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CONVERGENCE OF A SEQUENCE OF TRANSFORMATIONS OF DISTRIBUTION FUNCTIONS-II

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A previous paper of the present author was devoted to the study of the convergence properties of the iterates of a certain transformation of distribution functions (d.f.'s) of a random variable (r.v.). In this paper the definitions and some of the results are extended to the case of bivariate d.f.'s.

1. Definition and preliminaries. Throughout this paper $F(x, y)$ will denote the bivariate *d.f.* of a nonnegative random vector (X, Y). More precisely, (i) $F(x, y)$ is monotonic nondecreasing; i.e., for $a > c$, $b > d$ we have

$$[F(x, y)]_{c,d}^{a;b} = F(a, b) - F(a, d) - F(c, b) + F(c, d) \geqq 0 .$$

(ii) $F(x, 0) = F(0, y) = 0$ for all x and y. (iii) $F(+\infty, +\infty) = \lim_{x,y\to\infty} F(x, y) = 1$ and (iv) $F(x, y)$ is left continuous in each variable; i.e.,

$$\lim_{h\to 0-} F(x + h, y) = F(x, y)$$

for all x and y with a similar left continuity in y.

We shall let $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$ be the marginal *d.f.*'s of X and Y respectively and $\mu(i, j) = E(X^iY^j)$ be a product moment of order $i + j$ when it exists finitely. Hence $\mu(i, 0)$ and $\mu(0, i)$ are the i-th moments of the marginal *d.f.*'s F_1 and F_2 respectively. For brevity we let $\mu = \mu(1, 1)$.

Let us remark at this point that (1) all of the results of this paper (and more) follow immediately from the univariate case if F is the *d.f.* corresponding to a product measure; i.e., X and Y are independent and (2) although we are dealing explicitly with the bivariate case, the treatment and the results carry over in a direct way to distributions in the positive quadrant of R^n, $n \geqq 3$.

We develop now the requisite background material before introducing the bivariate transform in § 2.

The following two lemmas for integration by parts are basic. These formulas are known [11], but apparently not readily available, and so we give them in a form convenient for our use.

LEMMA 1.1. *Assuming the existence of the double Riemann-Stieltjes integral we have*

$$\int_0^a\int_0^b f(x, y)dg(x, y) = \int_0^a\int_0^b [1 - g(x, b) - g(a, y) + g(x, y)]df(x, y)$$
$$(1.1)\qquad - \int_0^a [g(x, b) - g(x, 0)]df(x, 0)$$
$$- \int_0^b [g(a, y) - g(0, y)]df(0, y)$$
$$+ [g(x, y)f(x, y)]_{0,0}^{a,b} - [f(x, y)]_{0,0}^{a,b}$$

where

$$[h(x, y)]_{0,0}^{a,b} = h(a, b) - h(a, 0) - h(0, b) + h(0, 0) .$$

LEMMA 1.2.

$$(1.2)\qquad \int_0^a\int_0^b f(x)dg(x, y) = \int_0^a f(x)d[g(x, b) - g(x, 0)] .$$

It is well known that the double Riemann-Stieltjes integral exists when, for example, one of the functions f and g is continuous and the other is of bounded variation (cf. [3]).

LEMMA 1.3. *If $G(x, y)$ is continuous and the bivariate d.f. of a nonnegative random vector except that $G(\infty, \infty)$ is arbitrary, then*

$$(1.3)\qquad \int_0^\infty\int_0^\infty G(x, y)dF(x, y) = \int_0^\infty\int_0^\infty [1 - F_1(x) - F_2(y) + F(x, y)]dG(x, y) .$$

Proof. Let $a > 0, b > 0$ and $S = [0, a] \times [0, b]$. Using (1.1) and simplifying we get

$$\int_S G(x, y)dF(x, y) = A + \int_0^a [F_1(x) - F(x, b)]dG(x, b)$$
$$(1.4)\qquad + \int_0^b [F_2(y) - F(a, y)]dG(a, y)$$
$$- G(a, b)[1 - F(a, b)]$$
$$= A + B$$

where

$$A = \int_S F^*(x, y)dG(x, y)$$

and

$$(1.5)\qquad F^*(x, y) = 1 - F_1(x) - F_2(y) + F(x, y)$$
$$= Pr(X \geqq x, Y \geqq y) \geqq 0 .$$

Now $B \leqq 0$. In fact, since $F_1(x) - F(x, b)$ and $F_2(y) - F(a, y)$ are nondecreasing functions in x and y respectively we have

$$\begin{aligned} B &\leqq G(a, b)[F_1(a) - F(a, b)] + G(a, b)[F_2(b) - F(a, b)] \\ &\qquad - G(a, b)[1 - F(a, b)] \\ &= -G(a, b)F^*(a, b) \leqq 0 . \end{aligned}$$

Next, noting (1.2) and integrating by parts

$$\begin{aligned} \int_0^a \int_b^\infty G(x, y)dF(x, y) &\geqq \int_0^a G(x, b)d[F_1(x) - F(x, b)] \\ &= -\int_0^a [F_1(x) - F(x, b)]dG(x, b) \\ &\qquad + G(a, b)[F_1(a) - F(a, b)] . \end{aligned}$$

