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HOMOTOPY GROUPS OF PL-EMBEDDING SPACES

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Let N be a compact PL-n-manifold, and let M be a PL-m-manifold without boundary. Two of the major problems in PL-topology are to determine conditions such that (1) any continuous map of N into M can be homotoped to a PL-embedding, and (2) two homotopic PL-embeddings are PL-isotopic.

If $C(N, M)$ is the space of continuous maps of N into M with the compact open topology, and if $PL(N, M)$ is the subspace of PL-embeddings, one can consider the map $i_{\#}: \Pi_0(PL(N, M)) \rightarrow \Pi_0(C(N, M))$ induced by inclusion. If (1) is true, then $i_{\#}$ is onto; if (2) is true, then $i_{\#}$ is one-to-one. In this paper, we investigate the higher homotopy groups of $PL(N, M)$ and $C(N, M)$.

Irwin has shown that if N is a closed manifold, $m \geqq n + 3$, then sufficient conditions for (1) are that N is $(2n - m)$—connected and M is $(2n - m + 1)$—connected. By raising the connectivities of N and M by one, Zeeman [7] proved (2).

By using Proposition 1 of Morlet [4] and Irwin [3], one can easily show the following theorem by using techniques similar to the proof of Theorem 2 below.

THEOREM 1. *Let N be a closed $(2n + s + 1 - m)$—connected PL-n-manifold and let M be a $(2n + s + 2 - m)$—connected PL-m-manifold without boundary, $m \geqq n + 3$. The homomorphism $i_{\#}$: $\Pi_s(PL(N, M)) \rightarrow \Pi_s(C(N, M))$ induced by inclusion is an isomorphism; if the connectivities of N and M are lowered by one, then $i_{\#}$ is onto.*

An analogous theorem in the differential case has been proved by J. P. Dax [1], [2].

If N has a nonempty boundary, then Dancis, Hudson and Tindell (independently and unpublished) have shown that if N has a k-dimensional spine with $m \geqq \{n + 3, n + k\}$, this is a sufficient condition for (1). If $m \geqq \{n + 3, n + k + 1\}$, they obtain (2). We generalize.

THEOREM 2. *Let N be a compact PL-n-manifold with k-spine K, $k < n$, and let M be a PL-m-manifold without boundary. If $m \geqq n + k + s + 1$, the homomorphism $i_{\#}$: $\Pi_s(PL(N, M)) \rightarrow \Pi_s(C(N, M))$ induced by inclusion is an isomorphism; if $m \geqq n + k + s$, $i_{\#}$ is onto.*

Note that the codimension 3 restriction is eliminated. In § 3, we obtain some consequences of this theorem and its proof.

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In this paper, we shall consider $PL(N, M)$ and $C(C, M)$ as Δ-sets (-i.e., as semisimplicial complexes in which the degeneracy maps are ignored). In § 1, we list the basic definitions and results on Δ-sets which we shall use. One may use either Rourke and Sanderson [6] or Morlet [5]. [Morlet uses the terminology "quasisimplicial" set.]

We shall assume familiarity with either [1] or [7] and shall use terminology therein with one exception. When referring to piecewise linear maps or manifolds, we shall always use the prefix "*PL*-".

Let X and Y be polyhedra. In this paper p_1 and p_2 will always denote projections of $X \times Y$ onto the first and second factors respectively. An isotopy between X and Y will be represented as a family of embeddings $f_t: X \to Y$, $t \in I = [0, 1]$.

1. Δ-sets. Let Δ^n denote the standard n-simplex with ordered vertices $v_0, v_1, \cdots, v_n$. The i-th *face map* $\partial_i: \Delta^{n-1} \to \Delta^n$ is the order preserving simplicial embedding which omits v_i. Δ is the category whose objects are Δ^n, $n = 0, 1, \cdots$ and whose morphisms are generated by the face maps. A *Δ-set* (*Δ-group*) is a contravariant functor from Δ to the category of sets (groups). A *Δ-map* between Δ-sets (Δ-groups) is a natural transformation between the functors.

