PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

$\mathscr{L}-2$ SUBSPACES OF GRASSMANN PRODUCT SPACES

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The subspaces of the second order Grassmann product space consisting of products of a fixed irreducible length k and zero are interesting not only for their own sake and their usefulness when determining the structure of linear transformations on the product space into itself which preserve the irreducible length k, but also because they are isomorphic to subspaces of skew-symmetric matrices of fixed rank $2k$. The structure of these subspaces and the corresponding preservers are known for $k=1$, when the underlying field F is algebraically closed. This paper gives a complete characterization of these subspaces when $k=2$ and F is algebraically closed. When F is not algebraically closed, these subspaces can be different.

Let $\mathscr{U}$ be an n-dimensional vector space over an *algebraically closed* field F. Let $\bigwedge^2 \mathscr{U}$ denote the $\binom{n}{2}$-dimensional space spanned by all Grassmann products $x_1 \wedge x_2$, $x_i \in F$. A vector $f \in \bigwedge^2 \mathscr{U}$ is said to have *irreducible length* k if it can be written as a sum of k, and *not less than* k, nonzero pure (decomposable) products in $\bigwedge^2 \mathscr{U}$. Let $\mathscr{L}_k$ denote the set of all vectors of irreducible length k in $\bigwedge^2 \mathscr{U}$, and $f \in \mathscr{L}_k$ if and only if $\mathscr{L}(f) = k$. A subspace of $\bigwedge^2 \mathscr{U}$ whose nonzero members are in $\mathscr{L}_k$ is called an $\mathscr{L} - k$ *subspace*.

An $\mathscr{L} - 2$ subspace H is a (1, 1)-*type subspace* if there exist fixed nonzero vectors $x \neq y$ such that each nonzero $f \in H$ can be written $f = x \wedge x_f + y \wedge y_f$. A basis of a (1, 1)-type subspace is called a (1, 1) basis. When $\dim \mathscr{U} = 4$, every $\mathscr{L}$-2 subspace has dimension one ([4], Th. 10).

It is shown here that (i) for $\dim \mathscr{U} = n \geqq 5$, there always exists an $\mathscr{L} - 2$ subspace of (1, 1)-type and dimension two; (ii) the 2-dimensional $\mathscr{L} - 2$ subspaces are of (1, 1)-type; (iii) every $\mathscr{L} - 2$ subspace of dimension at least four is of (1, 1)-type; (iv) the $\mathscr{L} - 2$ subspaces have dimension at most $(n-3)$ when $n \geqq 6$; and this maximum dimension is attained. Also the 3-dimensional $\mathscr{L} - 2$ subspaces are characterized, and these are the most varied.

From [4], Theorem 5, each $f \in \mathscr{L}_k$ can be uniquely associated with a $2k$-dimensional subspace $[f]$ of $\mathscr{U}$. The pair $\{f_1, f_2\}$ is said to be a P_m-pair in $\mathscr{L}_2$ if $[f_1] + [f_2]$ has dimension m; and the set $\{f_1, \cdots, f_k\}$ in $\mathscr{L}_2$ is pairwise-P_m if each pair is a P_m-pair, for $i \neq j$.

THEOREM 1. *Let* $\dim \mathscr{U} = n \geqq 5$. *Then there always exists a*

(1, 1)-type $\mathscr{L} - 2$ subspace of dimension two.

Proof. For $n = 5, u_1, \cdots, u_5$ independent in $\mathscr{U}$, the subspace $\langle u_1 \wedge u_2 + u_3 \wedge u_4, u_2 \wedge u_5 + u_1 \wedge u_3 \rangle$ is a (1, 1)-type $\mathscr{L} - 2$ subspace of dimension two. For $n = 6, u_1, \cdots, u_6$ independent in $\mathscr{U}$, the subspace $\langle u_1 \wedge u_2 + u_3 \wedge u_4, u_1 \wedge u_5 + u_3 \wedge u_6 \rangle$ is a (1, 1)-type $\mathscr{L} - 2$ subspace of dimension two.

THEOREM 2. *Every 2-dimensional $\mathscr{L} - 2$ subspace is a (1, 1)-type subspace.*

The theorem follows from the following Lemmas 1 to 4.

LEMMA 1. *Let f_1 and f_2 be a P_7-pair in $\mathscr{L}_2$, a, b be nonzero in F. Then $\mathscr{L}(af_1 + bf_2) = 3$.*

Proof. Let $[f_1] \cap [f_2] = \langle x_1 \rangle$. By Lemma 9 of [4], we can choose a basis $\{x_1, \cdots, x_4\}$ of $[f_1]$ such that $f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$ and a basis $\{x_1, x_5, x_6, x_7\}$ such that $f_2 = x_1 \wedge x_5 + x_6 \wedge x_7$, with $[f_1] + [f_2] = \langle x_1, \cdots, x_7 \rangle$. Then $z = af_1 + bf_2 = x_1 \wedge (ax_2 + bx_5) + ax_3 \wedge x_4 + bx_6 \wedge x_7$ and $\mathscr{L}(z) = 3$ by Theorem 7 of [4].

LEMMA 2. *Let f_1, f_2 be a basis of a 2-dimensional $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2\}$ is a P_k-pair where k is either 5 or 6.*

Proof. Each of $[f_1]$ and $[f_2]$ has dimension four. It is easy to see that k cannot be 4 (Theorem 10 of [4]). By Lemma 1, we conclude $k \neq 7$. If $k = 8$, Theorem 6 of [4] implies that $\mathscr{L}(f_1 + f_2) = 4$. Hence k is either 5 or 6.

DEFINITION. $f_1, f_2 \in \mathscr{L}_2$ can be expressed in (1, 1)-form if $\{f_1, f_2\}$ have representations $f_i = x \wedge u_i + y \wedge v_i$, $i = 1, 2$ and $\langle x, y \rangle$ is a fixed 2-dimensional subspace of $\mathscr{U}$.

LEMMA 3. *Let $\{f_1, f_2\}$ be a P_5-pair and a basis for an $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2\}$ have representations*

$$f_1 = y_4 \wedge u_1 + u_2 \wedge u_3 ,$$
$$f_2 = y_5 \wedge u_2 + u_1 \wedge u_3 ,$$

where $\{u_1, u_2, u_3, y_4, y_5\}$ is some basis of $[f_1] + [f_2]$.

Proof. Let $\mathscr{U}_0 = [f_1] \cap [f_2]$. By Lemma 9 of [4], there are representations

$$f_1 = x_1 \wedge v_1 + v_2 \wedge v_3 ,$$
$$f_2 = x_2 \wedge w_1 + w_2 \wedge w_3 ,$$

where $\langle v_1, v_2, v_3\rangle = \langle w_1, w_2, w_3\rangle = \mathscr{U}_0$. If v_1, w_1 are dependent then some combination of f_1 and f_2 has irreducible length $\leqq 1$. Hence they are independent. Moreover $\langle v_1, w_1\rangle \cap \langle v_2, v_3\rangle$ and $\langle v_1, w_1\rangle \cap \langle w_2, w_3\rangle$ are both nonnull, and hence, without loss of generality, both v_2 and w_2 are in $\langle v_1, w_1\rangle$. Thus $v_2 = av_1 + bw_1$ and $w_2 = cv_1 + dw_1$. Clearly $b \neq 0, c \neq 0$. Finally

$$w_3 = pv_1 + qw_1 + rv_3, r \neq 0 .$$

Setting $y_4 = br^{-1}c^{-1}(x_1 - av_3)$, $y_5 = x_2 - dw_3 + cqv_1$, $u_1 = b^{-1}rcv_1$, $u_2 = w_1$, $u_3 = bv_3$, we obtain the desired representations.