Similar lower bounds can be computed for $\int_a^\infty \int_0^b$ and $\int_a^\infty \int_b^\infty$. Combining these results we obtain

$$\left\{\int_0^a \int_b^\infty + \int_a^\infty \int_0^b + \int_a^\infty \int_b^\infty\right\} G(x, y)dF(x, y) \geqq -B \geqq 0 . \tag{1.6}$$

If now

$$c = \int_0^\infty \int_0^\infty G(x, y)dF(x, y) < \infty$$

the left side in (1.6) is

$$c - \int_S G(x, y)dF(x, y) \geqq -B \geqq 0 ,$$

and letting $a \to \infty$, $b \to \infty$ we get $B \to 0$. Hence $A \to c$ as a and b approach ∞. If, however, $c = +\infty$, since $B \leqq 0$ it follows from (1.4) that $A \geqq \int_S G(x, y)dF(x, y)$ and letting $a, b \to \infty$ we get $A = +\infty$. The lemma is proved in any case.

COROLLARY. *For* $m \geqq 1$, $n \geqq 1$

$$\mu(m, n) = mn \int_0^\infty \int_0^\infty x^{m-1} y^{n-1} F^*(x, y)dydx \tag{1.7}$$

where F^* *is defined in* (1.5). *In particular,*

$$\mu = \int_0^\infty \int_0^\infty F^*(x, y)dydx . \tag{1.8}$$

We now recall that the characteristic function (*c.f.*) $f(t, t')$ of a d.f. $F(x, y)$ is called an analytic c.f. if there exists a function $A(z, z')$ of two complex variables which is defined and holomorphic in a neighborhood of the origin and which coincides with f for real values of z

and z'. The lemma below is an extension of the necessity part of Theorem 7.2.1 in [5].

LEMMA 1.4. *If $F(x, y)$ has an analytic* c.f. *then there exists a positive constant R such that*

$$F^*(x, y) = o[e^{-(rx+r'y)}],\ x, y \longrightarrow \infty$$

for all positive r, r' smaller than R.

Proof. If f is holomorphic in $\{(z, z'): |z| < p, |z'| < p'\}$ for some $p > 0, p' > 0$, then it is holomorphic at least in the "band" $\{(z, z'): |\mathrm{Im}\, z| < p, |\mathrm{Im}\, z'| < p'\}$ (cf. [2], [8]). Put $R = \min(p, p')$ and $\mathrm{Im}\, z = t$, $\mathrm{Im}\, z' = t'$. Let $x > 0$, $y > 0$. Then

$$\int_x^\infty \int_y^\infty \exp(tu + t'v) dF(u, v)$$

exists finitely for $\max(|t|, |t'|) < R$. Pick positive numbers r, r' such that $r < R$, $r' < R$ and then s, s' such that $r < s < R$ and $r' < s' < R$. Then there exists a positive constant C such that

$$\begin{aligned} C &\geqq \int_x^\infty \int_y^\infty \exp(su + s'v) dF(u, v) \\ &\geqq \exp(sx + s'y) F^*(x, y)\ . \end{aligned}$$

Thus for $0 < r < R, 0 < r' < R$

$$\begin{aligned} 0 &\leqq F^*(x, y) \exp(rx + r'y) \\ &= F^*(x, y) \exp(sx + s'y) \exp[-(s - r)x - (s' - r')y] \\ &\leqq C \exp[-(s - r)x - (s' - r')y] \longrightarrow 0 \text{ as } x, y \longrightarrow \infty\ . \end{aligned}$$

2. The bivariate transform. We now define the bivariate transform and its iterates. Let $F(x, y)$ have finite moments $\mu(i, j)$ of all orders ($i \geqq 0, j \geqq 0$). Define the sequence $\{G_n\}$ of absolutely continuous d.f.'s as follows. Put

$$G_1(x, y) = \mu^{-1} \int_0^x \int_0^y F^*(u, v) dv du$$

for $x > 0, y > 0$ and zero elsewhere. For $n \geqq 1$ let

$$G_{n+1}(x, y) = [\alpha(n, 1)]^{-1} \int_0^x \int_0^y G_n^*(u, v) dv du$$

for $x > 0, y > 0$ and zero elsewhere. Here

$$\alpha(n, 1) = \int_0^\infty \int_0^\infty G_n^*(u, v) dv du$$

and $G_n^*(u, v) = 1 - G_n^{(1)}(u) - G_n^{(2)}(v) + G_n(u, v)$ where

$$G_n^{(1)}(u) = G_n(u, +\infty) \text{ and } G_n^{(2)}(v) = G_n(+\infty, v) .$$

In view of (1.8) $G_n(x, y)$ is indeed an absolutely continuous d.f. for $n \geqq 1$. Furthermore, if X and Y are independent so that $F(x, y) = F_1(x)F_2(y)$ we see that the bivariate transform of F is the product of the univariate transforms introduced in [10] of the marginal d.f.'s F_1 and F_2. In the general case, however, no such simple relationship exists. This is important to the understanding of why a separate treatment of the two dimensional case is necessary and also helps explain the difficulty in strengthening part (v) of Theorem 4.1.

