subset U$. Similarly $B \subset V$ and it follows that $U = A$ and $V = B$ as claimed. Hence A and B are open and the proof is complete.

LEMMA 12.4. *Let $a, b \in X$ and $a < b$. The following sets are nonempty and connected:*

$$\left\{ x \mid a \left\{\begin{matrix} \leqq \\ \leqq \\ < \\ < \end{matrix}\right\} x \left\{\begin{matrix} \leqq \\ < \\ \leqq \\ < \end{matrix}\right\} b \right\}, \qquad \left\{ x \mid x \left\{\begin{matrix} \leqq \\ \leqq \\ < \\ < \end{matrix}\right\} a \ \text{ or } \ b \left\{\begin{matrix} \leqq \\ < \\ \leqq \\ < \end{matrix}\right\} x \right\}, \qquad \{x \mid x \neq a\}\,.$$

Proof. Consider the sets $A = \{x \mid a < x < b\}$ and $B = \{x \mid x < a$ or $b < x\}$. We will show that $A \neq \varnothing$. Suppose $A = \varnothing$. Then $B \neq \varnothing$ for otherwise $X = \{a, b\}$ contradicting the fact that X is infinite. Let $c \in B$ and assume $c > b$. The case where $c < a$ can be handled similarly. Consider $U = \{x \mid a < x < c)$ and $V = \{x \mid x < b$ or $x > c\}$. The sets U and V are open by Lemma 12.3 and we have $U \cup V = X - \{c\}$. Also $U \cap V \subset A = \varnothing$, $a \in V$ and $b \in U$. By considering an $\alpha \in \mathscr{G}$ such that $\gamma = \{a, b, c\} \subset \alpha$ we see that $aR_\gamma b$ and consequently a cannot be separated from b in $X - \{c\}$. This is a contradiction. Thus $A \neq \varnothing$. Similarly we must have $B \neq \varnothing$. The other sets mentioned in the lemma are nonempty because they contain either A or B.

We will now show A is connected. Suppose W and Z are two open subsets of X such that $W \cap Z \cap A = \emptyset$, $W \cap A \neq \emptyset$, $Z \cap A \neq \emptyset$, $W \cup Z \supset A$. Let $d \in W \cap A$ and $e \in Z \cap A$ and we may assume without loss of generality that $a < d < e < b$. Let $f \in B$. We assume $f < a$. The case $f > b$ can be handled similarly. Now set $W_1 = (\{W \cap \{x \mid a < x < e\}) \cup \{x \mid f < x < d\}$ and

$$Z_1 = (Z \cap \{x \mid d < x < b\}) \cup \{x \mid e < x \text{ or } x < f\} .$$

Then W_1 and Z_1 are open in X,

$$W_1 \cap Z_1 \subset W \cap Z \cap \{x \mid d < x < e\} \subset W \cap Z \cap A = \emptyset ,$$

$d \in W_1$, $e \in Z_1$, and $W_1 \cup Z_1 = X - \{f\}$. Now by considering a $\beta \in \mathscr{G}$ such that $\delta = \{f, d, e\} \subset \beta$ we see that $dR_\delta e$ and so d cannot be separated from e in $X - \{f\}$. We have reached a contradiction. Thus A is connected. A similar argument shows that each of the other sets mentioned in the lemma are connected. This completes the proof.

The proofs of Lemma 10.1 and Corollary 10.4 are easily adapted to prove the following lemma and corollary.

LEMMA 12.5. *If C is a connected subset of X then either C is a singleton or $C = \emptyset$, X or C is of the form of one of the sets listed in Lemma* 12.4.

COROLLARY 12.6. *The cardinality of the set of all connected subsets of X is equal to the cardinality of X.*

LEMMA 12.7. *R_α for $\alpha \in \mathscr{F}$ is completely determined by $<$. In fact, for $x, y \in \alpha$, $x \leqq y$ the relation $xR_\alpha y$ holds if and only if $x \neq y$ and there does not exists z and t elements of α such that $x < z < y$, and either $t < x$ or $y < t$.*

Proof. The conclusion is obvious for $\alpha \in \mathscr{G}$. The conclusion follows for an arbitrary $\beta \in \mathscr{F}$ by picking an $\alpha \in \mathscr{G}$ such that $\beta \subset \alpha$ and then calculating R_β by $R_\beta = R_\alpha^\beta$.

LEMMA 12.8. *$X^m - GD_m$ has exactly $(m-1)!$ components for all $m \geqq 2$.*

Proof. We may proceed exactly as in the proof of Lemma 11.2 except for the determination of G. So we now address ourselves to the determination of G for the circular case.

