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ON $|C, 1|$ SUMMABILITY FACTORS OF FOURIER SERIES AT A GIVEN POINT

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Let $f(x)$ be a function integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and periodic with period 2π. Let its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$
$$\equiv \sum_{n=0}^{\infty} A_n(x) .$$

Whittaker proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^{\alpha} \qquad (\alpha > 0)$$

is summable $|A|$ almost everywhere. Prasad improved this result by showing that the series

$$\sum_{n=n_0}^{\infty} A_n(x)\Big/\Big(\prod_{\mu=1}^{k-1} \log^{\mu} n\Big)(\log^k n)^{1+\varepsilon} \qquad (\log^k n_0 > 0)$$

is summable $|A|$ almost everywhere.

In this note, the author is interested particularly in the $|C, 1|$ summability factors of the Fourier series at a given point x_0.

Write

$$\varphi(t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0) ,$$
$$\Phi(t) = \int_0^t |\varphi(u)|\, du .$$

The author establishes the following theorems.

THEOREM 1. **If**

$$\Phi(t) = O(t) \qquad (t \to +0) ,$$

then the series

$$\sum_{n=1}^{\infty} A_n(x_0)/n^{\alpha}$$

is summable $|C, 1|$ for every $\alpha > 0$.

THEOREM 2. **If**

$$\Phi(t) = O\left\{\frac{t}{\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}}\right\}$$

as $t \to +0$, then the series

$$\sum_{n=n_0}^{\infty} \frac{A_n(x_0)}{\Big(\prod_{\mu=1}^{k-1} \log^{\mu} n\Big)(\log^k n)^{1+\varepsilon}}$$

is summable $|C, 1|$ for every $\varepsilon > 0$.

A series $\sum a_n$ is said to be absolutely summable (A) or summable $|A|$, if the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is of bounded variation in the interval $\langle 0, 1 \rangle$. Let σ_n^α denote the nth Cesàro mean of order α of the series $\sum a_n$, i.e.,

$$\sigma_n^\alpha = \frac{1}{(\alpha)_n} \sum_{k=0}^{n} (\alpha)_k a_{n-k}, \; (\alpha)_k = \Gamma(k+\alpha+1)/\Gamma(k+1)\Gamma(\alpha+1) \; .$$

If the series

$$\sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha|$$

converges, then we say that the series $\sum a_n$ is absolutely summable (C, α) or summable $|C, \alpha|$. It is known that [2] *if a series is summable* $|C|$, *it is also summable* $|A|$, *but not conversely.*

2. Suppose that $f(x)$ is a function integrable in the sense of Lebesgue and periodic with period 2π. Let its Fourier series be

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &\equiv \sum A_n(x) \; . \end{aligned}$$

Whittaker [4] proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^\alpha \qquad (\alpha > 0)$$

is summable $|A|$ almost everywhere. Prasad [4] improved this result by showing that the series

$$\sum_{n=n_0}^{\infty} A_n(x) \Big/ \Big(\prod_{\mu=1}^{k-1} \log^\mu n \Big) (\log^k n)^{1+\varepsilon} (\log^k n_0 > 0) \; ,$$

where $\log^k n = \log(\log^{k-1} n)$, $\log^2 = \log(\log n)$, is summable $|A|$ almost everywhere.

Let (λ_n) be a convex and bounded sequence, Chow [1] demonstrated that the series

$$\sum A_n(x) \lambda_n$$

is summable $|C, 1|$ almost everywhere, if the series $\sum n^{-1}\lambda_n$ converges.

In this note, we are interested particularly in the $|C, 1|$ summability factors of the Fourier series at a given point. For a fixed point x_0, we write

$$\varphi(t) = \varphi_{x_0}(t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0) \; ,$$

and

$$\Phi(t) = \int_0^t |\varphi(u)|\, du\ .$$

We are going to establish the following

THEOREM 1. *If*

(i) $$\Phi(t) = O(t)$$

as $t \to +0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{A_n(x_0)}{n^\alpha}$$

is summable $|C, 1|$ *for every* $\alpha > 0$.

3. The following lemmas are required.

LEMMA 1 [3]. *Let* $\alpha > -1$ *and let* τ_n^α *be the nth Cesàro mean of order* α *of the sequence* $\{na_n\}$, *then*

$$\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)\ .$$

LEMMA 2. *Write*

$$S_n(t) = \sum_{k=0}^{n} (n + 2 - k) \cos (n + 2 - k)t\ ,$$

then

$$S_n(t) = O\begin{cases} nt^{-1} & (nt \geqq 1)\ , \\ n^2 & (\textit{for all } t)\ . \end{cases}$$