In this section we obtain the relation between the moments of F and of G_n for $n \geqq 1$.

THEOREM 2.1. *If the moment generating function* (m.g.f.) $M(t_1, t_2)$ *of* $F(x, y)$ *exists in a neighborhood* N *of the origin then the* m.g.f. $M^*(t_1, t_2)$ *of* $G_1(x, y)$ *exists in* N *and*

$$\begin{aligned}
&M^*(t_1, t_2) = (\mu t_1 t_2)^{-1}[M(t_1, t_2) - M_1(t_1) - M_2(t_2) + 1],\ t_1 t_2 \neq 0\\
&M^*(t_1, 0) = (\mu t_1)^{-1}[\partial M/\partial t_2|_{(t_1, 0)} - \partial M/\partial t_2|_{(0,0)}],\ t_1 \neq 0\\
&M^*(0, t_2) = (\mu t_2)^{-1}[\partial M/\partial t_1|_{(0, t_2)} - \partial M/\partial t_1|_{(0,0)}],\ t_2 \neq 0\\
&M^*(0, 0) = 1
\end{aligned} \tag{2.1}$$

where the arguments of M^* *are in* N *and* M_1 *and* M_2 *are the* m.g.f.*'s of the marginal* d.f.*'s* F_1 *and* F_2, *respectively.*

Proof. Clearly $M^*(0, 0) = 1$. Further, the existence of the m.g.f. M in N implies the existence of $M_1(u)$ and $M_2(u)$ for $(u, 0) \in N$ and $(0, u) \in N$ respectively. Consider first the case $t_1 > 0$, $t_2 > 0$, $(t_1, t_2) \in N$. The first assertion in the theorem follows at once from Lemma 1.3 by using $G(x, y) = (e^{t_1 x} - 1)(e^{t_2 y} - 1)$ and noting that

$$M^*(t_1, t_2) = \int_0^\infty\int_0^\infty \exp(t_1 x + t_2 y) F^*(x, y) dy dx .$$

The result follows similarly when $t_1 t_2 \neq 0$, t_1 and/or t_2 negative. We now turn to the second equation in (2.1) and merely sketch the proof. Since the m.g.f. M defines a holomorphic function in a "band" containing N, the integral

$$\int_0^\infty\int_0^\infty \exp(t_1 x + t_2 y) dF(x, y)$$

converges uniformly in compact subsets of N. Hence, for $(t_1, t_2) \in N$,

$$(\partial/\partial t_2)\int_0^\infty\int_0^\infty \exp(t_1x + t_2y)dF(x, y)$$
$$= \int_0^\infty\int_0^\infty (\partial/\partial t_2)\exp(t_1x + t_2y)dF(x, y)$$
$$= \int_0^\infty\int_0^\infty y\exp(t_1x + t_2y)dF(x, y) .$$

Thus, the quantity in square brackets on the right side of the second equation in (2.1) reduces to

$$\int_0^\infty\int_0^\infty y(e^{t_1x} - 1)dF(x, y) .$$

Use of Lemma 1.3 again gives us the result. The third equation in (2.1) is proved in the same way. The theorem is completely proved.

We shall write $\mu(i, j; n)$ to denote $E_{G_n}(X^iY^j)$, $i \geqq 0$, $j \geqq 0$, $n \geqq 1$. The following results are easily proved. If F has a m.g.f., these results are obtained as corollaries to Theorem 2.1.

THEOREM 2.2. *Let* $m \geqq 1$, $n \geqq 1$. *If* $\mu(i, j)$ *exists finitely for* $0 \leqq i \leqq m$, $0 \leqq j \leqq n$ *then* $\mu(i, j; 1)$ *exists finitely for* $0 \leqq i \leqq m - 1$, $0 \leqq j \leqq n - 1$. *In this case*

$$\mu(i, j; 1) = \mu(i + 1, j + 1)/(i + 1)(j + 1)\mu . \tag{2.2}$$

THEOREM 2.3. *If* $\mu(i, j)$ *exists finitely for all nonnegative integers* i *and* j, *then for all such* i *and* j *and* $n \geqq 1$,

$$\mu(i, j; n) = \binom{n+i}{i}^{-1}\binom{n+j}{j}^{-1}\mu(n + i, n + j)/\mu(n, n) . \tag{2.3}$$

3. A convergence theorem for d.f.'s on a finite rectangle. In this section we prove the following theorem:

THEOREM 3.1. *If* $F(x, y)$ *is a finite distribution on the rectangle* $[0, a] \times [0, b]$, *i.e.*, $F(a, b) = 1$, *but* $F(x, y) < 1$ *for* $x < a$ *or* $y < b$. *Then*

$$\lim_{n\to\infty} G_n(x/n, y/n) = G(x, y) = \begin{cases}[1 - \exp(-x/a)][1 - \exp(-y/b)], & \min(x, y) \geqq 0\\ 0 & \text{elsewhere.}\end{cases}$$