COROLLARY 1. *Let $\{f_1, f_2\}$ be a P_5-pair and $\langle f_1, f_2\rangle$ a 2-dimensional $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2\}$ can be expressed in* (1, 1)*-form.*

LEMMA 4. *Let $\{f_1, f_2\}$ be a P_6-pair and $\langle f_1, f_2\rangle$ a 2-dimensional $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2\}$ can be expressed in* (1, 1)*-form.*

Proof. By Lemma 9 of [4], there are representations

$$f_1 = x_1 \wedge u + v \wedge w , \qquad f_2 = x_1 \wedge u' + v' \wedge w' ,$$

where $\langle x_1\rangle \subset [f_1] \cap [f_2]$ and $\langle u, v, w\rangle, \langle u', v', w'\rangle$ are contained in

$$([f_1] + [f_2] - \langle x_1\rangle) .$$

If $\langle v, w\rangle \cap \langle v', w'\rangle = 0$, some linear combination of f_1, f_2 has irreducible length 3. If $\langle v, w\rangle = \langle v', w'\rangle$ some linear combination of f_1, f_2 has irreducible length $\leqq 1$. The result follows.

Lemma 2 implies the following lemma.

LEMMA 5. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, \cdots, f_k\}$ be an independent subset of H. Then*

(i) $3 \geqq [f_i] \cap [f_j] \geqq 2$ *for* $1 \leqq i < j \leqq k$;
(ii) $\dim \sum_{i=1}^{k-1} [f_i] \leqq \dim \sum_{i=1}^{k} [f_i] \leqq \dim \sum_{i=1}^{k-1} [f_i] + 2$.

Corollary 1 implies:

LEMMA 6. *Let $\{f_1, f_2, f_3\}$ be pairwise-P_6 and generate a 3-dimensional $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2, f_3\}$ is a* (1, 1) *basis for $\langle f_1, f_2, f_3\rangle$ if $[f_3] \supset [f_1] \cap [f_2]$.*

1. $\dim \mathscr{U} = 5$. It is not difficult to see that when $\dim \mathscr{U} = 5$, the basis of any $\mathscr{L} - 2$ subspace must consist of pairwise-P_5 vectors.

THEOREM 3. *Let* $\dim \mathscr{U} = 5$, *H an $\mathscr{L} - 2$ subspace. Let* $\{f_1, \cdots, f_k\}$ *be independent in H. Then* $k \leqq 3$.

Proof. Let $\{u_1, \cdots, u_5\}$ be a basis of $\mathscr{U}$. Then each f_l, $1 \leqq l \leqq k$, has the form $f_l = \sum a^l_{ij} u_i \wedge u_j (1 \leqq i < j \leqq 5)$, $a_{ij} \in F$. (*) Consider the vector $f = \sum_{i=1}^k \beta_i f_i$, $\beta_i \in F$ not all zero. Now $\mathscr{L}(z) \leqq 1$ if $k \geqq 4$ for some $\{\beta_i\}$ not all zero since the following is true. $f = \sum_{i=1}^k \beta_i f_i = \sum p(i_1, i_2) u_{i_1} \wedge u_{i_2} (1 \leqq i_1 < i_2 \leqq 5)$ where $p(k_{\sigma(1)}, k_{\sigma(2)}) = \operatorname{sgn} \sigma p(k_1, k_2)$, σ a permutation of $\{1, 2\}$, and $\{k_i\}$ are arbitrary integers $1 \leqq k_i \leqq 5$. Thus, using (*), it follows that $\{p(i_1, i_2)\}$ are linear homogeneous functions of $\{\beta_1, \cdots, \beta_k\}$. Then the quadratic p-relations

$$\sum_{\mu=0}^{r} (-1)^\mu p(i_1, \cdots, i_{r-1}, j_\mu) p(j_0, \cdots, j_{\mu-1}, j_{\mu+1}, \cdots, j_r) = 0$$

for all sequences $(i_1, \cdots, i_{r-1})$, $(j_0, \cdots, j_r)$ of integers taken from $\{1, \cdots, n\}$ define (for $n = 5$, $r = 2$ in this case) *five* nontrivial equations, which are in fact quadratic homogeneous equations in the indeterminates $\beta_1, \cdots, \beta_k$ in F. Moreover, of these five, exactly *three* are independent (see [3], pp. 289, 312). Hence, if $k \geqq 4$, then there exists a nontrivial solution for the five equations (see [6], chapter 11). For these values of $\beta_1, \cdots, \beta_k$ (not all zero), $\mathscr{L}(f) \leqq 1$. Hence $k < 4$. The following three vectors generate an $\mathscr{L} - 2$ subspace of dimension three:

$$\begin{aligned} f_1 &= u_4 \wedge u_1 + u_3 \wedge u_2 , \\ f_2 &= u_5 \wedge u_2 + u_3 \wedge u_1 , \\ f_3 &= (u_4 + u_5) \wedge u_3 + u_2 \wedge u_1 . \end{aligned}$$

The following theorem is true for all n.

THEOREM 4. *Let* $\dim \mathscr{U} = n$. *Let* $\{f_1, \cdots, f_k\}$ *be a* (1, 1) *basis for an $\mathscr{L} - 2$ subspace. Then* $k \leqq n - 3$.

Moreover, when $n \geqq 5$, there always exists a (1, 1)*-type $\mathscr{L} - 2$ subspace of dimension* $(n - 3)$.

Proof. Suppose $k = n - 2$. Each f_i can be written $f_i = u_1 \wedge y_i + u_2 \wedge z_i$, $1 \leqq i \leqq n - 2$, where $\langle u_1, u_2, y_1, \cdots, y_{n-2}, z_1, \cdots, z_{n-2} \rangle \subseteqq \mathscr{U}$. Now $\{u_1, u_2, y_1, \cdots, y_{n-2}\}$ must be independent for, if not, some linear combination of $\{f_i\}$ has irreducible length $\leqq 1$. Hence $\mathscr{U} = \langle u_1, u_2, y_1, \cdots, y_{n-2} \rangle$. Thus $z_j = \sum_{i=1}^{n-2} \alpha_{ij} y_i + \beta_j u_1$, $1 \leqq j \leqq n - 2$. If $\beta_j \neq 0$, write

$$f_j = u_1 \wedge (y_j - \beta_j u_2) + u_2 \wedge \left(\sum_{j=1}^{n-2} \alpha_{ij} y_i \right) .$$

Hence, without loss of generality, we can assume $\{z_i\}$ is dependent on $\{y_i\}$. Using a similar argument, $\{y_i\}$ is dependent on $\{z_i\}$. Hence $\langle y_1, \cdots, y_{n-2}\rangle = \langle z_1, \cdots, z_{n-2}\rangle$. Hence, for some $\{\alpha_i\} \in F$, not all zero, we have $\sum_{i=1}^{n-2}\alpha_i y_i = \lambda\sum_{i=1}^{n-2}\alpha_i z_i = y$ for some $0 \neq \lambda \in F$; and $f = \sum_{i=1}^{n-2}\alpha_i f_i$ has irreducible length $\leqq 1$. Hence $k \leqq n-3$.