First we set up a little machinery. For each $\sigma \in S_m$ set

$$A(\sigma) = \{(x_1, \cdots, x_m) \in X^m - GD_m \mid x_{\sigma^{-1}(1)} < x_{\sigma^{-1}(2)} < \cdots < x_{\sigma^{-1}(m)}\} .$$

Let τ be the element of S_m given by $\tau(i) = i + 1$ for $i = 1, \cdots, m - 1$ and $\tau(m) = 1$. Let H be the subgroup of S_m generated by τ, i.e., $H = \{\tau, \tau^2, \cdots, \tau^m = \text{identity}\}$. Now set $U = \bigcup_{\sigma \in H} A(\sigma)$ and $V = \bigcup_{\sigma \notin H} A(\sigma)$.

Now we claim that $G = H$. The inclusion $H \subset G$ follows easily from 3.5. To see the reverse inclusion we will separate y (see proof of Lemma 11.2) from σy for each $\sigma \notin H$. We claim that the sets U and V do separate y from all σy with $\sigma \notin H$. Clearly $y \in U$ and $\sigma y \in V$ for all $\sigma \notin H$. It is also easy to see that $U \cap V = \varnothing$ and $U \cup V = X^m - GD_m$. Finally using Lemmas 12.3 and 12.4 one can readily establish that each point of U or V is an interior point of U or V respectively and thus U and V are open. Therefore U and V produce the desired separation as claimed. Consequently $G \subset H$ and so $G = H$ as claimed. It only remains for us to note that ${}^{\#}G = m$ and so ${}^{\#}Q = {}^{\#}S_m / {}^{\#}G = m!/m = (m - 1)!$.

13. In this section and all the following sections we do not assume a priori that X is circular or noncircular.

This section is devoted to presenting simple characterizations of the circular case, the noncircular case, $\mathscr{M}$, $\mathscr{E}$, $\mathscr{E}^+$, $\mathscr{E}^-$, and $\{x \in X \mid x$ is either a terminal or initial point of $\mathscr{M}$ under $<\}$. (The sets $\mathscr{M}$, $\mathscr{E}$, $\mathscr{E}^+$, and $\mathscr{E}^-$ are of course defined only in the noncircular case.) We also delineate here the nature and number of cut points of X.

LEMMA 13.1. *X is circular or noncircular depending respectively on whether X has none or at least one cut point.*

Proof. This lemma follows readily from Lemmas 9.1 and 12.4 and the fact $\mathscr{M}$ is infinite. This latter fact comes from the observation that X is infinite, $\mathscr{E}$ is finite and $\mathscr{M} = X - \mathscr{E}$.

NOTATION 13.2. Let $\mathscr{N}$ stand for the set of noncut points of X and $T = \{x \mid x$ is either a terminal or initial point of $\mathscr{M}$ under $<\}$.

LEMMA 13.3. *$\mathscr{N} = \mathscr{E} \cup T$ provided X is noncircular.*

Proof. This follows from Lemmas 9.1, 9.2 and 10.3.

We now wish to characterize the sets $\mathscr{E}^+$, $\mathscr{E}^-$ and T.

LEMMA 13.4. *Suppose that x is a cut point of X. Let U and V be nonempty disjoint open sets such that $U \cup V = X - \{x\}$. Set*

$$\bar{U} = \begin{cases} U \cap \mathscr{N} & \text{if } U \cap \mathscr{N} \text{ is not a singleton} \\ \varnothing & \text{if } U \cap \mathscr{N} \text{ is a singleton} \end{cases} .$$

Define $\bar{V}$ similarly. Then

(a) $\{\mathscr{E}^-, \mathscr{E}^+\} = \{\bar{U}, \bar{V}\}$ *and*

(b) $T = \{P \mid \{P\} = U \cap \mathscr{N}$ *or* $\{P\} = V \cap \mathscr{N}\}$.

Proof. It follows from Lemma 10.3 that

$$\{U, V\} = \{\{t \mid t < x\}, \{t \mid t > x\}\} .$$

Now the conclusion follows easily from Definition 8.12 and Lemmas 13.3, 9.2 and 9.7.

With the above characterizations of $\mathscr{E}^-$, $\mathscr{E}^+$ and T in mind, the formulas $\mathscr{M} = X - \mathscr{E}$, $\mathscr{E} = \mathscr{N} - T$, and $\mathscr{E} = \mathscr{E}^- \cup \mathscr{E}^+$ provide the desired characterizations of $\mathscr{M}$ and $\mathscr{E}$.

In the following theorem we state some facts about the cut points of X which follow readily from the theory we have developed but do not involve that theory in their statement.

THEOREM 13.5. *Either X has no cut points or all points of X except for at most n points are cut points. If x is a cut point then $X - \{x\}$ has exactly two components. If X has no cut points and S is a subset of X with exactly m elements, $m \geqq 1$ then $X - S$ has exactly m components. The set of cut points of X is a connected Hausdorff space.*

Proof. The first statement follows easily from Lemmas 13.1, 13.3, 9.2, and 9.6. The second follows from Lemmas 13.3, 9.1, and 10.3. The third conclusion follows from Lemmas 13.1, 12.3, and 12.4. The last statement follows from Lemmas 13.3 and 10.3 (for the connectedness) and Lemma 9.1 (for being Hausdorff).