To prove the theorem we need several inequalities concerning the growth rates of moments which we now obtain. For every nonnegative real number m, n, p, q and real number t, we have

$$\mu(2m, 2n) + 2t\mu(m + p, n + q) + t^2\mu(2p, 2q) = E(X^m Y^n + tX^p Y^q)^2 \geqq 0$$

so that, if the moments are positive and finite we get

$$\mu(2m, 2n)\mu(2p, 2q) \geqq \mu^2(m + p, n + q) . \tag{3.1}$$

Let r, s be positive integers. Letting $2m = r + 1, 2n = s + 1$, $2p = r - 1, 2q = s - 1$ in (3.1) and then $s + 1 = r$ we obtain

$$\mu(r + 1, s + 1)/\mu(r, s) \geqq \mu(r, s)/\mu(r - 1, s - 1) \tag{3.2}$$

$$\mu(r + 1, r)/\mu(r, r - 1) \geqq \mu(r, r - 1)/\mu(r - 1, r - 2) . \tag{3.3}$$

Similarly,

$$\mu(s, s + 1)/\mu(s - 1, s) \geqq \mu(s - 1, s)/\mu(s - 2, s - 1) . \tag{3.4}$$

Setting $2m = 2p = r, 2n = 2q = s$ in (3.1) we get

$$\mu(r + 1, s)/\mu(r, s) \geqq \mu(r, s)/\mu(r - 1, s) \tag{3.5}$$

and its dual

$$\mu(r, s + 1)/\mu(r, s) \geqq \mu(r, s)/\mu(r, s - 1) . \tag{3.6}$$

Lemma 3.1 through 3.4 are proved under the hypothesis of Theorem 3.1.

LEMMA 3.1.

$$\lim_{n\to\infty} \mu^{1/n}(n, n) = ab . \tag{3.7}$$

$$\lim_{n\to\infty} \mu^{1/n}(n + i, n + j) = ab, \; i \geqq 0, j \geqq 0 . \tag{3.8}$$

Proof. Similar to Boas [1].

COROLLARY.

$$\lim_{n\to\infty} \mu(n + 1, n + 1)/\mu(n, n) = ab . \tag{3.9}$$

LEMMA 3.2.

$$\lim_{n\to\infty} \mu(n + i, n)/\mu(n + i - 1, n) = a, \; i \geqq 1 \tag{3.10}$$

$$\lim_{n\to\infty} \mu(n + i, n)/\mu(n, n) = a^i, \; i \geqq 0 . \tag{3.11}$$

Proof. It suffices to prove (3.10) since (3.11) follows from it. Let $i = 1$. Clearly $\limsup_{n\to\infty} \mu(n + 1, n)/\mu(n, n) \leqq a$. Since

$$\mu(n, n + 1)/\mu(n, n) \leqq b ,$$

we have from (3.5), for $n \geqq 2$,

$$\mu(n+1, n)/\mu(n, n) \geqq b^{-1}\mu(n, n)/\mu(n-1, n-1)$$

which implies that the lim inf of the left side is at least $b^{-1}ab = a$. For a general i we use (3.5) and induction on i to get

$$\begin{aligned} a &\geqq \mu(n+i+1, n)/\mu(n+i, n) \\ &\geqq \mu(n+i, n)/\mu(n+i-1, n) \to a, \text{ as } n \to \infty . \end{aligned}$$

Similarly we have the dual results

$$\lim_{n\to\infty} \mu(n, n+j)/\mu(n, n+j-1) = b, j \geqq 1 . \tag{3.12}$$

$$\lim_{n\to\infty} \mu(n, n+j)/\mu(n, n) = b^j, j \geqq 0 . \tag{3.13}$$

LEMMA 3.3.

$$\lim_{n\to\infty} \mu(n-i, n)/\mu(n-i-k, n) = a^k, i \geqq 0, k \geqq 0 . \tag{3.14}$$

Proof. It suffices to consider $k = 1$, $i \geqq 1$.

$$\begin{aligned} &\mu(n-i, n)/\mu(n-i-1, n) \\ &= \frac{\mu(n-i, n)}{\mu(n-i, n-i)} \frac{\mu(n-i-1, n-i-1)}{\mu(n-i-1, n)} \frac{\mu(n-i, n-i)}{\mu(n-i-1, n-i-1)} \\ &\sim \frac{\mu(n, n+i)}{\mu(n, n)} \frac{\mu(n, n)}{\mu(n, n+i+1)} \frac{\mu(n+1, n+1)}{\mu(n, n)} \\ &\to b^i(b^{-1})^{i+1}ab = a, \text{ as } n \to \infty \end{aligned}$$

in view of (3.10)-(3.13).

In a similar fashion

$$\lim_{n\to\infty} \mu(n, n-i)/\mu(n, n-i-k) = b^k, i \geqq 0, k \geqq 0 . \tag{3.15}$$

LEMMA 3.4.

$$\lim_{n\to\infty} \mu(n+i, n+j)/\mu(n, n) = a^i b^j, i \geqq 0, j \geqq 0 . \tag{3.16}$$

Proof.

$$\begin{aligned} &\mu(n+i, n+j)\mu(n, n) \\ &\quad = [\mu(n+i, n+j)/\mu(n, n+j)][\mu(n, n+j)/\mu(n, n)] \\ &\quad \to a^i b^j, \text{ as } n \to \infty \end{aligned}$$

by (3.14) and (3.15).