Now let $f_i = u_1 \wedge u_{i+2} + u_2 \wedge u_{i+3}$ for $i = 1, \cdots, (n-3)$, where $\langle u_1, \cdots, u_n\rangle = \mathscr{U}$. Then $\{f_i\}$ generate an $\mathscr{L}-2$ subspace of dimension $(n-3)$.

COROLLARY 2. *Let* $\dim \mathscr{U} = 5$, H *an* $\mathscr{L}-2$ *subspace of* $(1, 1)$-*type. Then, if* $\dim H > 1$, $\dim H = 2$.

We pause here to introduce some notation.

DEFINITION 1. For subsets S, T of $\mathscr{U}$, $[S; T] = \langle S \cup T\rangle - \langle T\rangle$. In the case where $S = \{x_1, \cdots, x_s\}$ and $T = \{x_{s+1}, \cdots, x_k\}$, we use the convention $[S; T] = [x_1, \cdots, x_s; x_{s+1}, \cdots, x_k]$. Note that in this case if $y \in [S; T]$, then $y = \sum_{i=1}^{k}\alpha_i x_i$, $\alpha_i \in F$, and *at least one* of $\alpha_1, \cdots, \alpha_s$ is nonzero.

DEFINITION 2. For subsets S, T of $\mathscr{U}$, $S \wedge T = \{x \wedge y : x \in S$ and $y \in T\}$. In the case where S is the singleton $\{x\}$, we shall write $S \wedge T$ as $x \wedge T$. Similarly for T. Also, if S is the space $\langle x_1, \cdots, x_k\rangle$, then we shall regard S as a set and write $S \wedge T$ as $[x_1, \cdots, x_k] \wedge T$. Similarly for T.

The three-dimensional $\mathscr{L}-2$ ***subspace when*** $\dim \mathscr{U} = 5$. In this context, a basis $\{f_1, f_2, f_3\}$ of an $\mathscr{L}-2$ subspace H is necessarily pairwise P_5. It is not a $(1, 1)$ basis. However, either there exists a three-dimensional subspace $\mathscr{U}_0$ of $\mathscr{U}$ contained in each $[f_i]$, or there exists a exists a five-dimensional subspace $\mathscr{W} \subseteqq \mathscr{U}$ which contains each $[f_i]$ (see [1], p. 14). In fact, $\mathscr{W} = \mathscr{U}$. Moreover, since $\dim \mathscr{U} = 5$, $\dim [f_1] \cap [f_2] = 3$, and $\dim [f_3] = 4$, then $\dim \bigcap_{i=1}^{3} [f_i] \geqq 2$. Consequently this intersection has dimension two or three.

THEOREM 5. *Let* $\dim \mathscr{U} = 5$. *Let* $\{f_1, f_2, f_3\}$ *be a basis for an* $\mathscr{L}-2$ *subspace* H *such that* $[f_i] \supset \mathscr{U}_0$, $i = 1, 2, 3$, *where* $\mathscr{U}_0$ *is a three-dimensional subspace of* $\mathscr{U}$. *Then* $\mathscr{U}$ *has a basis* $\{u_1, u_2, u_3, x_4, x_5\}$ *such that there are representations*

$$f_1 = x_4 \wedge u_1 + u_2 \wedge u_3 ,$$
$$f_2 = x_5 \wedge u_2 + u_1 \wedge u_3 ,$$
$$f_3 = y \wedge u_3 + u_2 \wedge u_1 ,$$

where $y \in [x_4; x_5 \cdot u_1, u_2] \cap [x_5; x_4, u_1, u_2]$.

Proof. $\mathscr{U}$ has a basis $\{w_1, w_2, w_3, y_4, y_5\}$ such that $\mathscr{U}_0 = \langle w_1, w_2, w_3\rangle$ and there are representations $f_1 = y_4 \wedge w_1 + w_2 \wedge w_3$, $f_2 = y_5 \wedge w_2 + w_1 \wedge w_3$ (see Lemma 3). Now there exists $y' \in [f_3]$ such that $y' \notin \mathscr{U}_0$ and $y' \in [y_4, y_5; w_1, w_2, w_3]$. Since $\{f_1, f_2, f_3\}$ is pairwise-P_5, it is easy to see $y' \in [y_4; y_5, w_1, w_2, w_3] \cap [y_5; y_4, w_1, w_2, w_3]$. Hence f_3 has a representation

$$f_3 = y' \wedge u + v \wedge w; \mathscr{U}_0 = \langle u, v, w\rangle ,$$

(see [4], Lemma 9). Now if $u \in \langle w_1, w_2\rangle$, it is possible to find representations of f_1, f_2, f_3 such that they form a (1, 1) basis for H. This contradicts Corollary 2. Hence $u \notin \langle w_1, w_2\rangle$, but $u \in [w_3; w_1, w_2]$. In fact, without loss of generality, we can take $u = w_3 + cw_1 + c'w_2$.

Now $\langle w_2, u\rangle$, $\langle w_1, u\rangle$, $\langle v, w\rangle$ intersect pairwise in dimension at least one. Also $u \notin \langle v, w\rangle$. Therefore we may suppose $v \in [w_2; u]$, $w \in [w_1; u]$. We set

$$v = aw_2 + a'u,\ w = bw_1 + b'u .$$

Then

$$f_3 = (y' + ab'w_2 - a'bw_1) \wedge u + \gamma w_2 \wedge w_1,\ 0 \neq \gamma \in F .$$

Let

$$\begin{aligned} &\alpha^2 = \gamma , \\ &w_2 = \alpha^{-1}u_2,\ w_1 = \alpha^{-1}u_1,\ u = \alpha u_3 . \end{aligned}$$

Then

$$\begin{aligned} f_1 &= (y_4 - cw_2) \wedge \alpha^{-1}u + u_2 \wedge u_3 , \\ f_2 &= (y_5 - c'w) \wedge \alpha^{-1}u_2 + u_1 \wedge u_3 , \\ f_3 &= x \wedge \alpha u_3 + u_2 \wedge u_1 . \end{aligned}$$

We have the result on setting $x_4 = \alpha^{-1}(y_4 - cw_2)$, $x_5 = \alpha^{-1}(y_5 - c'w_1)$, $y = \alpha x$, and noting that $y \in [x_4; x_5, u_1, u_2] \cap [x_5; x_4, u_1, u_2]$.

THEOREM 6. *Let* $\dim \mathscr{U} = 5$. *Let* $\{f_1, f_2, f_3\}$ *be a basis for an* $\mathscr{L} - 2$ *subspace H such that* $\dim \bigcap_{i=1}^{3} [f_i] = 2$. *Then* $\mathscr{U}$ *has a basis* $\{u_1, u_2, u_3, x_4, x_5\}$ *such that* f_1, f_2, f_3 *have representations given by either* (i) *or* (ii) *below.*

(i) $f_1 = x_4 \wedge u_1 + u_2 \wedge u_3$, $f_2 = x_5 \wedge u_2 + u_1 \wedge u_3$, $f_3 = u \wedge y + u_3 \wedge y'$, $y, y' \in [x_4, x_5; u_1, u_2, u_3]$, $u \in \langle u_1, u_2\rangle$,

(ii) f_1, f_2 *as in* (i). *With* $u \in \langle u_1, u_2\rangle$, $u' \in \langle u_1, u_2, u_3\rangle$, $f_3 = \gamma u \wedge u' + y \wedge y'$, $y, y' \in [x_4, x_5; u_1, u_2, u_3]$, $0 \neq \gamma \in F$.