14. This section is concerned with the concept of local connectivity at a point. Recall that a topological space Y is locally connected at a point p if for each open set U containing p there is a connected open set V such that $p \in V \subset U$. A space is locally connected if it is locally connected at each of its points.

If $<$ is a partial ordering of a set Y then we distinguish two topologies on Y induced by $<$. The first is the linear topology, denoted by $<^l$, and has as a sub-base the sets $\{y \in Y \mid y < b\}$ and

$\{y \in Y \mid y > b\}$ where b is an arbitrary element of Y. The second is the circular topology, denoted by $<^c$, and has as a base the sets $\{y \in Y \mid a < y < b\}$ and $\{y \in Y \mid y < a$ or $b < y\}$ where a and b are arbitrary elements of Y.

Let τ be a topology for Y and $p \in Y$. We will mean by *τ at p* the neighborhood system of τ at p, i.e., the set $\{A \mid p \in \operatorname{int} A\}$. Let τ be the given (original) topology of X.

LEMMA 14.1. *If X is circular then X is locally connected at p if and only if τ at $p = <^c$ at p. If X is noncircular then X is locally connected at $p \in \mathscr{M}$ if and only if τ at $p = <^l$ at p.*

Proof. The key observation is that most intervals with a closed condition at one or both ends, i.e., sets like $\{t \mid a \leqq t < b\}$ are *not* open. We will consider an example to display the technique. Suppose X is noncircular and $a, b \in \mathscr{M}$, $a < b$, a is not an initial point of $\mathscr{M}$, and $A = \{t \mid a \leqq t < b\}$. We will show that A is not open. Assume the contrary, A is open. Then $U = \{t \mid t < a\}$ and $V = A \cup \{t \mid a < t\}$ are disjoint nonempty open sets such that $U \cup V = X$. This is impossible since X is connected and so A is not open. The theorem now follows easily in the circular case from Lemmas 12.3 and 12.5. In the noncircular case we make the observation that the sets $\{x \mid x < a\}$, $\{x \mid x > a\}$, and $\{x \mid a < x < b\}$ where a and b are arbitrary elements of X form a base for $<^l$ and then the theorem follows readily from Lemmas 9.1, 9.2, and 10.3.

LEMMA 14.2. *Suppose X is noncircular and $x, y \in \mathscr{E}^-(\mathscr{E}^+)$. If there are disjoint open sets U and V such that $x \in U$ and $y \in V$ then X is not locally connected at x.*

Proof. Let $c \in \mathscr{M}$ and set $U' = U \cap \{t \mid t < c\}$. Suppose A is a connected open set such that $x \in A \subset U'$. Then from Lemma 10.3 and $A \cap \mathscr{E}^- \neq \varnothing$ we conclude that $A = I - E$ where $I = \{t \mid t < a\}$ or $I = \{t \mid t \leqq a\}$ for some $a \in \mathscr{M}$ and $E \subset \mathscr{E}$. Now since

$$U' \cap V \subset U \cap V = \varnothing$$

we have $V' = V \cap \{t \mid t < a\} \subset \mathscr{E}^-$. Thus V' is a finite nonempty ($y \in V'$) open subset of X which contradicts the connectedness of X. Therefore no such A exists and so X is not locally connected at x.

COUNTER EXAMPLE 14.3. The following example shows that the Hausdorff type of separation assumed in Lemma 14.2 cannot be dropped. Let $X = \{-1\} \cup [0, 1]$ with the topology generated by all sets of the form $\{t \mid t < x\}$ with $x > 0$, and $\{t \mid t < x\} - \{0\}$ with $x > 0$, and $\{t \mid t > x\}$ with x arbitrary. Then X is T_1, connected, $X^n - GD_n$ is disconnected for $n > 2$, $\{-1, 0\} \subset \mathscr{E}^-$ (provided we make the right choice in ordering X) but X is locally connected at -1.

THEOREM 14.4. *If X is locally connected and Hausdorff then there is a total ordering $<$ of X such that*

(a) *if X has any cut points then all points of X are cut points except an initial or terminal point of X under $<$ (which if one of both exist are not cut points), and the topology of $X = <^l$, and*

(b) *if X has no cut points then the topology of $X = <^c$.*

Proof. This theorem follows readily from Lemmas 14.1, 14.2, 9.7, 13.3, and 13.1.

15. In this section we consider the concept of local compactness and obtain results very analogous to those of § 14. Let τ be the given topology of X.

LEMMA 15.1. *If X is circular and locally compact at p then τ at $p = <^c$ at p. If X is noncircular and locally compact at $p \in \mathscr{M}$ then τ at $p = <^l$ at p.* (Note that the implications in Lemma 15.1 are only one way in contrast to the two way implications of Lemma 14.1.)

