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ON SECONDARY CHARACTERISTIC CLASSES IN COBORDISM THEORY

WEI-LUNG TING

This paper introduces into cobordism theory a new notion borrowed from ordinary cohomology theory. Specifically, let ξ be a $U(n)$-bundle over the CW-complex X. Let E and E_0 be the total spaces of the associated bundles whose fibers are respectively the unit disc $E^{2n} \subset C^n$ and the unit sphere $S^{2n-1} \subset C^n$. The classifying map for ξ gives rise to an element $U_\xi \in \Omega_U^{2n}(E, E_0)$. One defines the Thom isomorphism φ: $\Omega_U^q(X) \to \Omega_U^{q+2n}(E, E_0)$ by $\varphi(x) = (p^*x)U_\xi$ and Euler class, $e(\xi)$ of ξ, by $e(\xi) = p^{*-1}j^*(U_\xi)$. For each $\alpha = (\alpha_1, \alpha_2, \cdots)$, let $cf_\alpha(\xi) \in \Omega_U^{2|\alpha|}(X)$ be the Conner-Floyd Chern class of ξ, and S_α: $\Omega_U^q(X, Y) \to \Omega_U^{q+2|\alpha|}(X, Y)$ be the operation defined by Novikov. Then one has the relation, $S_\alpha(e(\xi)) = cf_\alpha(\xi)\cdot e(\xi)$. Now if ξ is a bundle such that $e(\xi) = 0$, then one can define a secondary characteristic class

$$\Sigma_\alpha(\xi) \in \Omega_U^*(X) \text{ mod } (S_\alpha - cf_\alpha(\xi))\Omega_U^*(X)$$

by using the above relation. The object of this paper is to study some of the properties of such secondary characteristic classes.

Secondary characteristic classes adapt particularly to the study of embedding and immersion problems. Massey and Peterson and Stein developed secondary characteristic classes in ordinary cohomology theory [4][7][8], and Lazarov has studied secondary characteristic classes in K-theory [3]. We hope the secondary characteristic classes given here, and the operations on cobordism, defined by Novikov, will have some applications on embedding and immersion problems.

The organization of the papers is as follows. In §1 we collect some results on cobordism theory and give the definition of secondary characteristic classes of cobordism theory. In §2 we give an example and carry out some computations of these characteristic classes.

1. Definition of secondary characteristic classes.

Let ξ be a $U(n)$-bundle over the CW-complex X. Let E and E_0 be the total spaces of the associated bundles whose fibres are respectively the unit disc $E^{2n} \subset C^n$ and the unit sphere $S^{2n-1} \subset C^n$. Then the Thom complex is the quotient space E/E_0. In particular, if we take ξ to be the universal $U(n)$-bundle over $BU(n)$, then the resulting Thom complex $M(\xi)$ is written $MU(n)$. The sequence of spaces

$$(MU(0), MU(1), \cdots, MU(n), \cdots)$$

is a spectrum. Associated with this spectrum we have a cohomology functor, the groups of this cohomology functor are written $\Omega_U(X, Y)$ and called complex cobordism groups. We know that $\Omega_U^*(.)$ is a multiplicative cohomology theory and $\Omega_U(P)$, where P is a point, is a polynomial ring $Z[x_1, x_2, \cdots, x_i, \cdots]$ where $x_i \in \Omega_U^{-2i}(P)$.

Next for each $U(n)$-bundle ξ over X the classifying map for ξ induces a map

$$\gamma\colon M(\xi) \longrightarrow MU(n) .$$

The map γ represents an element $U_\xi \in \Omega_U^{2n}(E, E_0)$. We define the Thom isomorphism

$$\varphi\colon \Omega_U^q(X) \longrightarrow \Omega_U^{q+2n}(E, E_0)$$

by $\varphi(x) = (p^*x)U_\xi$.

Now we need the following known theorems:

THEOREM 1 (*Conner-Floyd*) [1]. *To each ξ over X and each $\alpha = (\alpha_1, \alpha_2, \cdots)$ we can assign classes $cf_\alpha(\xi) \in \Omega_U^{2|\alpha|}(X)$, called the Conner-Floyd classes, with the following properties:*

(i) $cf_0(\xi) = 1$;

(ii) $cf_\alpha(g^*\xi) = g^*cf_\alpha(\xi)$;

(iii) *Whitney sum formula* $cf_\alpha(\xi \oplus \eta) = \sum_{\beta+\gamma=\alpha} (cf_\beta\xi)(cf_\gamma\eta)$;

(iv) *Let ξ be a $U(1)$-bundle* over X, *classified by a map* $X \longrightarrow BU(1)$, *and let the composite* $X \xrightarrow{f} BU(1) \longrightarrow MU(1)$ *represent the element* $w \in \Omega_U^2(X)$. *Then* $cf_1(\xi) = w$.