Proof. The proof involves a suitable choice of a basis of $\mathscr{U}$, as in the proof of Theorem 5, and the use of the following lemma.

LEMMA 7. *Let* $f \in \mathscr{L}_2$ *and* $\langle u_1, u_2\rangle$ *any two-dimensional subspace* *of* $[f]$. *Then either*

(i) *there exist* $v, w \in [f]$ *such that* $f = \gamma u_1 \wedge u_2 + v \wedge w$, $0 \neq \gamma \in F$,

or (ii) *there exist* $v', w' \in [f]$ *such that* $f = u_1 \wedge v' + u_2 \wedge w'$.

Proof. Let $\{u_1, \cdots, u_4\}$ be any basis of $[f]$. By Lemma 9 of [4], f has a representation $f = u_1 \wedge u + v \wedge w$, where $\langle u, v, w \rangle = \langle u_2, u_3, u_4 \rangle$. If $\underline{u_1 \wedge u_2 \wedge f = 0}$, then $\langle u_1, u_2 \rangle \cap \langle v, w \rangle \neq 0$, and it is easy to see $u_2 \in \langle v, w \rangle$ since $u_1 \notin \langle u, v, w \rangle$. If $u_1 \wedge u_2 \wedge f \neq 0$, then $\langle u_1, u_2, v, w \rangle = [f]$, and $u = au_1 + bu_2 + cv + dw$ with $b \neq 0$. Then $f = bu_1 \wedge u_2 + [u_1 \wedge (cv + dw) + v \wedge w]$. By Corollary 8 of [4] and since $\mathscr{L}(f) = 2$, the term in square brackets has irreducible length one.

We can in fact replace the basis $\{f_1, f_2, f_3\}$ in Theorem 3 by the basis $\{f_1 + f_2, f_2, f_3\}$. Then $[f_1 + f_2] \cap [f_2] \cap [f_3]$ has dimension two. We obtain:

THEOREM 7. *Let* $\dim \mathscr{U} = 5$, H *an* $\mathscr{L} - 2$ *subspace of dimension three. Then* H *has a basis which is either of type* (i) *or type* (ii) *in Theorem* 6.

Examples of such bases are the following:

EXAMPLE 1. $f_1 = x_4 \wedge u_1 + u_2 \wedge u_3$, $f_2 = x_5 \wedge u_2 + u_1 \wedge u_3$,
$f_3 = u_2 \wedge x_4 + u_3 \wedge x_5$.

EXAMPLE 2. f_1, f_2 as in Example 1. $f_3 = u_2 \wedge (u_1 + u_3) + x_4 \wedge x_5$.

2. $\dim \mathscr{U} = 6$.

The three-dimensional $\mathscr{L} - 2$ *subspaces*. If H is an $\mathscr{L} - 2$ subspace with a basis $\{f_1, f_2, f_3\}$ and $\dim \mathscr{U} = 6$, then $\dim \sum_{i=1}^{3} [f_i] = 5$ or 6. The first case was discussed in § 1. We show that, in the second case, H has a basis of pairwise-P_6 vectors, and there are three possibilities for such a basis.

Suppose $\dim \sum_{i=1}^{3} [f_i] = 6$. Now each pair in $\{f_1, f_2, f_3\}$ is either a P_5-or a P_6-pair. Thus either $\{f_1, f_2, f_3\}$ is pairwise-P_5 or at least one pair is a P_6-pair. The first case is then reduced to the second.

THEOREM 8. *Let* H *be an* $\mathscr{L} - 2$ *subspace, and let* $\{f_1, f_2, f_3\}$ *be pairwise-*P_5, *independent in* H *such that* $\dim \sum_{i=1}^{3} [f_i] = 6$. *Then* $(\sum_{i=1}^{3} [f_i])$ *has a basis* $\{u_1, u_2, u_3, x_4, x_5, x_6\}$ *such that there are representations*

$$f_1 = x_4 \wedge u_1 + u_2 \wedge u_3 ,$$

$$f_2 = x_5 \wedge u_2 + u_1 \wedge u_3 ,$$
$$f_3 = x_6 \wedge u + v \wedge u_3 ,$$
$$\langle u, v\rangle = \langle u_1, u_2\rangle,\ u \notin \langle u_1\rangle,\ u \notin \langle u_2\rangle .$$

Proof. There exists a three-dimensional subspace $\mathscr{U}_0$ of $\mathscr{U}$ contained in each $[f_i]$ (see [1], p. 14). The proof is similar to that of Theorem 5. We choose a basis $\{u_1, u_2, v_3, y_4, y_5, y_6\}$ of $\sum_{i=1}^{3}[f_i]$ in order to obtain representations $f_1 = y_4 \wedge u_1 + u_2 \wedge v_3$, $f_2 = y_5 \wedge u_2 + u_1 \wedge v_3$, $f_3 = y_6 \wedge w_1 + w_2 \wedge w_3$, and $\langle w_1, w_2, w_3\rangle = \langle u_1, u_2, u_3\rangle = \mathscr{U}_0$. Without loss of generality, we can assume $w_2 \in \langle u_1, u_2\rangle$. Then $w_1 \in \langle u_1, u_2\rangle$, for, if not, $\langle u_1, u_2, w_1\rangle = \mathscr{U}_0$ and $(f_1 + f_2 + f_3)$ has irreducible length 3 (see [4], Th. 7). Moreover $u \notin \langle u_1\rangle$ and $u \notin \langle u_2\rangle$ (see proof of Lemma 3). Thus $\langle w_1, w_2\rangle = \langle u_1, u_2\rangle$ and $w_3 = \lambda(v_3 + \bar{u})$ for some $0 \neq \lambda \in F$ and $\bar{u} \in \langle u_1, u_2\rangle$. Then $f_1 = y_4' \wedge u_1 + u_2 \wedge (v_3 + \bar{u})$, $f_2 = y_5' \wedge u_2 + u_1 \wedge (v_3 + \bar{u})$, and $f_3 = y_6 \wedge w_1 + \lambda w_2 \wedge (v_3 + \bar{u})$. The appropriate choice of new basis vectors gives the required representations.

COROLLARY 3. *Let H be an $\mathscr{L} - 2$ subspace, and let $\{f_1, f_2, f_3\}$ be pairwise-P_5, independent in H such that* $\dim \sum_{i=1}^{3}[f_i] = 6$. *Then $\{f_1, f_2, f_3\}$ is a* (1, 1) *basis for $\langle f_1, f_2, f_3\rangle$.*

Proof. Choose a suitable representation of f_3.