If X is a Δ-set, $X^k = X(\Delta^k)$ is the set of *k-simplexes* and the maps $\partial_i = X(\partial_i)$ are called *face maps*. We shall be interested in pointed Δ-sets in which we distinguish a simplex $*^k \in X^k$ for each k and designate $* \subset X$ as the sub-Δ-set of X consisting of these simplexes and maps ∂_i defined by $\partial_i *^k = *^{k-1}$.

With each ordered simplicial complex K, we associate a Δ-set, also designated by K, whose k-simplexes are order-preserving simplicial embeddings of Δ^k into K.

Let $\Lambda_{n,i} = \mathrm{Cl}\,(\mathrm{bdry}\ \Delta^n - \partial_i \Delta^{n-1})$. A Δ-set X is called a *Kan Δ-set* if every Δ-map $f: \Lambda_{n,i} \to X$ can be extended to a Δ-map $f_1: \Delta^n \to X$.

If X is a Kan Δ-set and P is a polyhedron, a *map* $f: P \to X$ is a Δ-map $f: K \to X$ where K is an ordered triangulation of P. $f_0, f_1: P \to X$ are *homotopic* if there is a map $F: P \times I \to X$ such that $F \mid P \times \{i\} = f_i$, $i = 0, 1$. $[P; X]$ denotes the set of homotopy classes. We shall need the following two propositions which are proved by Rourke and Sanderson.

PROPOSITION 1. *Any homotopy class in $[P; X]$ is represented by a Δ-map $f: K \to X$ where K is any ordered triangulation of P.*

PROPOSITION 2. *Let Q be a subpolyhedron of P and let*

$h: Q \times I \cup P \times \{0\} \to X$ be a Δ-map to a Kan Δ-set X; then h extends to a Δ-map $h': P \times I \to X$.

If X is a pointed Kan Δ-set, then the n-th *homotopy group* of X, $\Pi_n X = [I^n, \text{bdry } I^n; X, *]$, the homotopy classes of Δ-maps of pairs, where I^n is the PL-n-cell.

$C(N, M)(PL(N, M))$ is made into a Δ-set by defining the k-simplexes to be maps (PL-embeddings) $f: N \times \Delta^k \to M \times \Delta^k$ such that $p_2 f = p_2$ and defining $\partial_i f = f \mid N \times \partial_i \Delta^k$.

PROPOSITION 3. *$C(N, M)$ and $PL(N, M)$ are Kan Δ-sets.*

Proof. Let $f: \Lambda_{n,i} \to PL(N, M)$ be a Δ-map. f can then be considered as a PL-embedding

$$f: N \times \Lambda_{n,i} \longrightarrow M \times \Lambda_{n,i}$$

such that $p_2 f = p_2$. Using the fact that the pair $(\Lambda_{n,i} \times I, \Lambda_{n,i} \times \{0\})$ is PL-homeomorphic to $(\Delta^n, \Lambda_{n,i})$, one can easily construct the desired extension.

2. Proof of Theorem 1. The following two propositions are generalizations to product spaces of the simplicial approximation and general position theorems. They can be proved similarly.

PROPOSITION 4. *Let M and Y be PL-manifolds and let $P \subseteqq Q$ be compact polyhedra. Suppose $f: Q \to M \times Y$ is a continuous map such that $f \mid P$ is PL. There exists a homotopy $h_t: M \times Y \to M \times Y$, $t \in I$, such that*

(i) *$p_2 h_t = p_2$ for $t \in I$;*
(ii) *$h_t f \mid P = f$ for $t \in I$;*
(iii) *$h_1 f: Q \to M \times Y$ is PL.*

PROPOSITION 5. *Let M and Y be PL-manifolds and let $P \subseteqq Q$ be compact polyhedra. Suppose $f: Q \to M \times Y$ is a PL-map such that $f \mid P$ is a PL-embedding. There exists a PL-homotopy $h_t: M \times Y \to M \times Y$, $t \in I$, such that*

(i) *$p_2 h_t = p_2$ for $t \in I$;*
(ii) *$h_t f \mid P = f$ for $t \in I$;*
(iii) *the singular set of $h_1 f$ has dimension $\leqq 2 \dim Q - \dim (M \times Y)$;*
(iv) *the branch set of $h_1 f$ has dimension $< 2 \dim Q - \dim (M \times Y)$.*

The following two constructions are needed frequently in the following propositions.