Proof. Consider the case where X is noncircular and locally compact at $p \in \mathscr{M}$ and p is not an initial or terminal point of $\mathscr{M}$. Let C be a compact subset of X such that $p \in$ interior of $C = \operatorname{int} C$. We wish to show that τ at $p = <^l$ at p. The inclusion $<^l$ at $p \subset \tau$ at p follows immediately from Lemma 9.1. Now let $B \in \tau$ at p. Then $p \in \operatorname{int} B$. We must show that there exists $a, b \in X$ such that $p \in \{t \mid a < t < b\} \subset B$.

Consider the open set $U = \operatorname{int} C \cap \operatorname{int} B$. Then $p \in U$ and it is sufficient to show that $p \in \{t \mid a < t < b\} \subset U$ for some $a, b \in X$.

First we claim that either $\{t \mid z < t < p\} \cap U \neq \varnothing$ for all $z < p$ or $\{t \mid p < t < y\} \cap U \neq \varnothing$ for all $y > p$. Suppose this were not so. Then we would have a $z < p$ and a $y > p$ such that $\{t \mid z < t < y\} \cap U = \{P\}$. We now have a finite open subset of X which is impossible. Thus the claim is established.

For definiteness we will assume $\{t \mid z < t < p\} \cap U \neq \varnothing$ for all $z < p$. We now claim that $\{t \mid z < t < p\} \subset U$ for some $z < p$. Suppose the contrary. Consider the open covering of C consisting of all the sets $\{t \mid t < z\}$ with $z < p$ and the set $U \cup \{t \mid p < t\}$. Since C is compact there is a finite subcovering and we thus conclude that there is a $z_0 < p$ such that $\{t \mid z_0 \leqq t \leqq p\} \cap C \subset U$. By our assumption of the contrary to the claim we know that there must be a z_1 such that $z_0 < z_1 < p$ and $z_1 \notin U$. Next, there must be a z_2 such that $z_1 < z_2 < p$ and $z_2 \in U$. Finally there is a z_3 such that $z_2 < z_3 < p$ and $z_3 \notin U$. It follows that $V = \{t \mid z_1 \leqq t \leqq z_3\} \cap C = \{t \mid z_1 \leqq t \leqq z_3\} \cap U = \{t \mid z_1 \leqq t \leqq z_3\} \cap U$. Thus V is a nonempty open subset of X which does not equal X ($p \notin V$). We now assert that V is also closed. To see this, first note that since $z_0 < z_1 < z_3 < p$ we must have $z_1, z_3 \varepsilon \mathscr{M}$. Thus $\{t \mid z_1 \leqq t \leqq z_3\} = X - (\{t \mid t < z_1\} \cup \{t \mid t > z_3\})$ showing that $\{t \mid z_1 < t < z_3\}$ is closed. V is consequently a closed subset of C and is therefore compact. It follows from Theorem 13.5 that $\{t \mid z_1 \leqq t \leqq z_3\}$ is a Hausdorff space and thus V is a closed subset of $\{t \mid z_1 \leqq t \leqq z_3\}$ in the relative topology of $\{t \mid z_1 \leqq t \leqq z_3\}$. But, since $\{t \mid z_1 \leqq t \leqq z_3\}$ is closed in X, V must be closed in X as we asserted. We have thus contradicted the connectedness of X. Therefore $\{t \mid z < t < p\} \subset U$ for some $z < p$ as claimed above. Let a be such a z.

Next we claim that $\{t \mid p < t < y\} \cap U \neq \varnothing$ for all $y > p$. Assume the contrary. Then $\{t \mid p < t < y_0\} \cap U = \varnothing$ for some $y_0 > p$. Now consider the set $A = \{t \mid a < t < y_0\} \cap U = \{t \mid a < t \leqq p\}$. A is clearly open. Consequently $Z = \{t \mid a < t \leqq p\} \cup \{t \mid t < p\}$ and $W = \{t \mid t > p\}$ are nonempty ($p \in Z$, $y_0 \in W$) disjoint open sets such that $Z \cup W = X$. This is impossible and thus $\{t \mid p < t < y\} \cap U \neq \varnothing$ for all $y > p$ as claimed.

Now we claim that $\{t \mid p < t < y\} \subset U$ for some $y > p$. The proof of this claim is completely analogous to the proof of the second claim above and so will be left to the reader. Let b be such a y. We then have $p \in \{t \mid a < t < b\} \subset U \subset \operatorname{int} B$ as desired.

The cases where X is circular or p is an initial or terminal point of $\mathscr{M}$ can be handled in a manner very similar to the above argument and so will be left to the reader. This completes the proof.