LEMMA 8. *Let $\{f_1, f_2, f_3\}$ be a* (1, 1) *basis of an $\mathscr{L} - 2$ subspace satisfying* (i) $\dim \sum_{i=1}^{3}[f_i] = 6$, (ii) *$\{f_1, f_2\}$ is a P_6-pair. Then $\{f_1, f_2\}$ can be extended to a* (1, 1) *basis of pairwise-P_6 vectors of $\langle f_1, f_2, f_3\rangle$.*

Proof. We choose a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ of $\sum_{i=1}^{3}[f_i]$ so that

$$f_1 = u_1 \wedge x_3 + u_2 \wedge x_4,\ f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$$

(Lemma 4). Also $f = u_1 \wedge y + u_2 \wedge y'$, and we can take

$$\langle y, y'\rangle \subset \langle u_2, x_3, \cdots, x_6\rangle$$

([4], Lemma 9). Let $y = u + \sum_{i=3}^{6}\alpha_i x_i$, $y' = u' + \sum_{i=3}^{6}\beta_i x_i$ where $\{u, u'\} \in \langle u_2\rangle$. We can choose $\lambda, \mu \in F$ such that

$$\begin{vmatrix} \alpha_3 + \lambda & \alpha_4 \\ \beta_3 & \beta_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha_5 + \mu & \alpha_6 \\ \beta_5 & \beta_6 \end{vmatrix}$$

are both nonzero. Then $g_3 = (\lambda f_1 + \mu f_2 + f_3)$ extends $\{f_1, f_2\}$ to a basis of $\langle f_1, f_2, f_3\rangle$ and $[g_3] \cap \langle x_3, x_4\rangle = 0$, $[g_3] \cap \langle x_5, x_6\rangle = 0$.

In Lemma 8, we can in fact take

$$f_1 = u_1 \wedge x_3 + u_2 \wedge x_4 \,,$$
$$f_2 = u_1 \wedge x_5 + u_2 \wedge x_6 \,,$$
$$f_3 = u_1 \wedge y + u_2 \wedge y', \langle y, y' \rangle \subset \langle u_2, x_3, \cdots, x_6 \rangle$$

and does not intersect each $[f_i]$, $i \neq 3$.

THEOREM 9. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be pairwise-P_5, independent in H such that* $\dim \sum_{i=1}^{3} [f_i] = 6$. *Then $\langle f_1, f_2, f_3 \rangle$ has a* (1, 1) *basis of pairwise-P_6 vectors.*

Proof. Using the representations of f_1, f_2, f_3 obtained in Theorem 8 and Corollary 3, we take $g_1 = (f_1 + f_3)$. Then $\{g_1, f_2, f_3\}$ is a (1, 1) basis $\{g_1, f_2\}$ a P_6-pair, and $[g_1] \cap [f_2] \cap [f_3] = \langle u_1, u_2 \rangle$. The result follows by Lemma 8.

COROLLARY 4. *Let $\{f_1, f_2, f_3\}$ be a* (1, 1) *basis for an $\mathscr{L} - 2$ subspace such that* $\sum_{i=1}^{3} [f_i] = 6$. *Then there exist a* (1, 1) *basis of pairwise-P_6 vectors for $\langle f_1, f_2, f_3 \rangle$.*

THEOREM 10. *Let H be an $\mathscr{L} - 2$ subspace,* $\dim H \geqq 3$. *Let $\{f_1, f_2, f_3\}$ be independent in H such that* (i) $\dim \sum_{i=1}^{3} [f_i] = 6$, (ii) $\bigcap_{i=1}^{3} [f_i] = 0$. *Then $\{f_1, f_2, f_3\}$ are pairwise-P_6 and for any basis $\{u_1, u_2\}$ of $[f_1] \cap [f_2]$, $(\sum_{i=1}^{3} [f_i])$ has a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ such that $\{f_1, f_2, f_3\}$ have representations* $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, $f_3 = x_3 \wedge w_1 + x_4 \wedge w_2 = x_5 \wedge v_1 + x_6 \wedge v_2$, $\langle w_1, w_2 \rangle = \langle x_5, x_6 \rangle$, $\langle v_1, v_2 \rangle = \langle x_3, x_4 \rangle$.

Proof. If $\{f_1, f_2, f_3\}$ were not pairwise-P_6, we would have a contradiction of (ii). Since $\{f_1, f_2\}$ is a P_6-pair, the choice of representations of f_1, f_2 is immediate (Lemma 4). Let

$$[f_3] = \langle x'_3, x'_4, z_1, z_2 \rangle, x'_3 \in [x_3; u_1, u_2,], x'_4 \in [x_4; u_1, u_2] \,.$$

It is not difficult to show we can represent $f_3 = x'_3 \wedge w_1 + x'_4 \wedge w_2$, where $\langle w_1, w_2, x'_4 \rangle = \langle x'_4, z_1, z_2 \rangle$, and thus $\{w_1, w_2\} \in [z_1, z_2; x'_4]$, and $f_1 = u_1 \wedge x'_3 + u_2 \wedge x'_4$ (using Lemma 9 of [4] and proof of Lemma 4).

In a similar fashion, *without altering* u_1 or u_2, we can choose

$$x'_5 \in [x_5; u_1, u_2], x'_6 \in [x_6; u_1, x_2], \langle u'_5, x'_6 \rangle = \langle z_1, z_2 \rangle \,,$$

so that $f_2 = u_1 \wedge x'_5 + u_2 \wedge x'_6$, $f_3 = x'_5 \wedge v_1 + x'_6 \wedge v_2$, where $\langle v_1, v_2, x'_6 \rangle = \langle x'_6, x'_3, x'_4 \rangle$. Thus $\{v_1, v_2\} \in [x'_3, x'_4; x'_6]$. From above, f_3 is also $x'_3 \wedge w_1 + x'_4 \wedge w_2$, and $\{w_1, w_2\} \in [x'_5, x'_6; x'_4]$. With respect to the independent set $\{x'_i \wedge x'_j\}$, the coefficient of $x'_3 \wedge x'_4$ is zero in the second expression obtained for f_3, and the coefficient of $x'_5 \wedge x'_6$ is zero in the first. It follows that neither term appears in f_3. We have the result on placing x_i for x'_i, $i = 3, \cdots, 6$.

LEMMA 9. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H satisfying*

(i) $\dim \sum_{i=1}^{3} [f_i] = 6$,

(ii) *$\{f_1, f_2\}$ is a P_6-pair,*

(iii) $\dim \bigcap_{i=1}^{3} [f_i] = 1$.

Then there exists $g_3 \in \langle f_1, f_2, f_3\rangle$ such that $\{f_1, f_2, g_3\}$ is a basis of pairwise-P_6 vectors for $\langle f_1, f_2, f_3\rangle$ and $\bigcap_{i=1}^{3} [f_i] = [g_3] \cap [f_1] \cap [f_2]$.

Proof. There are representations $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, and $\sum_{i=1}^{3} [f_i] = \langle u_1, u_2, x_3, \cdots, x_6\rangle$. Let $\bigcap_{i=1}^{3} [f_i] = \langle u\rangle$. Then $u \in \langle u_1, u_2\rangle$. Without loss of generality, we can take $u = u_1$. By Lemma 9 of [4], $f_3 = u_1 \wedge w + w' \wedge v$, $\langle w, w', v\rangle \subset \langle u_2, x_3, \cdots, x_6\rangle$. If $\{f_1, f_2, f_3\}$ are pairwise-P_6, we have the result.