In fact, we have

$$\begin{aligned} S_n(t) &= I\left\{\frac{d}{dt} e^{i(n+2)t} \sum_{k=0}^{n} e^{-ikt}\right\} \\ &= I\left\{\frac{d}{dt}\left(\frac{e^{i(n+2)t}}{1 - e^{-it}} - \frac{e^{it}}{1 - e^{-it}}\right)\right\} \\ &= I\left\{(n + 2)\frac{ie^{i(n+2)t}}{1 - e^{-it}} - \frac{ie^{i(n+2)t}}{(1 - e^{-it})^2} \right. \\ &\qquad \left. - \frac{ie^{it}}{1 - e^{-it}} + \frac{i}{(1 - e^{-it})^2}\right\} \\ &= O(nt^{-1}) + O(t^{-2}) \\ &= O(nt^{-1})\ , \end{aligned}$$

if $nt \leqq 1$. This proves the lemma. From this lemma, we can easily derive the following

LEMMA 3.

$$\left|\frac{1}{n+1}\left\{\sum_{\nu=1}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\}\right| \leqq \begin{cases} \dfrac{A}{th^\alpha} + \dfrac{A}{nt^{2-\alpha}} & (t \geqq 1)\,, \\ An^{1-\alpha} & (\textit{for all } t)\,. \end{cases}$$

By Lemma 2, for $nt \geqq 1$, we write

$$\begin{aligned} \frac{1}{n+1}\left\{\sum_{\nu=1}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\} &= \frac{1}{n+1}\left\{\sum_{\nu=1}^{[t^{-1}]-1} + \sum_{\nu=[t^{-1}]+1}^{n}\right\} + O\left(\frac{1}{nt^{2-\alpha}}\right) \\ &= \frac{1}{n}O\left(\sum_{\nu=1}^{[t^{-1}]}\nu^{1-\alpha}\right) + \frac{1}{nt}O\left(\sum_{\nu=1}^{n}\frac{1}{\nu^\alpha}\right) \\ &\quad + O\left(\frac{1}{nt^{2-\alpha}}\right) \\ &= O\left(\frac{1}{nt^{2-\alpha}}\right) + O\left(\frac{1}{tn^\alpha}\right)\,, \end{aligned}$$

and for all t,

$$\begin{aligned} \frac{1}{n+1}\left\{\sum_{\nu=1}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\} &= \frac{1}{n+1}O\left\{\sum_{\nu=1}^{n}\nu^2\frac{1}{\nu^{1+\alpha}}\right\} \\ &= \frac{1}{n+1}O\left\{\sum_{\nu=1}^{n}\nu^{1-\alpha}\right\} \\ &= O(n^{1-\alpha})\,. \end{aligned}$$

This proves the lemma.

4. We have

$$A_n(x_0) = \frac{2}{\pi}\int_0^\pi \varphi(t)\cos ntdt\,.$$

Let $\tau_n(x_0)$ be the nth Cesàro mean of first order of the sequence $\{nA_n(x_0)/n^\alpha\}$, then

$$\frac{\pi}{2}\tau_n(x_0) = \int_0^\pi \varphi(t)\frac{1}{n+1}\sum_{\nu=0}^{n}\frac{(\nu+2)\cos(\nu+2)t}{(\nu+2)^\alpha}dt\,.$$

Abel's transformation gives

$$\begin{aligned} \frac{\pi}{2}\tau_n(x_0) &= \int_0^\pi \varphi(t)\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\}dt \\ &\quad + \int_0^\pi \varphi(t)\frac{1}{n+1}\cdot\frac{S_n(t)}{(n+3)^\alpha}dt \\ &= I_{1n} + I_{2n}\,, \end{aligned}$$

say. Thus, on writing

$$I_{1n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{3n} + I_{4n} ,$$

say, we see that

$$I_{3n} = O\Big(n^{1-\alpha}\int_0^{1/n} |\varphi|\, dt\Big) = O(n^{-\alpha}) ,$$

by condition (i) of the theorem.

$$I_{4n} = O\Big\{\frac{1}{n^{\alpha}}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\Big\} + O\Big\{\frac{1}{n}\int_{1/n}^{\pi} \frac{|\varphi|}{t^{2-\alpha}} dt\Big\} .$$

Now,

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \Big(\frac{\Phi}{t}\Big)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt = O(1) + O\Big\{\int_{1/n}^{\pi} \frac{dt}{t}\Big\} = O(\log n) ,$$

and

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t^{2-\alpha}} dt \leqq n^{1-\alpha}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = O(n^{1-\alpha} \log n) .$$

It follows that

$$I_{4n} = O\{\log n/n^{\alpha}\} .$$

As before, we write

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{5n} + I_{6n} ,$$

say. Then,

$$I_{5n} = O\Big(n^{1-\alpha}\int_0^{1/n} |\varphi|\, dt\Big) = O(n^{-\alpha}) .$$

And

$$I_{6n} = O\Big\{n^{-\alpha}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\Big\} = O\{\log n/n^{\alpha}\} ,$$

by the similar arguments as in the estimation of the integral I_{4n}. By Lemma 1, we have to establish the convergence of $\sum |\tau_n(x_0)|/n$. And from the above analysis, it concludes that

$$\sum_{n=1}^{\infty} \frac{|\tau_n(x_0)|}{n} \leqq \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\{|I_{3n}| + |I_{4n}| + |I_{5n}| + |I_{61}|\}$$
$$= O\Big\{\sum_{n=1}^{\infty} \frac{\log n}{n^{1+\alpha}}\Big\} = O(1) .$$

This proves Theorem 1.