LEMMA 15.2. *Suppose X is noncircular and $p, q \in \mathscr{E}^+(\mathscr{E}^-)$. If there are disjoint open sets A and B such that $p \in A$ and $q \in B$ then X is not locally compact at p.*

Proof. Suppose C is a compact set such that $p \in \operatorname{int} C$. Consider the open set $U = A \cap \operatorname{int} C$. In order to avoid the absurdity of a nonempty finite open subset of X we must admit that $\{t \mid z < t < p\} \cap U \neq \varnothing$ for all $z < p$. Next it follows, just as in the proof of the second claim in the proof of Lemma 15.1, that $\{t \mid a < t < p\} \subset U \subset A$ for some $a < p$. Similarly there must be an $a' < q$ such that $\{t \mid a' < t < q\} \subset B$. Lemma 9.2 implies that $\mathscr{M}$ has no maximum and hence

$$\varnothing \neq \{t \mid a < t < p\} \cap \{t \mid a' < t < q\} \subset A \cap B = \varnothing$$

which is absurd. Therefore no such C exists which shows that X is not locally compact at p.

THEOREM 15.3. *If X is locally compact and Hausdorff then X is locally connected and so the conclusions of Theorem* 14.4 *hold.*

Proof. This follows easily from Lemmas 15.2, 15.1, 14.1, and 14.4 used in that order.

16. In this section we prove the major results of the paper. They are obtained from the preceding results with the help of the following well known result. We give a brief proof since our statement of it may not coincide exactly with the statements of it in the standard references.

LEMMA 16.1. *Let $\prec$ be a simple order on a set S. Let S have the topology $\prec^t$ and suppose S is connected, D is a countable dense subset of S, and a and b are minimum and maximum elements of S respectively, $a \neq b$. Then there is an order preserving homeomorphism of S with the unit interval $I = [0, 1]$.*

Proof. Observe first that there is a one-to-one order preserving function f from $D' = D - \{a, b\}$ onto D'' = diatic rationals in $(0, 1)$. (see Hocking and Young [1], Th. 2-22.) Next observe that the connectedness of S implies that S has the least upper bound property (see the proof of Lemma 10.1). Thus we can define the order preserving functions $\varphi\colon S \to I$ and $\psi\colon I \to S$ by

$$\varphi(x) = \operatorname*{lub}_{\substack{y \in D' \\ y < x}} f(y) \qquad \psi(t) = \operatorname*{lub}_{\substack{s \in D'' \\ s < t}} f^{-1}(s) .$$

It is easy to verify that ψ is the inverse of φ and thus φ is one-to-one and onto. Since the topologies of S and I are determined by their respective orders, φ must be a homeomorphism.

DEFINITION 16.2 We will say that a topological space S is locally separable at a point $p \in S$ provided there is a neighborhood U of p and a countable set D which is dense in U. If S is locally separable at each of its points then S is locally separable.

THEOREM 16.3. *If X is Hausdorff, locally connected (or locally compact), and locally separable then X is locally homeomorphic to $\boldsymbol{R}^+ = \{t \in \boldsymbol{R} \mid t \geqq 0\}$ (i.e., X is a 1-dimensional manifold with boundary).*

Proof. First note that the locally compact case reduces to the locally connected case by Theorem 15.3. There are four cases which depend upon X and the point p in question: Case 1, X circular and p the initial point u of $<$; Case 2, X circular and p not the initial point of X; Case 3, X noncircular and p either an initial or terminal point of $\mathscr{M}$; Case 4, X noncircular and p not an initial or terminal point of $\mathscr{M}$. In the last three cases p has a closed neighborhood of the form $S = \{t \mid a \leqq t \leqq b\}$ with the property that $<$ is a simple order on S and S has a countable dense subset. In the first case we can adjust $<$ in an obvious way so that the preceding statement holds for p. In each case it follows immediately from either Lemma 12.5 or Lemma 10.1 that S is connected. Besides, by Theorem 14.4 S has the $<^l$ topology. The desired conclusion now follows from Lemma 16.1.

REMARK. Note that the local homeomorphism in the above proof also preserves the order (adjusted order in Case 1).

COROLLARY 16.4. *Under the hypothesis of Theorem* 16.3, *X must be locally compact.*

THEOREM 16.5. *If X is Hausdorff, locally connected* (*or locally compact*) *and separable then X is homeomorphic to one of the following spaces*:

(a) *the closed interval* $[0, 1]$,
(b) *the open interval* $(0, 1)$,
(c) *the half open, half closed interval* $(0, 1]$,
(d) *the circle*, $\{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 = 1\}$.

Proof. This is a corollary to the proofs of Lemma 16.1 and Theorem 16.3. If D is a countable dense subset of X then $D \cap S$ is a countable dense subset of S if S is as in Theorem 16.3. Now by lining up D'' ($D'' = D -$ possible initial or terminal points) with the diatic rationals in $(0, 1)$ once and for all, the local homeomorphism we get in Theorem 16.3 will be all coherent. (In fact we need only consider at most an appropriately chosen pair of sets like S.) The four possibilities (a)—(d) are determined by whether X is noncircular or circular and if noncircular whether X has 0, 1, or 2 end points.

THEOREM 16.6. *If X is a compact metric space then X is homeomorphic to either the closed interval* $[0, 1]$ *or the circle* $\{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 = 1\}$.