Case 1. Suppose $\{f_1, f_3\}$ is a P_6-pair and $\{f_2, f_3\}$ is a P_5-pair. Then we can take $f_3 = u_1 \wedge w + x_4 \wedge v'$ (use Lemma 6 and (iii)), where

$$\langle w, v_4, v'\rangle \subset \langle u_2, x_3, \cdots, x_6\rangle .$$

Let $[f_2] \cap [f_3] = \langle u_1, y, y'\rangle$. Then $\{y, y'\} \in [x_5, x_6; u_2]$. Therefore

$$f_3 = u_1 \wedge w + x_4 \wedge v', \; w \in [x_5, x_6; u_2, x_4], \; v' \in [x_5, x_6; u_2] .$$

Let $v' = ax_5 + bx_6 + cu_2$. Choose $\gamma \neq 0$ such that $\gamma + c \neq 0$. Let $g_3 = f_3 + \gamma f_1$. Then $\{g_3, f_1\}$ and $\{f_2, g_3\}$ are P_6-pairs.

Case 2. Suppose $\{f_1, f_3\}$, $\{f_2, f_3\}$ are both P_5-pairs. This and (iii) imply $\dim ([f_1] \cap [f_3]) + ([f_2] \cap [f_3]) = 5$, which exceeds the dimension of $[f_3]$. Hence this case is not possible.

LEMMA 10. *If $f \in \mathscr{L}_2$ and $f \in x_1 \wedge [x_2, x_3, x_4] + [x_4; x_2] \wedge [x_3; x_2]$ where $[f] = \langle x_1, \cdots, x_4\rangle$, then $f \in x_1 \wedge [x_2] + [x_4; x_1, x_2] \wedge [x_3; x_1, x_2]$.*

Proof. Apply Lemma 7 to $\langle x_1, x_2\rangle$ and notice that the coefficient of $x_4 \wedge x_3$ is nonzero in f.

THEOREM 11. *Let H be an $\mathscr{L} - 2$ subspace, $\dim H \geqq 3$. Let $\{f_1, f_2, f_3\}$ be pairwise-P_6 and independent in H satisfying*

(i) $\dim \sum_{i=1}^{3} [f_i] = 6$,

(ii) $\dim \bigcap_{i=1}^{3} [f_i] = 1$.

Then for $\langle u_1\rangle = \bigcap_{i=1}^{3} [f_i]$ and any vector u_2 such that $\langle u_1, u_2\rangle = [f_1] \cap [f_2]$, there exists a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ such that $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, $f_3 = u_1 \wedge y + x_4 \wedge x_6$, where $y \in \langle u_2, x_3, \cdots, x_6\rangle$,

$y \notin \langle u_1, x_3, x_5\rangle$, $y \notin [f_i]$, $i = 1, 2$. Furthermore, there exists g_3 such that $\langle f_1, f_2, g_3\rangle = \langle f_1, f_2, f_3\rangle$ and $g_3 = u_1 \wedge u_2 + v \wedge w$, $v \in [x_4; u_1, u_2]$, $w \in [x_6; u_1, u_2]$ and $g_3 = v' \wedge w' + \gamma x_4 \wedge x_6$, $0 \neq \gamma \in F$, $v' \in [u_1; x_4, x_6]$, $w' \in [u_2; x_4, x_6]$.

Proof. The proof involves choosing a suitable basis of $\sum_{i=1}^{3} [f_i]$ and the use of Lemma 6 and 7. To obtain the form of g_3, we use Lemma 10.

LEMMA 11. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H such that*

(i) $\dim \sum_{i=1}^{3} [f_i] = 6$,

(ii) *$\{f_1, f_2\}$ is a P_6-pair,*

(iii) $\dim \bigcap_{i=1}^{3} [f_i] = 2$;

then $\{f_1, f_2\}$ can be extended to a basis of pairwise-P_6 vectors for $\langle f_1, f_2, f_3\rangle$.

Proof. By a suitable choice of basis vectors for $\sum_{i=1}^{3} [f_i]$, and the application of Lemma 7, we have two possible cases. One case implies $\{f_1, f_2, f_3\}$ is a (1, 1) basis and the result follows by Lemma 8. This case is when either $\{f_1, f_3\}$ or $\{f_2, f_3\}$ is a P_6-pair. Thus, the other possible case is when both $\{f_1, f_3\}$ and $\{f_2, f_3\}$ are P_5-pairs. Then $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$ with $\sum_{i=1}^{3} [f_i] = \langle u_1, u_2, x_3, \cdots, x_6\rangle$. By Lemma 7, f_3 is either $u_1 \wedge v + u_2 \wedge w$ or $u_1 \wedge u_2 + v' \wedge w'$. The first case implies $\{f_1, f_2, f_3\}$ is $a(1, 1)$ basis and Lemma 8 applies. In the second case, we can take $v' \in [f_1]$, $w' \in [f_2]$; i.e., $v' \in [x_3, x_4; u_1, u_2]$, $w' \in [x_5, x_6; u_1, u_2]$. In fact, we can take $v' \in [x_3; x_4, u_1, u_2]$, and $v' = x_3 + au_1 + bu_2 + cx_4$. Now $w' = dx_5 + a'u_1 + b'u_2 + c'x_4$. We then show $c' - cd = 0$, by considering the determinant of (a_{ij}), where a_{ij} is defined as follows. Let $z = f_1 + f_2 + f_3$. We can express

$$z = w_1 \wedge w_2 + w_3 \wedge w_4 + w_5 \wedge w_6 .$$

For $i = 1, 2$, a_{ij} is the coefficient of u_i in w_j. For $i = 3, \cdots, 6$, a_{ij} is the coefficient of x_i in w_j. This determinant is $\pm(c' - cd)$. If it is nonzero, $\mathscr{L}(z) = 3$. Hence it must equal zero. Then a suitable choice of basis vectors of $\sum_{i=1}^{3} [f_i]$ will allow us to assume that $c = 0$ in v' and $c' = 0$ in w'. Then $g_3 = (f_3 - f_1 + f_2)$ will extend $\{f_1, f_2\}$ to a pair wise-P_6 basis for $\langle f_1, f_2, f_3\rangle$.

We have sufficient reason now to assert the following theorem.

THEOREM 12. *Let $\{f_1, f_2, f_3\}$ generate a three-dimensional $\mathscr{L} - 2$ subspace H, and $\dim \sum_{i=1}^{3} [f_i] = 6$. Then H has a basis of pairwise-P_6 vectors $\{g_1, g_2, g_3\}$ which either form a (1, 1) basis of H or have intersection $\bigcap_{i=1}^{3} [g_i]$ with dimension 0 or 1. Moreover, if $\{f_1, f_2\}$ is a*

P_6-pair, then this pair can be extended to a basis of pairwise-P_6 vectors of H.

EXAMPLES. H is generated by $\{f_1, f_2, f_3\}$ where

(i) $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, $f_3 = u_1 \wedge (u_2 + x_3 + x_5) + x_4 \wedge x_6$;

(ii) f_1, f_2 as in (i), $f_3 = u_1 \wedge x_4 + u_2 \wedge x_5$;

(iii) f_1, f_2 as in (i), $f_3 = x_3 \wedge x_5 + x_4 \wedge x_6$.