We are now ready to prove Theorem 3.1. The moment $E(X^iY^j)$, $i, j \geqq 0$, of $G_n(x/n, y/n)$ is

$$n^in^j\binom{n+i}{i}^{-1}\binom{n+j}{j}^{-1}\mu(n+i, n+j)/\mu(n, n) \quad \text{(Theorem 2.3)}$$

which converges to $a^ii!\,b^jj!$ (Lemma 3.4). This last quantity is the moment of order (i, j) of $G(x, y)$ given in the statement of the theorem. The result now follows by the bivariate moment convergence theorem. We observe that the limit distribution is the product of two univariate distributions; i.e., the limiting random variables are independent.

Examples 5.1 and 5.2 illustrate this theorem.

4. *D.F.*'s on an infinite range.

In this section let F be distributed on the whole positive quadrant of the plane; i.e., $F(x, y) < 1$ for all real x and y.

Let $\{c_n\}$, $\{d_n\}$ be sequences of positive real numbers and use the following abbreviations. (Superscripts indicate the appropriate marginal d.f.'s) $H_n(x, y) = G_n(c_nx, d_ny)$,

$$H_n^*(x, y) = 1 - H_n^{(1)}(x) - H_n^{(2)}(y) + H_n(x, y),$$
$$G^*(x, y) = 1 - G^{(1)}(x) - G^{(2)}(y) + G(x, y)$$

$b_n = \int_0^\infty\int_0^\infty H_n^*(x, y)dydx$, and $b = \int_0^\infty\int_0^\infty G^*(x, y)dydx$. We note that $b_n = E_{H_n}(XY)$ and $b = E_G(XY)$. We further recall that a d.f. is proper if there is no straight line in the xy-plane which contains the whole mass of the distribution. The main result of this section is the following theorem.

THEOREM 4.1. *Let positive real numbers c_n and d_n exist such that $\lim_{n\to\infty} H_n(x, y) = G(x, y)$ and $\lim_{n\to\infty} H_n^*(x, y) = G^*(x, y)$ where $G(x, y)$ is a proper* d.f. *Let* $\limsup_{n\to\infty} c_n/c_{n-1} = l_1 < \infty$ *and* $\limsup_{n\to\infty} d_n/d_{n-1} = l_2 < \infty$. *Then*

(i) *$\{b_n\}$ is a bounded sequence.*

(ii) $\lim_{n\to\infty} b_n = b < \infty$.

(iii) *$\lim_{n\to\infty} c_n/c_{n-1} = l_1$ and $\lim_{n\to\infty} d_n/d_{n-1} = l_2$ exist.*

(iv) *$l_1l_2 \geqq 1$ and equality holds if F has an analytic* c.f.

(v) *For $i \geqq 0$, $\mu_{i,i}(H_n) \to \mu_{i,i}(G)$ as $n \to \infty$ where*

$$\mu_{i,j}(\varphi) = E_\varphi(X^iY^j)\ .$$

(vi) *If $a_n \to a$, $a'_n \to a'$ as $n \to \infty$ where a_n, a'_n, a, a', are all positive, then $\lim_{n\to\infty} H_n(a_nx, a'_ny) = G(ax, a'y)$.*

(vii) *$\lim_{n\to\infty} H_n^{(1)}(x) = G^{(1)}(x)$ and $\lim_{n\to\infty} H_n^{(2)}(y) = G^{(2)}(y)$ uniformly in x and y.*

(viii) *$G(x, y)$ is continuous and the convergence $H_n(x, y) \to G(x, y)$ is uniform in x and y.*

Proof. The first five parts of the theorem follow as in Theorem 4.1 in [10]. As for the remainder, we first prove that $G(x, y)$ is continuous. This involves several steps.

Step 1.

$$G(x, y) = b^{-1}\int_0^{l_1 x}\int_0^{l_2 y} G^*(u, v)dvdu,\ x > 0,\ y > 0\ . \tag{4.1}$$

This is easily proved.

Step 2. $\sup_{y>0} y\int_0^\infty H_n^*(x, y)dx$ is uniformly bounded for n sufficiently large.

Proof. Since $b_n \to b < \infty$, there exists N and $M > 0$ such that $n > N$ implies $\int_0^\infty\int_{y/2}^y H_n^*(u, v)dvdu \leqq M$ for all $y > 0$. Since $H_n^*(u, v)$ is monotonic decreasing in v, we have for $n > N$ and all $y > 0$

$$M \geqq \int_0^\infty\int_{y/2}^y H_n^*(u, v)dvdu \geqq \frac{y}{2}\int_0^\infty H_n^*(u, y)du$$

which proves our result.

Step 3. $\sup\limits_{x>0} x\int_0^\infty H_n^*(x, y)dy$ is uniformly bounded for n sufficiently large.