Proof. Since a compact metric space is separable and locally compact the present theorem follows immediately from Theorem 16.5.

LEMMA 16.7. *If either X is circular or X is noncircular and $\mathscr{M}$ has an initial and a terminal point then local compactness and local separability for X implies compactness and separability.*

Proof. Assume X is noncircular, locally compact and locally separable, $a = \min \mathscr{M} = \min X$, $b = \max \mathscr{M} = \max X$. Let A be the set of all x such that $\{y \mid a \leqq y \leqq x\}$ is compact and separable. We will show that A is nonempty, open and closed.

First of all $a \in A$ so A is nonempty. Secondly, by Theorem 16.3 the remark that follows we see that A is open (X is obviously Hausdorff under the hypothesis). Finally, let $x_0 \in$ closure of A. Consider a neighborhood of x_0 of the form $S = \{t \mid c \leqq t \leqq d\}$ which is order preservingly homeomorphic to $[0, 1]$. Since $x_0 \in$ closure of A there is some $x \in A \cap S$. It then follows that $\{t \mid a \leqq t \leqq x_0\} = \{t \mid a \leqq t \leqq x\} \cup \{t \mid x \leqq t \leqq x_0\}$ is a compact and separable set being the union of two compact and separable sets. Thus $x_0 \in A$ which shows that A is closed.

Since X is connected $A = X$. Thus $X = \{t \mid a \leqq t \leqq b\}$ is compact and separable. The circular case can be handled very similarly using a point p as both a and b simultaneously. The details are left to the reader.

THEOREM 16.8. *If X is Hausdorff, locally compact and locally separable and X has at least two noncut points then X is homeomorphic to either the closed interval $[0, 1]$ or the circle $\{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 = 1\}$.*

Proof. This follows immediately from Theorem 15.3, § 13, Lemma 16.7 and Theorem 16.5 used in that order.

17. In this section we will present an example of an X like we have been studying and then show how this example is rather typical of a large class of possible X's

The example is pictured in Figure 10. It is not hard to show

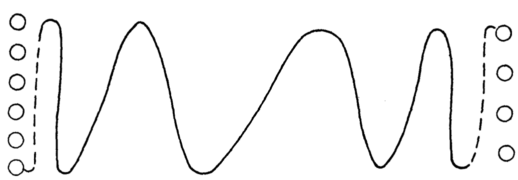

Figure 10

that this space really is an example of an X with $n \geqq 10$, and with a proper choice of $<$ we ave $^{\#}\mathscr{C}^{+} = 4$ and $^{\#}\mathscr{C}^{-} = 6$. This space is clearly metric and separable and locally compact at points of $\mathscr{M}$. The following theorem is a sort of converse to this example.

THEOREM 17.1. *If X is metric, separable, and locally compact at cut points and there are cut points then X can be embedded in $\boldsymbol{R}^3$. In fact, X is homeomorphic to the union of a finite number of points of $\boldsymbol{R}^3$ with the graph of a continuous one-to-one map of* $(0, 1)$ *into $\boldsymbol{R}^3$.*

Proof. First note that the set L of cut points of X is homeomorphic to $(0, 1)$ by the general arguments presented in § 16, where $L = \mathscr{M}$ – possible end points. We will assume $L = (0, 1)$. Consider $\boldsymbol{R}^3 = \boldsymbol{R} \times \boldsymbol{C}$ where $\boldsymbol{C}$ is the complex numbers. Let $^{\#}\mathscr{E}^- = N$, $^{\#}\mathscr{E}^+ = M$, $\mathscr{E}^- = \{e_1, \cdots, e_N\}$ and $\mathscr{E}^+ = \{e_{N+1}, \cdots, e_{N+M}\}$. Set $\tilde{e}_j = (0, \exp 2\pi i(j/(N+M))) \in \boldsymbol{R} \times \boldsymbol{C}$ for $1 \leqq j \leqq N+M$. Now for $t \in \boldsymbol{R}$ set $t^+ = \begin{cases} t & \text{if } t \geqq 0 \\ 0 & \text{if } t < 0 \end{cases}$. Finally define $\varphi: X \to \boldsymbol{R}^3 = \boldsymbol{R} \times \boldsymbol{C}$ by

$$\begin{aligned} \varphi(e_j) &= \tilde{e}_j \text{ for } 1 \leqq j \leqq N \\ \varphi(e_j) &= \tilde{e}_j + (1, 0) \text{ for } N+1 \leqq j \leqq N+M \\ \varphi(t) &= \sum_{j=1}^{N+M} (1 - \varepsilon^{-1} d(t, e_j))^+ \tilde{e}_j + (t, 0) \end{aligned}$$

where d is the metric for X and

$$\varepsilon = \frac{1}{2} \min_{j \neq k} d(e_j, e_k) .$$

The verification that φ is the desired homeomorphism is left to the reader.

18. In this section we show how one of our fundamental hypotheses may be weakened and draw from this a theorem on "connectedness" in the deleted product $X^n - GD_n$.