THEOREM 2 (*Novikov*) [1]. *For each $\alpha = (\alpha_1, \alpha_2, \cdots)$ there exists an operation*

$$S_\alpha\colon \Omega_U^q(X, Y) \longrightarrow \Omega_U^{q+2|\alpha|}(X, Y)$$

with the following properties:

(i) $S_0 = 1$;

(ii) $S_\alpha f^* = f^*S_\alpha$;

(iii) S_α *is stable*: $S_\alpha\delta = \delta S_\alpha$;

(iv) *Cartan formula*

$$S_\alpha(xy) = \sum_{\beta+\gamma=\alpha} (S_\beta x)(S_\gamma y);$$

(vi) *If* $w \in \mathrm{Map}\,(X, MU(1)) \subset \Omega_U^2(X)$ *then* $S_{(k)}(w) = w^{k+1}$, *and*

$$S_\alpha(w) = 0 \text{ if } \alpha \neq (k) ;$$

(vii) *Suppose that ξ is an $U(n)$-bundle over X then we have*

$$cf_\alpha(\xi) = \varphi^{-1}S_\alpha\varphi(1);$$

where φ is the Thom isomorphism for Ω_U^*.

DEFINITION 3. The Euler class of a $U(n)$-bundle ξ over X, denoted $e(\xi)$, is $p^{*-1}j^*(U_\xi)$, where $j^*\colon \Omega_U^i(E_1, E_0) \longrightarrow \Omega_U^i(E)$ is induced by the inclusion $j\colon E \longrightarrow (E, E_0)$, and the isomorphism $p^*\colon \Omega_U^i(X) \text{——} \Omega_U^i(E)$ is induced by the projection $p\colon E \longrightarrow X$.

The following propositions are not difficult to prove:

PROPOSITION 4. *If ξ is a trivial, then $e(\xi) = 0$.*

PROPOSITION 5. *For the Euler class, the relation*

$$e(\xi \oplus \eta) = e(\xi)e(\eta)$$

holds.

PROPOSITION 6. *If a $U(n)$-bundle has an nonzero cross section, then $e(\xi) = 0$.*

From Theorem 2 we have $cf_\alpha(\xi) = \varphi^{-1}S_\alpha\varphi(1)$ so that

$$S_\alpha U_\xi = \varphi cf_\alpha(\xi) = p^*(cf_\alpha(\xi))U_\xi\ .$$

Therefore we have $S_\alpha e(\xi) = cf_\alpha(\xi)e(\xi)$.

Now let ξ be a bundle such that $e(\xi) = 0$, then the long exact sequence for (E, E_0) breaks up into short exact sequences.

$0 \longrightarrow \Omega_U^i(X) \longrightarrow \Omega_U^i(E_0) \xrightarrow{\delta} \Omega_U^{i+1}(E, E_0) \longrightarrow 0.$ Let $a_\xi \in \Omega_U^{2n-1}(E_0)$ such that $\delta(a_\xi) = U_\xi$. Then every element in $\Omega_U^i(E_0)$ can be written uniquely as $xa_\xi + y$ where $x \in \Omega_U^{i-(2n-1)}(X)$ and $y \in \Omega_U^i(X)$. In particular, write $S_\alpha(a) = xa_\xi + y$. Then we apply δ and find that $x = cf_\alpha(\xi)$. If a^1 is another element with $(a^1) = U_\xi$, then $S_\alpha(a^1) = cf_\alpha(\xi)a^1 + y^1$. Then $y - y^1 \in (S_\alpha = cf_\alpha(\xi))\Omega_U^{2n-1}(X)$. Thus we can define a natural transformation Σ_α, from $U(n)$-bundle whose $e(\xi)$-class vanishes, to a natural quotient of Ω_U^*. If ξ is a such bundle $\Sigma_\alpha(\xi)$ takes values in $\Omega_U^*(X)$ mod $(S_\alpha - cf_\alpha(\xi))\Omega_U^*(X)$ and is the coset of y.

The following property can be easily proved:

PROPOSITION 7. *If ξ has a nonzero cross section then $\Sigma_\alpha(\xi) = 0$.*

2. Example. Consider $U(n+1)$ as a principal $U(n)$-bundle over S^{2n+1} for $n > 1$. Let ξ be the associated complex vector bundle. Then the sphere bundle is the complex Stiefel manifold $U(n+1)/U(n-1)$. Since $\Omega_U^{2n}(S^{2n+1}) = 0$, then $\Sigma_\alpha(\xi)$ is defined.