Step 4. Let

$$g_n(x, y) = \int_a^x\int_a^y H_n^*(u, v)dvdu,\ (x, y) \in [a, \infty)\times[a, \infty),\ a > 0\ .$$

Then there exists a subsequence $\{g_{n_k}(x, y)\}$ converging uniformly to

$$g(x, y) = \int_a^x\int_a^y G^*(u, v)dvdu\ .$$

Proof. It is clear by the bounded convergence theorem that $g_n \to g$ pointwise. To obtain a subsequence converging uniformly we shall show that $\{g_n\}$ is uniformly bounded and equicontinuous and then appeal to the Arzela-Ascoli theorem [6, *p.* 242].

First $\{g_n\}$ is uniformly bounded since $|g_n(x, y)| \leqq b_n \leqq M$. Now we prove that it is equicontinuous. Let ε be given. ($\varepsilon < 1$). Choose N and $M > 0$ such that for $n > N$

$$\sup_{x>0} x\int_0^\infty H_n^*(x, y)dy < M \text{ and } \sup_{y>0} y\int_0^\infty H_n^*(x, y)dx < M\ .$$

This is possible by Steps 2 and 3. Next, pick $\delta < \min(\varepsilon, \varepsilon a/M)$, $\delta > 0$.

Let $|x - x'| < \delta$, $|y - y'| < \delta$ and for definiteness let $x' < x$, $y' < y$. (Other cases are similarly handled.) Then, for $n > N$

$$|g_n(x, y) - g_n(x', y')| \leqq A + B + C$$

where

$$C = \left|\int_{x'}^{x}\int_{y'}^{y} H_n^*(u, v)dvdu\right| \leqq (x - x')(y - y') < \delta < \varepsilon .$$

$$A = \left|\int_{a}^{x'}\int_{y'}^{y} H_n^*(u, v)dvdu\right|$$

$$\leqq \frac{(y - y')}{y'}\, y'\int_{a}^{x'} H_n^*(u, y')du \leqq (y - y')M/a < \delta M/a < \varepsilon$$

using Step 2. In a similar fashion

$$B = \left|\int_{x'}^{x}\int_{a}^{y'} H_n^*(u, v)dvdu\right| < \varepsilon .$$

Step 4 is proved.

We now turn to the proof of the continuity of $G(x, y)$. Clearly, G is continuous at $(c, 0)$, $c > 0$ since by Step 1 $G(x, y) \leqq Mxy$ and $G(c, 0) = \lim_{n\to\infty} H_n(c, 0) = 0$. Similarly, G is continuous at $(0, 0)$ and at $(0, d)$, $d > 0$. Hence let $c > 0$, $d > 0$ and consider continuity at (c, d). Let $\varepsilon > 0$ be given, $\varepsilon < 1$. Choose $a > 0$, $4a < \min(c, d, \varepsilon, 1/l_1, 1/l_2, l_1c, l_2d)$ and let

$$g(x, y) = b^{-1}\int_{a}^{l_1x}\int_{a}^{l_2y} G^*(u, v)dvdu .$$

Note that $g(c, d)$ is defined. Since $H_n^*(u, v)$ is continuous for each $n \geqq 1$, it follows from Step 4 that $g(x, y)$ is continuous in $[a, \infty) \times [a, \infty)$. Let $\eta > 0$ be the delta needed for the given ε and (c, d) in the definition of continuity of g. Further by Step 1,

$$G(x, y) = g(x, y) + \left\{\int_{0}^{a}\int_{a}^{l_2y} + \int_{a}^{l_1x}\int_{0}^{a} + \int_{0}^{a}\int_{0}^{a}\right\} b^{-1}G^*(u, v)dvdu,\ l_1x > a,\ l_2y > a .$$

This equation is also true for $x = c$, $y = d$. Choose $\delta < \min(\eta, a)$. Then, for $|x - c| < \delta$, $|y - d| < \delta$ we have that (x, y) belongs to the domain of g and

$$A = |g(x, y) - g(c, d)| < \varepsilon$$

$$B = \left|\int_{0}^{a}\int_{l_2y}^{l_2d} G^*(u, v)dvdu\right| < al_2|y - d| < \varepsilon$$

$$C = \left|\int_{l_1x}^{l_1c}\int_{0}^{a} G^*(u, v)dvdu\right| < al_1|x - c| < \varepsilon .$$

Hence, $|G(x, y) - G(c, d)| \leqq A + B + C < 3\varepsilon$. The proof of the continuity of $G(x, y)$ is completed. Since $H_n(x, y)$ converges to $G(x, y)$ and these are all continuous d.f.'s, the bivariate version of a familiar result [9, p. 438] asserts that the convergence $H_n \to G$ is uniform. This uniform convergence now yields parts (vi) and (vii) of the theorem immediately. Theorem 4.1 is completely proved.