DEFINITION 18.1. We define the relation T in $X^n - GD_n$ by setting $\bar{x} T \bar{y}$ for $\bar{x}, \bar{y} \in X^n - GD_n$ if and only if $\bar{y} = (y_1, \cdots, y_n) = (x_1, \cdots, x_{i-1}, y_j, x_{j+1}, \cdots, x_n)$ for some j where $\bar{x} = (x_1, \cdots, x_n)$, and $\bar{x}$ cannot be separated from $\bar{y}$ in

$$\{x_1\} \times \{x_2\} \times \cdots \times \{x_{j-1}\} \times X \times \{x_{j+1}\} \times \cdots \times \{x_n\} \cap \{X^n - GD_n) .$$

We observe that the fundamental hypothesis of this paper that (h): "$X^n - GD_n$ is not connected" can be replaced by the apparently weaker hypothesis (h'): "$\bar{x}(\bar{T})'\bar{y}$ for some $\bar{x}, \bar{y} \in X^n - GD_n$" (see Definition 2.4 for $\bar{T}$). In fact hypothesis (h) was used only to prove

Lemma 3.10 and the proof of Lemma 3.10 essentially uses only hypothesis (h'). We can now prove the following theorem.

THEOREM 18.2. *If Y is a connected T_1-space and $\bar{x} = (x_1, \cdots, x_n)$ and $\bar{y} = (y_1, \cdots, y_n)$ cannot be separated in $Z = Y^n - GD_n(Y)$ then $\bar{x}\bar{T}\bar{y}$ where $GD_n(Y)$ and T are the same as GD_n and T above except defined for Y instead of X.*

Proof. Assume $x\bar{T}'y$. Then from above we know that all the analysis of this paper holds for Y in place of X. In the proof of Lemmas 11.2 and 12.8 we saw that the set of $\bar{z} \in Z$ such that $\bar{x}\bar{T}\bar{z}$ coincided with the set of $\bar{z} \in Z$ such that $\bar{x}S'_Z\bar{z}$ when $\bar{x}$ was of the form $\bar{x} = (x_1, x_2, \cdots, x_n)$ with $x_1 < x_2 < \cdots < x_n$ and $x_i \in \mathscr{M}$, $i = 1, \cdots, n$, in the noncircular case. This can be seen to hold for any $\bar{x} \in Z$ by noting the following two facts. Fact 1: The proof of Lemma 3.9 really shows that if $\bar{u}, \bar{v} \in Z$ then $\bar{u}\bar{T}\sigma\bar{v}$ for some $\sigma \in S_n$. Fact 2: For each $\sigma \in S_n$ the map σ: $Z \rightarrow Z$ preserves all the structure (e.g., T, S_Z) for which we are concerned. It follows that $\bar{x}S_Z\bar{y}$ a contradiction. Thus $\bar{x}\bar{T}\bar{y}$ as we wished to prove.

REMARK. Theorem 18.2 holds without the hypothesis that Y is connected. The proof consists of reducing the general case to Theorem 18.2 by seeing how the partition of Y^n induced by the partition of Y into components behaves with respect to $GD_n(Y)$. The details are left to the reader.

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In the following list D = definition, L = lemma, C = corollary, and T = theorem.