Let t_n be the Thom space of S^{2n+1} with respect to ξ, we have the short exact sequence

$$0 \longrightarrow \Omega_U^{2n-1}(S^{2n+1}) \longrightarrow \Omega_U^{2n-1}(U(n+1)/U(n-1)) \longrightarrow \Omega_U^{2n}(t_n) \longrightarrow 0 .$$

Since $H^*(U(n+1)/U(n-1)) = \Lambda[\gamma_{2n-1}, \gamma_{2n+1}]$ be the exterior algebra generated by γ_{2n-1} and γ_{2n+1} of dimensions $2n-1, 2n+1$ respectively. Therefore by [2] we have $\Omega_U^*(U(n+1)/U(n-1))\Lambda[\gamma_{2n-1}, \gamma_{2n+1}] \otimes \Omega_U^*(P)$. Let $\tilde{\gamma}_{2n-1} \in \Omega_U^*(U(n+1)/U(n-1))$, $\tilde{\gamma}_{2n+1} \in \Omega_U^*(U(n+1)/U(n-1))$ such that $\mu_z(\tilde{\gamma}_{2n-1}) = \gamma_{2n-1}$, $\mu_z(\tilde{\gamma}_{2n+1}) = \gamma_{2n+1}$, where $\mu_z\colon \Omega_U^* \longrightarrow H^*(\ , Z)$ is the map defined by the Thom class (see [2] for definition), the group $\Omega_U^{2n-1}(U(n+1)/U(n-1))$ is $Z+Z$ with generators $\tilde{\gamma}_{2n-1}$ and $\tilde{\gamma}_{2n+1}[CP^1]$ where $[CP^1] \in \Omega_U^{-2}(P)$ is a generator of $\Omega_U^*(P)$. The group $\Omega_U^{2n-1}(S^{2n+1})$ is infinite cyclic with generator $\tilde{\gamma}_{2n+1}[CP^1]$, and so $\delta(\tilde{\gamma}_{2n-1}) = \pm U_\xi$. We know that $S_{(1)}\tilde{\gamma}_{2n-1} = cf_{(1)}\tilde{\gamma}_{2n-1} + b\tilde{\gamma}_{2n+1}$ where $\pm b\tilde{\gamma}_{2n+1}$ represents $\Sigma_{(1)}(\xi)$. Since $\Omega_U^*(U(n+1)/U(n-1))$ injects into $\Omega_U^*(U(n+1))$, we can compute it in $\Omega_U^*(U(n+1))$. Now by using the notation of [9, p. 40] we have the monomorphism

$$\mu^*\colon \Omega_U^*(U(n+1)) \longrightarrow \Omega_U^*(Q_{n+1} \times U(n)) .$$

By induction, we can determine $S_{(1)}$ if we know $S_{(1)}$ in Q_{n+1} and its behavior under cross products. By [9] we have $Q_{n+1} = SCP^n VS^1$ and since $S_{(1)}$ commutes with the suspension map

$$s\colon \Omega_U^i(CP^n) \longrightarrow \Omega_U^{i+1}(SCP^n) ,$$

so we need only know $S_{(1)}$ in $\Omega_U^*(CP^n)$. By [2, p. 52] we know that $\Omega_U^*(CP^n)$ is a free $\Omega_U^*(P)$-module with basis $1, w_n, \cdots, (w_n)^n$ where $w_n \in \text{Map}\,[CP^n, MU(1)] \subset \Omega_U^2(CP^n)$. Moreover, the inclusion

$$i\colon CP^{n-1} \subset CP^n$$

has $i^* w_n = w_{n-1}$. By Theorem 2 we have $S_{(1)}(w_n)^j = j(w_n)^{j+1}$, hence $S_{(1)}s(w_n)^j = sS_{(1)}(w_n)^j = sj(w_n)^{j+1} = js(w_n)^{j+1}$, here $s(w_n)^j$, $s(w_n)^{j+1}$ in $\Omega_U^*(SCP^n)$ are the images of $(w_n)^i$, $(w_n)^{i+1}$ under the suspension map s respectively. From above data and an argument, similar to [9, p. 53], we obtain $S_{(1)}\tilde{\gamma}_{2n-1} = (n-1)\tilde{\gamma}_{2n+1}$, hence $cf_{(1)} = 0$ and $b = n-1$. Now we compute $(S_{(1)} - cf_{(1)})\,\Omega_U^{2n-1}(S^{2n+1}) = S_{(1)}\Omega_U^{2n-1}(S^{2n+1})$, which is generated by $S_{(1)}(\tilde{\gamma}_{2n+1}[CP^1])$. By [5] we have $S_{(1)}(\tilde{\gamma}_{2n+1}[CP^1]) = 2\tilde{\gamma}_{2n+1}$. Therefore $\Sigma_{(1)}(\xi) \neq 0$ if $n-1 \neq 0 \bmod (2)$.

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