REMARK 1. A consequence of (ii) is the asymptotic equivalence: $c_n d_n \sim \mu(n+1, n+1)/b(n+1)^2\mu(n, n)$ where $b = E_G(XY)$. Thus, the theorem gives the asymptotic nature of only the product of the normalizing sequences in terms of the rate of growth of the moments of F. It might be natural to seek conditions under which the normalizers will be given by

$$\begin{aligned} c_n &\sim k\mu(n+1, n)/(n+1)\mu(n, n) \\ d_n &\sim \mu(n, n+1)/bk(n+1)\mu(n, n) \end{aligned} \tag{4.2}$$

for some constant $k > 0$. If (4.2) holds, it is natural to expect $1/k$ and bk to correspond to the first moments of the marginal d.f.'s G_1 and G_2 of the limiting d.f. G. This is true and is seen as follows. By a straightforward calculation

$$\mu_{1,0}(H_n) \sim \mu(n+1, n)/c_n(n+1)\mu(n, n) \to 1/k$$

under (4.2). Letting

$$f_n(x) = 1 - H_n^{(1)}(x),\ f(x) = 1 - G^{(1)}(x),\ g_n(x) = \int_0^x f_n(u)du$$

and $g(x) = \int_0^x f(u)du$ we have $g_n \to g$ by the bounded convergence theorem. In fact, applying the Arzela-Ascoli theorem to $\{g_n\}$, it is easy to conclude that $g_{n'} \to g$ uniformly in x, where n' is a suitable subsequence of the natural numbers. It now follows by the Moore-Osgood theorem [7, p. 285] that

$$\lim_{n'\to\infty}\lim_{x\to\infty} g_{n'}(x) = \lim_{x\to\infty}\lim_{n'\to\infty} g_{n'}(x);\ \text{i.e.,}$$

$$\lim_{n'\to\infty} \mu_{1,0}(H_{n'}) = \int_0^\infty [1 - G^{(1)}(u)]du = \mu_{1,0}(G)\,.$$

Since $\mu_{1,0}(H_n) \to 1/k$ it follows that $\mu_{1,0}(G) = 1/k$. Similarly, $\mu_{0,1}(G) = bk$. Incidentally, we have proved that $\mu_{1,0}(H_n) \to \mu_{1,0}(G)$, and $\mu_{0,1}(H_n) \to \mu_{0,1}(G)$ under the condition (4.2).

REMARK 2. Part (v) of the theorem asserts the convergence of $\mu_{i,j}(H_n)$ to $\mu_{i,j}(G)$ only for $i = j$. Remark 1 above extends this to the case $i = 0, j = 1$ and $i = 1, j = 0$ under the condition (4.2). It might be interesting to investigate if the general moment convergence is a consequence of (4.2) but we shall not pursue that in this paper.

REMARK 3. Under the conditions of the theorem and (4.2) the following relations for the growth rates of the moments of F are easily obtained:

(i) $\mu(n+1, n+1) \sim \mu(n+1, n)\mu(n, n+1)/\mu(n, n)$
(ii) $\mu(n+2, n+1)\mu(n, n)/\mu(n+1, n+1)\mu(n+1, n) \sim l_1$
(iii) $\mu(n+1, n+2)\mu(n, n)/\mu(\mu+1, n+1)\mu(n, n+1) \sim l_2$.

We observe that (4.2) is valid if, for example, X and Y are independent and the c_n and d_n are normalizers satisfying the conditions of Theorem 4.1 in [10] corresponding to the d.f.'s of X and Y respectively. Theorems 4.2 and 4.3, below, illustrate situations where X and Y are dependent and (4.2) holds.

THEOREM 4.2. *Let U and V be independent positive* r.v'*s having analytic* c.f.'*s. Then the* n-*th iterated transform of the joint* d.f. *of $X = UV$ and $Y = V$, suitably normalized converges to the product of simple exponential* d.f.'*s.*

Proof. Uuder the stated conditions all the moments λ_n and σ_n respectively of U and V are finite and the moments of the d.f. of (X, Y) and of its nth iterated transform are given by

$$\mu(i, j) = E(X^i Y^j) = E(U^i)E(V^{i+j}) = \lambda_i \sigma_{i+j}$$

$$\mu(i, j; n) \sim i!\, n^{-i-j}(\lambda_{n+i}/\lambda_n)(\sigma_{2n+i+j}/\sigma_{2n}) .$$

Choosing the c_n and d_n as in (4.2) with $b = 1$, $k = 1$ we see after some simplification that $\mu_{i,j}(H_n) \rightarrow i!\,j!$ as $n \rightarrow \infty$. (Here we have used Lemmas 4.2 and 4.3 in [10]). Such a choice of c_n and d_n is valid since c_{n+1}/c_n and d_{n+1}/d_n are bounded. Indeed they approach 1. The theorem is proved.

REMARK. If U and V have independent exponential distributions then the joint probability density function (p.d.f.) of X and Y is the one considered in Example 5.3.

We close this section with the following result illustrating a situation where the normalizers are as in (4.2) but the limit d.f. is not necessarily a d.f. of independent r.v.'s. To prove the theorem we merely need to verify that the mements of $G(x, y)$ determine it uniquely. This follows readily from the following sufficient condition for the determinateness of a moment sequence $\{m_{ij}\}$, namely, that the series $\sum_{i,j=0}^{\infty} m_{ij}x^i y^j/i!\,j!$ have a nonvanishing radius of convergence (cf. [4, p. 217]).

In the present case

$$m_{ij}(G) = \frac{ai!\,j!}{k_1^i k_2^j} + \frac{bi!\,j!}{k_3^i k_4^j}$$

and this clearly satisfies the sufficiency condition.