The maximal $\mathscr{L} - 2$ subspaces, $\dim \mathscr{U} = 6$. We shall now obtain this main theorem:

THEOREM 13. *Let H be an $\mathscr{L} - 2$ subspace and* $\dim \mathscr{U} = 6$. *Then* $\dim H \leqq 3$.

We prove this theorem by a series of lemmas, which show $\dim H \not> 3$, in fact, $\dim H \neq 4$. We take two three-dimensional $\mathscr{L} - 2$ subspaces $\langle f_1, f_2, f_3 \rangle$ and $\langle f_1, f_2, f_4 \rangle$ and show their sum is not an $\mathscr{L} - 2$ subspace. Theorem 12 allows us to take $\{f_1, f_2, f_3\}$ and $\{f_1, f_2, f_4\}$ to be pairwise-P_6, and there are 6 cases to consider since $\dim \bigcap_{i=1}^3 [f_i] = 0, 1, 2$ and a similar intersection property holds for the second set.

The following results are true for any dimension n of $\mathscr{U}$ unless otherwise specified.

LEMMA 12. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent pairwise-P_6 in H satisfying*

(i) $\dim \sum_{i=1}^3 [f_i] = 6$,

(ii) $\bigcap_{i=1}^3 [f_i] = 0$.

If $f_4 \in \mathscr{L}_2$, independent of $\{f_1, f_2, f_3\}$, satisfying

(a) $\dim \sum_{i=1}^4 [f_i] = 6$

(b) *$\{f_1, f_2, f_4\}$ is pairwise-P_6*

(c) $\dim \bigcap_{i=1,2,4} [f_i] = 1$,

then $\langle f_1, \cdots, f_4 \rangle$ is not an $\mathscr{L} - 2$ subspace.

Proof. By Lemma 10, $\sum_{i=1}^3 [f_i]$ has a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ such that $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, $f_3 = x_5 \wedge z + x_6 \wedge z'$, $\langle z, z' \rangle = \langle x_3, x_4 \rangle$. Let $\langle u \rangle = \bigcap_{1,2,4} [f_i]$. Then $u \in \langle u_1, u_2 \rangle$. We can take $u_1 = u$.

By Theorem 11, there exists $g_3 \in \langle f_1, f_2, f_4 \rangle$ such that $g_3 = v' \wedge w' + \gamma x_4 \wedge x_6$, $0 \neq \gamma \in F$ and $\langle f_1, f_2, g_3 \rangle = \langle f_1, f_2, f_4 \rangle$. Since $\{v', w', x_6, x_5, z, z'\}$ is independent and $\{x_4 + \alpha z', z\}$ is independent for some $\alpha \in F$, then $z = g_3 - \alpha f_3$ has irreducible length 3 for some α. Hence $\langle f_1, \cdots, f_4 \rangle$ is not an $\mathscr{L} - 2$ subspace.

Since the proofs of the lemmas involving the other cases are similar to the proof of Lemma 8 in the sense that in each case, we exhibit a vector of irreducible length 3 or less than 2 *except* in the 0-0 case, which we can reduce to one of the other cases, we shall simply state the final lemma.

LEMMA 13. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H such that* $\dim \sum_{i=1}^{3} [f_i] = 6$. *If $f_4 \in \mathscr{L}_2$, independent $\{f_1, f_2, f_3\}$ such that* $\dim {}_{i=1}^{4} [f_i] = 6$, *then $\langle f_1, \cdots, f_4 \rangle$ is not an $\mathscr{L} - 2$ subspace.*

We have to check one more case before we obtain Theorem 13.

LEMMA 14. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H,* $\dim \sum_{i=1}^{3} [f_i] = 5$. *If $f_4 \in \mathscr{L}_2, f_4 \notin \langle f_1, f_2, f_3 \rangle$, and* $\dim \sum_{i=1}^{4} [f_i] = 6$, *then $\langle f_1, \cdots, f_4 \rangle$ is not an $\mathscr{L} - 2$ subspace.*

Proof. We note $\dim \sum_{1,2,4} [f_i] = 6$ and apply Lemma 13.

We have now:

LEMMA 15. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, \cdots, f_k\}$ be independent in H,* $\dim \sum_{i=1}^{k} [f_i] = 6$. *Then $k \leqq 3$. For $k = 3$, $\langle f_1, f_2, f_3 \rangle$ has a basis of pairwise-P_6 vectors.*

Theorem 13 follows from Lemma 15

3. $\dim \mathscr{U} = 7$.

The three dimensional $\mathscr{L} - 2$ subspaces.

THEOREM 14. *Let H be an $\mathscr{L} - 2$ subspace of dimension $\geqq 3$. Let $\{f_1, f_2, f_3\}$ be independent in H such that* $\dim \sum_{i=1}^{3} [f_i] = 7$. *Then $\{f_1, f_2, f_3\}$ contains a P_6-pair, say $\{f_1, f_2\}$, which can be extended to a pairwise-P_6 basis $\{f_1, f_2, g_3\}$ of $\langle f_1, f_2, f_3 \rangle$. Moreover, either this basis is a $(1, 1)$ basis or* $\dim ([f_1] \cap [f_2] \cap [g_3]) = 1$; *and any basis $\{u_1, u_2\}$ of $[f_1] \cap [f_2]$ can be extended to a basis $\{u_1, u_2, x_3, \cdots, x_7\}$ of $[f_1] + [f_2] + [g_3]$ such that $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4, f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$; and $g_3 = u_1 \wedge x_7 + u_2 \wedge v$, $v \in \langle u_2, x_3, \cdots, x_6 \rangle$, $v \notin \langle u_2, x_4, x_6 \rangle$, and $v \notin [f_1]$ and $v \notin [f_2]$ in the first case; $g_3 = u_1 \wedge x_7 + x_4 \wedge x_6$ in the second case.*

Proof. A consideration of the various intersections and sums of $[f_i]$, $i = 1, 2, 3$ shows $\dim \bigcap_{i=1}^{3} [f_i]$ is either 1 or 2, and that there are at least two P_6-pairs in $\{f_1, f_2, f_3\}$. In the *first* case this independent set is in fact pairwise-P_6. The *second* case implies $\{f_1, f_2, f_3\}$ is a $(1, 1)$ basis for $\langle f_1, f_2, f_3 \rangle$. If this basis is not pairwise-P_6 but $\{f_1, f_2\}$ and $\{f_2, f_3\}$ are P_6-pairs, and $\{f_1, f_3\}$ a P_5-pair, we can choose a basis

$\{u_1, u_2, x_3, \cdots, x_7\}$ to give $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, $f_3 = u_1 \wedge x_7 + u_2 \wedge v$, $v \in \langle u_2, x_3, \cdots, x_6 \rangle$. Then we can take $g_3 = f_2 + f_3$. To obtain the desired representations of $\{f_1, f_2, f_3\}$ in the first case, we use an argument similar to the ones used earlier to obtain basis representations.

The maximal $\mathscr{L} - 2$ *subspaces*, $\dim \mathscr{U} = 7$. We obtain the following theorem.

THEOREM 15. *Let H be an $\mathscr{L} - 2$ subspace, $\dim \mathscr{U} = 7$. Then $\dim H \leqq 4$. When $\dim H = 4$, H has a (1, 1) basis, three of whose members are pairwise-P_6.*

The proof is contained in Lemmas 16, 17, and 18 which follow.