PROPOSITION 6. *Let N be a PL-n-manifold with k-spine K. Let*

P be a polyhedron in N such that $\dim P + \dim K + 1 \leqq \dim N$. *There exists a PL-isotopy H_t of N, $t \in I$, such that $H_0 =$ identity and $H_1(N) \cap P = \emptyset$.*

Proof. By general position, we can find a PL-ambient isotopy L_t of N so that $L_1K \cap P = \emptyset$. Let N' be a regular neighborhood of L_1K in N such that $N' \cap P = \emptyset$. Note that L_1K is also a spine of N. Hence, by the uniqueness theorem of regular neighborhoods, there is a PL-isotopy H_t of $N, t \in I$, such that $H_0 =$ identity and $H_1(N) = N'$.

CONSTRUCTION α. Let I^s_+ be a PL-cell in the interior of I^s and let U be a neighborhood of $\mathrm{Cl}\,(I^s - I^s_+)$ in I^s. Let U_0, U_1 be regular neighborhoods of $\mathrm{Cl}\,(I^s - I^s_+)$ in I^s such that $U_0 \subseteqq \mathrm{int}\, U_1$ and $U_1 \subseteqq U$. Let $\varphi: S^{s-1} \times I \rightarrow \mathrm{Cl}\,(U_1 - U_0)$ be a PL-homeomorphism such that $\varphi(S^{s-1} \times \{i\}) = \mathrm{bdry}\, U_i \cap \mathrm{int}\, I^s$, $i = 0, 1$.

PROPOSITION 7. *Let N, K, M be as in Theorem 2 with $m \geqq n + k + s$. Let $f: N \times I^s \rightarrow M \times I^s$ be a PL-map such that $p_2 f = p_2$ and such that there exists a neighborhood U of* $\mathrm{Cl}\,(I^s - I^s_+)$ *such that $f \mid N \times U$ is a PL-embedding, then there exists a PL-homotopy $f_t: N \times I^s \rightarrow M \times I^s$ and a neighborhood V of* $\mathrm{Cl}\,(I^s - I^s_+)$ *in I^s such that*

(i) $f_0 = f$, $p_2 f_t = p_2$, $t \in I$;

(ii) $f_t \mid V = f$, $t \in I$;

(iii) *$f_1: N \times I^s \rightarrow M \times I^s$ is a PL-embedding.*

Proof. By Proposition 5, we can assume that the singular set T of f has dimension $\leqq 2(n + s) - (m + s)$, the branch set $B \subset T$ of f has dimension $< 2(n + s) - (m + s)$, and that $f \mid K \times I^s$ is a PL-embedding. By Proposition 6, there is a PL-isotopy H_t of N such that $H_0 =$ identity and $H_1(N) \cap p_1 B = \emptyset$. Hence there is no loss of generality in assuming that $f \mid p_1^{-1}(H_1(N)) \times I^s$ is a PL-embedding.

Let U_0, U_1 and φ be as in construction α. Define $F_t: N \times I^s \rightarrow N \times I^s$, $t \in I$, by

$$F_t(x, y) = \begin{cases} (H_t(x), y) & y \in \mathrm{Cl}\,(I^s - U_1) \\ (x, y) & y \in U_0 \\ (H_{tt_0}(x), y) & y \in \mathrm{Cl}\,(U_1 - U_0),\ y = \varphi(y_0, t_0). \end{cases}$$

Let $f_t = fF_t$ and $V = U_0$.

The following is the theorem of Dancis, Hudson and Tindell mentioned in the introduction. We include the proof for completeness.

PROPOSITION 8. *Let N, K, M be as in Theorem 2 with $m \geqq n + k$. There exists a PL-embedding $f: N \rightarrow M$.*

Proof. Let $f': N \to M$ be a continuous map and approximate f' by a PL-map f'' such that f''/K is a PL-embedding and f'' is in general position. Let $B \subset S$ be the branch and singular set of f'' respectively. By Proposition 6, there is a PL-isotopy H_t, $t \in I$, of N such that $H_1(N) \cap S = S \cap K$. Let $f = f''H_1$.

REMARK. We shall make $PL(N, M)$ and $C(N, M)$ into pointed Δ-sets by defining the basepoint complex $*$ as follows. Let $*^s(x, y) = (f(x), y)$, $x \in N$, $y \in \Delta^s$ where f is defined in Proposition 8. The face operators are defined naturally.