THEOREM 4.3. *Let X and Y be independent positive* r.v.'s *with* d.d.f.'s $f_1(x)$ *and* $f_2(y)$ *respectively and having analytic* c.f.'s *(so that the moments λ_n and σ_n of X and Y respectively are all finite). Let further $\lambda_{n+1}\sigma_n/\lambda_n\sigma_{n+1} \sim \alpha$ where $0 < \alpha < \infty$. Define the* p.d.f.

$$f(x, y) = af_1(x)f_2(y) + bf_1(y)f_2(x)$$

where $a + b = 1$ and a, b are positive real numbers. Then the normalizers (4.2) *lead to the limiting* d.f.

$$G(x, y) = \int_0^x\int_0^y (Ae^{-uk_1-vk_2} + Be^{-uk_3-vk_4})dvdu$$

where $k_1 = (a + b/\alpha)$, $k_2 = (a + b\alpha)$, $k_3 = (b + a\alpha)$, and $k_4 = (b + a/\alpha)$ $A = ak_1k_2$ *and* $B = bk_3k_4$.

COROLLARY. *If $\alpha \neq 1$, $G(x, y)$ is not the* d.f. *of independent* r.v.'s. *If $\alpha = 1$, $G(x, y)$ is the product of exponential distributions.*

The hypothesis of the theorem are satisfied if, for example,

$$f_1(x) = \exp(-x),\ x > 0;\ f_2(y) = \alpha \exp(-\alpha y),\ y > 0$$

where $\alpha > 0$.

5. Examples. This section contains three examples. The first two examples illustrate Theorem 3.1; the third one illustrates Theorem 4.2.

EXAMPLE 5.1. Let a, b, c be positive real numbers such that $a + b + c < 1$. Then the d.f. of a bivariate Bernoullian random vector is:

$$F(x, y) = \begin{cases} a & 0 < x, y \leqq 1 \\ a + b & 0 < x \leqq 1,\ y > 1 \\ a + c & 0 < y \leqq 1,\ x > 1 \\ 1 & \max(x, y) > 1 \\ 0 & \min(x, y) \leqq 0 \end{cases}$$

with $\mu = E(XY) = \alpha$ where $\alpha = 1 - a - b - c$. It is easily verified that the n-th iterated transform of $F(x, y)$ is the joint d.f. of two independently distributed random variables with a common d.f. given by $[1 - (1 - x)^n]$ for $0 < x < 1$ and one for $x > 1$. Thus $G_n(x, y)$ converges to the degenerate distribution (degenerate at the origin). But $G_n(x/n, y/n)$ converges to the product of exponential d.f.'s.

EXAMPLE 5.2. Consider the bivariate distribution with p.d.f. $f(x, y) = x + y$ for $0 < x, y < 1$ and zero elsewhere. The computation of G_n is unwieldy but

$$\mu(i, j; n) = \frac{i!\,j!}{2}(n + 1)^{-1}[(n + j + 1)(n + 2)_{(i)}^{-1}(n + 3)_{(j)}^{-1} + (n + i + 1)(n + 3)_{(i)}^{-1}(n + 2)_{(j)}^{-1}]$$

where $(a)_{(r)} = a(a + 1)\cdots(a + r - 1)$ for a positive integer r and $(a)_{(0)} = 1$. It follows that the moment of order (i, j) of $G_n(x/n, y/n)$ converges to $i!\,j!$ and hence the limiting d.f. is the product of simple exponential d.f.'s.

EXAMPLE 5.3. Let $f(x, y) = y^{-1}\exp(-y - x/y)$, $\min(x, y) > 0$ and zero elsewhere be a joint p.d.f. Here Theorem 4.2 applies and the limiting d.f. is again the product of simple exponential d.f.'s if we choose $c_n \sim 2n$, $d_n \sim 2$ as given by (4.2).

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REFERENCES

1. R. P. Boas, *Entire functions*, Academic Press, New York, 1954.
2. R. Cuppens, *Decomposition des functions caracteristiques analytiques*, C. R. Acad. Sci., Paris **263** (1966), A86-A88.
3. T. H. Hildebrandt and I. J. Schoenberg, *On linear functional operations and the moment problem for a finite interval in one or several dimensions*, Ann. of Math. (2), **34** (1933).
4. M. Loeve, *Probability theory*, D. Van Nostrand Co., Princeton 1963.
5. E. Lukacs, *Characteristic functions*, Hafner Publishing Co., New York, 1960.
6. I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar Publishing Co., New York, 1955.
7. J. M. H. Olmsted, *Real variables*, Appleton-Century-Crofts Inc., New York, 1959.
8. I. V. Ostrovskii, *Decompositions of multidimensional probability laws*, Soviet Math. **7** (1966), 1052-55.
9. E. Parzen, *Modern probability theory and its applications*, John. Wiley. and Sons, Inc., New York, 1960.

10. R. Shantaram and W. L. Harkness, *Convergence of a sequence of transformations of distribution functions*, Pacific J. Math. **31** (1969), 403-415.

11. W. H. Young. *On multiple integration by parts and the second theorem of the mean*, Proc. Lond. Math. Soc. (2) **16** (1917).

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