LEMMA 16. *Let $\{f_1, f_2, f_3\}$ be a (1, 1) basis for the $\mathscr{L} - 2$ subspace $\langle f_1, f_2, f_3 \rangle$, such that $\dim \sum_{i=1}^{3} [f_i] = 7$. If $f_4 \in \mathscr{L}_2$, independent of $\{f_1, f_2, f_3\}$ such that*

(i) $\dim \sum_{i=1}^{4} [f_i] = 7$,

(ii) *$\langle f_1, \cdots, f_4 \rangle$ is an $\mathscr{L} - 2$ subspace, then $\langle f_1, \cdots, f_4 \rangle$ has a (1, 1) basis, three of whose members are pairwise-P_6.*

Proof. By Theorem 14, $\{f_1, f_2, f_3\}$ can be assumed to be pairwise-P_6 with the representations given. Then it is easy to see that some pair in $\{f_1, f_2, f_3\}$, say $\{f_1, f_2\}$, is such that $\dim \sum_{i=1,2,4} [f_i] = 7$, and $\{f_1, f_2, f_4\}$ can be assumed pairwise-P_6. The two cases given in Theorem 14, apply to $\{f_1, f_2, f_4\}$. One case gives the desired result immediately. We can eliminate the other case by showing the presence of a vector in $\mathscr{L}_3$ in $\langle f_1, \cdots, f_4 \rangle$; in fact we can take the vector $f_1 + f_2 + f_3 + \alpha f_4$ for some suitable $0 \neq \alpha \in F$.

LEMMA 17. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H, $\dim \sum_{i=1}^{3} [f_i] = 7$. If $f_4 \in \mathscr{L}_2$, $f_4 \notin \langle f_1, f_2, f_3 \rangle$ such that*

(i) $\dim \sum_{i=1}^{4} [f_i] = 7$,

(ii) *$\langle f_1, \cdots, f_4 \rangle$ is an $\mathscr{L} - 2$ subspace,*

then $\langle f_1, \cdots, f_4 \rangle$ has a (1, 1) basis, three of whose members are pairwise-P_6.

Proof. In view of Theorem 14 and Lemma 16, it is sufficient to eliminate the case $\dim \bigcap_{i=1}^{3} [f_i] = 1$. We use a similar procedure as in the proof of Lemma 16, and the representations of $\{f_i\}$ in Theorem 14. We have two cases: (a) $\bigcap_{i=1,2,4} [f_i] = \langle u_1 \rangle$, (b) $\bigcap_{i=1,2,4} [f_i] = \langle u_2 \rangle$. In (a), $\langle f_1, \cdots, f_4 \rangle$ contains a vector of irreducible length one. In (b), $\langle f_1, \cdots, f_4 \rangle$ contains a vector or irreducible length at least three.

In addition to these two lemmas, we note that if H is an $\mathscr{L}-2$ subspace, $\{f_1, f_2, f_3\}$ independent in H and (i) $\dim \sum_{i=1}^{3}[f_i] = 6$, then $\{f_i\}$ can be taken to be pairwise-P_6 (Lemma 15) and if $f_4 \notin \langle f_1, f_2, f_3\rangle$, $\dim \sum_{i=1}^{4}[f_i] = 7$, then $\dim \sum_{i=1,2,4}[f_i] = 7$; (ii) $\sum_{i=1}^{3}[f_i] = 5$, and if $f_4 \notin \langle f_1, f_2, f_3\rangle$, $\dim \sum_{i=1}^{4}[f_i] = 7$, then $\dim \sum_{i=2}^{4}[f_i] = 7$. Hence both these cases reduce to the case considered in Lemma 17.

LEMMA 18. *Let H be an $\mathscr{L}-2$ subspace, and $\{f_1, \cdots, f_4\}$ be independent in H, $\dim \sum_{i=1}^{4}[f_i] = 7$. If $f_5 \in \mathscr{L}_2$, $f_5 \notin \langle f_1, \cdots, f_4\rangle$, and $\dim \sum_{i=1}^{5}[f_i] = 7$, then $\langle f_1, \cdots, f_5\rangle$ is not an $\mathscr{L}-2$ subspace.*

Proof. Apply Lemma 17 to $\{f_1, \cdots, f_4\}$ and $\{f_2, \cdots, f_4\}$ taking $\{f_1, f_2, f_3\}$ pairwise-P_6. Then $\langle f_1, \cdots, f_5\rangle$ has a (1, 1) basis, contradicting Theorem 4.

4. The main results.

LEMMA 19. *If H is an $\mathscr{L}-2$ subspace and $\{f_1, f_2, f_3\}$ is independent in H, $\dim \sum_{i=1}^{3}[f_i] = 8$, then $\{f_1, f_2, f_3\}$ is a (1, 1), pairwise-P_6 basis of $\langle f_1, f_2, f_3\rangle$, and we can represent*

$$\begin{aligned} f_1 &= u_1 \wedge x_3 + u_2 \wedge x_4 \,, \\ f_2 &= u_1 \wedge x_5 + u_2 \wedge x_6 \,, \\ f_3 &= u_1 \wedge x_7 + u_2 \wedge x_8 \,; \\ \sum_{i=1}^{3}[f_i] &= \langle u_1, u_2, x_3, \cdots, x_8\rangle \,. \end{aligned}$$

If $f_4 \in \mathscr{L}_2$, $f_4 \notin \langle f_1, f_2, f_3\rangle$, and $\langle f_1, \cdots, f_4\rangle$ is an $\mathscr{L}-2$ subspace, then $\{f_1, \cdots, f_4\}$ is a (1, 1) basis for $\langle f_1, \cdots, f_4\rangle$.

Proof. The first part is not difficult to see. Using Lemma 5 we obtain $\dim [f_4] \cap \langle u_1, u_2\rangle \geqq 1$. This intersection will have dimension 2, and f_4 forms a P_6-pair with one of $\{f_1, f_2, f_3\}$ since $\dim [f_4] = 4$.

Lemma 19 is extremely *important* as the second part states that presence of a 3-subset $\{f_1, f_2, f_3\}$ of any basis of an $\mathscr{L}-2$ subspace H such that $\dim \sum_{i=1}^{3}[f_i] = 8$ will guarantee that the basis will be a (1, 1) basis. We know that if $\dim \mathscr{L} \geqq 8$, then in any basis of H, we can find a 3-subset $\{g_1, g_2, g_3\}$ such that $\dim \sum_{i=1}^{3}[g_i] = 6, 7$ or 8. It is by now a more or less routine, and somewhat tedious, procedure to show the existence of a 3-subset $\{f_1, f_2, f_3\}$ in such a basis of H for $\dim \mathscr{U} = 8$, and then by induction for $\dim \mathscr{U} \geqq 9$. We shall simply state the main result and remark here that Theorem 4 provides the value of the maximal dimension of a (1, 1) basis.

THEOREM 16. *Let* $\dim \mathscr{U} = n \geqq 6$. *If H is an $\mathscr{L}-2$ subspace,* *then* $\dim H \leqq n-3$. *If* $\dim H \geqq 4$, *then* H *has a* $(1, 1)$ *basis, and is hence a* $(1, 1)$*-type subspace.*

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Received April 14, 1969, and in revised form April 14, 1969.

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