The proof of the following proposition is well known.

PROPOSITION 9. *Let N, M, K be as in Theorem 2 with $m \geqq n + k$. Let $g: N \times I^s \to M \times I^s$ represent an s-simplex in $PL(N, M)(C(N, M))$ such that*

$$g \mid N \times \text{bdry}\, I^s = *^s \mid N \times \text{bdry}\, I^s ,$$

g is homotopic rel bdry I^s *in $PL(N, M)(C(N, M))$ to g': $N \times I^s \to M \times I^s$ such that for some neighborhood U of* $\text{Cl}\,(I^s - I^s +)$ *in I^s, $g' \mid N \times U = *^s \mid N \times U$.*

PROPOSITION 10. *Let N, M, K be as in Theorem 2 with $m \geqq n + k + s + 1$ and let $F_t: N \times I^s \to M \times I^s$ be a PL-homotopy such that*

(i) *F_i are PL-embeddings, $i = 0, 1$;*

(ii) *$p_2F_t = p_2$, $t \in I$:*

(iii) *there exists a neighborhood U of* $\text{Cl}\,(I^s - I^s_+)$ *in I^s such that $F_t \mid N \times U = *^s$.*

Then there exists a PL-isotopy G_t: $N \times I_s \to M \times I^s$ such that

(i) *$G_i = F_i$ for $i = 0, 1$;*

(ii) *$p_2G_t = p_2$, $t \in I$:*

(iii) *there exists a neighborhood V of* $\text{Cl}\,(I^s - I^s_+)$ *in I^s such that $G_t \mid N \times V = *^s$.*

Proof. Note that there is no loss of generality in assuming that there is an $\varepsilon > 0$ so that F_t are PL-embeddings, $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$. However, now this is a restatement of Proposition 7.

The proof of Theorem 2 now follows easily from the above propositions.

3. Applications. One of the immediate consequences of Theorem 2 is a partial generalization of Hudson's "concordance implies isotopy" theorem [2]. (See also Proposition 1 of [4].)

COROLLARY 1. *Let N be a compact PL-n-manifold with k-spine K, $k < n$, and let M be a PL-m-manifold without boundary. Let $f: N \times I^s \to M \times I^s$ be a PL-embedding such that $p_2 f \mid N \times \text{bdry}\, I^s = p_2$. Then if $m \geqq n + k + s$, there exists a PL-embedding $F: N \times I^s \to M \times I^s$ such that $F \mid N \times \text{bdry}\, I^s = f$ and $p_2 F = p_2$. If $m \geqq n + k + s + 1$, f and F can be chosen to be isotopic* rel $N \times \text{bdry}\, I^s$.

Let X be an s-dimensional polyhedron and let $p: E \to X$ and $q: F \to X$ be PL-fiber bundles with fibers N and M respectively with structure groups Aut(N) and Aut(M) where

(i) N is a PL-n-manifold with k-spine, $k < n$;

(ii) M is a PL-m-manifold without boundary;

(iii) Aut(N) and Aut(M) are the groups of PL-automorphisms of N and M, respectively.

By triangulating X and by using the propositions above together with induction on the dimension of the simplexes of X, one can easily prove the following.

COROLLARY 2. *If $f: E \to F$ is a continuous bundle map* (-i.e., *$qf = p$) and $m \geqq n + k + s$, then f is homotopic through bundles maps to a PL-bundle map which is an embedding of E into F. If $m \geqq n + k + s + 1$; any two PL-bundle embeddings of E into F are isotopic through bundle maps.*

A PL_m-bundle is a PL-bundle $q: F \to X$ whose fiber is Euclidean m-space R^m and whose structural group is the PL-automorphisms of R^m mod the origin.

COROLLARY 3. *Let N be a PL-n-manifold with k-spine, $k < n$; let $p: E \to X^s$ be a PL-fiber bundle with N as fiber and* Aut(N) *as structural group. If $m \geqq n + k + s$, then for any PL_m-bundle $q: F \to X$, there exists a PL-bundle map $f: E \to F$ which is an embedding. If $m \geqq n + k + s + 1$, then any such two PL-bundle embeddings are isotopic through bundle maps.*

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