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1.2 D R'
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2.1 L$\cdots$ $\alpha - \beta = \{z\} \cdots$ then $xR_\alpha z$ and $zR_\alpha y$
2.2 D R^T
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3.5 Dots and markers
3.6 D aSb
3.7 D $\sigma(x_1, \cdots, x_n)$
3.8 D simple R-chain
3.9 L $xS'\sigma y$ some $\sigma \in S_n$
3.10 L $aS\sigma a$ some $\sigma \in S_n$
4.1 D $sp_R x$
4.2 L $sp_{R_\alpha} x \leqq n$
4.3 L $\cdots {}^{\#}Y \leqq M^{N+1}$
4.4 L $\cdots sp_{R_\alpha} x = 2$
5.1 L $C(\alpha)$ exists and is unique
5.2 C ${}^{\#}\alpha \geqq n^{8n+10} + 1 \Rightarrow {}^{\#}C(\alpha) \geqq 6n + 1$
5.3 L $C(\alpha) = \alpha \Leftrightarrow C(\beta) = \beta$
6 D noncircular case, $\mathscr{F}'(n^{10n+1} + 1)$
6.1 D presentation
6.2 L only two presentations
6.3 L $x_1 R'_\alpha x_m$
6.4 D x_0, x_{m+1}, $\mathscr{L}(\alpha)$, $\mathscr{E}(\alpha)$
6.5 L $x_i R_\alpha x_j \Rightarrow |i - j| = 1$ for $0 \leqq i, j \leqq m + 1$
7.1 L $\mathscr{E}(\alpha) \subset \mathscr{E}(\beta)$ if $\alpha \subset \beta$, $\beta - \alpha = \{z\}$ analysis
8.1 L ${}^{\#}X = \infty$
8.2 L $x \in \mathscr{E}(\alpha) \Rightarrow \exists R_\alpha$-chain $\cdots$ either x_0 or x_{m+1}
8.3 D $\mathscr{E}$
8.4 L ${}^{\#}\mathscr{E} < \infty$
8.5 D $\mathscr{M}$, $\mathscr{G}$, $\overline{\mathscr{C}}(\alpha) \sim (x_0, x_1, \cdots, x_m, x_{m+1})$
8.6 D $I_\alpha(x)$
8.7 D $<$
8.8 L $<$ well defined
8.9 D $\mathscr{E}^{\pm}(\alpha)$
8.10 L $\mathscr{E}^{\pm}(\alpha) \subset \mathscr{E}^{\pm}(\beta)$ if $\alpha \subset \beta$
8.11 D $\mathscr{E}^{\pm}$
8.12 D $<$ on X
9.1 L $\cdots \{y \mid y < x\}$ is open
9.2 $\exists$L min (max) $\mathscr{M} \Rightarrow \mathscr{E}^{-}(\mathscr{E}^{+}) = \emptyset$
9.3 L $x \in \mathscr{E}^{-}(\alpha)(\mathscr{E}^{+}(\alpha)) \Rightarrow xR_\alpha x_0 (xR_\alpha x_{m+1})$
9.4 L $x, y \in \mathscr{E}^{-}(\alpha)(\mathscr{E}^{+}(\alpha)) \Rightarrow xR_\alpha y$
9.5 L $<$ determines R_α
9.6 L ${}^{\#}\mathscr{E} \leqq n$, ${}^{\#}\mathscr{E}^{\pm} < n$
9.7 L ${}^{\#}\mathscr{E}^{\pm} \neq 1$
10.1 L All connected subsets of $\mathscr{M}$ are $\cdots$
10.2 C $\cdots$ are connected
10.3 L A is connected if and only if $A = I - E \cdots$
10.4 C ${}^{\#}\{$connected subsets of $X\} = {}^{\#}X$

11.1 D quasicomponent

11.2 L ${}^{\#}\{$components of $X^m \quad GD_m\} = m!/(N!M!) \cdots$

12 Circular case Definition of $\mathscr{G}$, u, v, w,

$\alpha \sim (x_1, \cdots, x_m)$, $I_\alpha(x)$

12.1 D $<$

12.2 L $<$ well defined

12.3 L intervals are open

12.4 L $\cdots$ are $\neq \varnothing$ and connected

12.5 L all connected subsets of X are $\cdots$

12.6 L ${}^{\#}\{$connected subsets of $X\} = {}^{\#}X$

12.7 L $<$ determines R_α

12.8 L ${}^{\#}\{$components of $X^m - GD_m\} = (m-1)!$

13 General case

13.1 L X circular if and only if $\mathscr{N} = X$

13.2 D $\mathscr{N}$, T

13.3 L $\mathscr{N} = \mathscr{E} \cap T$ if X noncircular

13.4 L Characterize $\mathscr{E}^-$, $\mathscr{E}^+$, and T

13.5 T cut points—either none or all but n; $X - \{x\}$ has two components $\cdots$; $X - S$ has ${}^{\#}S$ components $\cdots$; set of cut points is Hausdorff space.

14 D locally connected at p, $<^l$, $<^c$, τ at p

14.1 L locally connected at $p \Leftrightarrow$ order top. at $p \cdots$

14.2 L X not locally connected at $x \in \mathscr{E}^-$ if $\cdots$

14.3 Counter Example

14.4 T Hausdorff and locally connected $\Rightarrow$ nice $<$, $\tau = <^l$ $(<^c)$ $\cdots$ cut points.

15.1 L locally compact at $p \Rightarrow$ order top. at p

15.2 L X not locally compact at $x \in \mathscr{E}^-$ if $\cdots$

15.3 T Hausdorff and locally compact $\Rightarrow$ locally connected $\Rightarrow$ 14.4

16.1 L $\cdots (S, <) \cong (I, <)$

16.2 D Locally separable at p

16.3 T X Hansdorff and locally connected (locally compact) and locally separable $\Rightarrow X$ manifold with boundary

16.4 C $\cdots \Rightarrow X$ locally compact

16.5 T X Hausdorff, locally connected and separable $\Rightarrow X \cong (0, 1)$, $(0, 1]$, $[0, 1]$, or circle

16.6 T X compact metric $\Rightarrow X \cong [0, 1]$ or circle

16.7 L locally separable and locally compact $\Rightarrow$ separable and compact if $\cdots$

16.8 T X locally compact Hausdorff and separable and $\cdots \Rightarrow X \cong [0, 1]$ or circle

17 Example

17.1 T $\cdots$ can be embedded in $\boldsymbol{R}^3$

18.1 D T

18.2 T $\cdots \bar{x}\bar{T}\bar{y}$

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