5. Let $\tau_n(x_0)$ be the nth Cesàro mean of first order of the sequence

$$\left\{nA_n(x_0)\Big/\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}\right\} \qquad (\varepsilon > 0) ,$$

where k is a positive integer. Abel's transformation gives

$$\frac{\pi}{2}\tau_n(x_0) = \int_0^{\pi}\varphi(t)\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} S_\nu(t)\Delta\frac{1}{\left\{\prod_{\mu=1}^{k-1}\log^{\mu}(\nu+2)\right\}\{\log^k(\nu+2)\}^{1+\varepsilon}}dt\right.$$
$$+ \int_0^{\pi}\varphi(t)\frac{1}{n+1}\cdot\frac{S_n(t)}{\left\{\prod_{\mu=1}^{k-1}\log^{\mu}(n+3)\right\}\{\log^k(n+3)\}^{1+\varepsilon}}dt$$
$$= I_{1n} + I_{2n} ,$$

say. As before, we write

$$I_{1n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{3n} + I_{4n} ,$$

say, and

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{5n} + I_{6n} ,$$

say. Since, for $\nu \geqq n_0$,

$$\left|\Delta\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^{\mu}\nu\right)(\log^k \nu)^{1+\varepsilon}}\right| \leqq \frac{A}{\nu\left(\prod_{\mu=1}^{k-1}\log^{\mu}\nu\right)(\log^k \nu)^{1+\varepsilon}} ,$$

we obtain

$$\left|\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} S_\nu(t)\Delta\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^{\mu}(\nu+2)\right)(\log^k(\nu+2))^{1+\varepsilon}}\right.\right|$$
$$\leqq \begin{cases} \dfrac{A}{t\left(\prod_{\mu=0}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} + \dfrac{A}{t^2\left(\prod_{\mu=1}^{k-1}\log^{\mu}\frac{1}{t}\right)\left(\log^k\frac{1}{t}\right)^{1+\varepsilon}} & (nt \geqq 1) , \\[2ex] \dfrac{An}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} & (\text{for all } t) . \end{cases}$$

Now, if

$$\Phi(t) = O\left\{\frac{t}{\left(\prod_{\mu=1}^{k}\log^{\mu}\frac{1}{t}\right)}\right\}$$

as $t \to +0$, then

$$I_{3n} = O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\text{long}^k n)^{1+\varepsilon}}\int_0^{1/n} |\varphi|\, dt\right\}$$

$$= O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\}.$$

$$I_{4n} = O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\}$$

$$+ O\left\{\frac{1}{n}\int_{1/n}^{\pi} \frac{|\varphi|}{t^2\left(\prod_{\mu=1}^{k-1}\frac{1}{t}\right)\left(\log^k \frac{1}{t}\right)^{1+\varepsilon}}\right\}.$$

But since

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt$$

$$= O(1) + O\left\{\int_{1/n}^{\pi} \frac{dt}{t\left(\prod_{\mu=1}^{k}\log^{\mu}\frac{1}{t}\right)}\right\}$$

$$= O(1) + O\{\log^{k+1} n\},$$

and

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t^2\left(\prod_{\mu=1}^{k-1}\log^{\mu}\frac{1}{t}\right)\left(\log^k \frac{1}{t}\right)^{1+\varepsilon}} dt = O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\}$$

$$= O\left\{\frac{n\log^{k+1} n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\},$$

we obtain

$$I_{4n} = O\left\{\frac{\log^{k+1} n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\}.$$

Finally,

$$I_{5n} = O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\int_0^{1/n} |\varphi|\, dt\right\}$$

$$= O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\},$$

$$I_{6n} = O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\}$$

$$= O\left\{\frac{\log^{k+1} n}{\left(\prod_{\mu=1}^{k=1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{|\tau_n(x_0)|}{n} = O\left\{\sum_{n=n_0}^{\infty} \frac{\log^{k+1} n}{n\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\} + O(1)$$

$$= O(1) .$$

Hence, we establish

THEOREM 2. *If*

(ii) $$\Phi(t) = O\left\{\frac{t}{\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}}\right\}$$

as $t \rightarrow +0$, *then the series*

$$\sum_{n=n_0}^{\infty} \frac{A_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \qquad (\log^{k} n_0 > 0)$$

is summable $|C, 1|$ *for every* $\varepsilon > 0$.

6. For the conjugate series

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x) ,$$

we can derive two analogous theorems. Write, for a fixed $x = x_0$,

$$\Psi(t) = \int_0^t |\psi(u)| \, du \equiv \int_0^t |f(x_0 + u) - f(x_0 - u)| \, du .$$

We have the following

THEOREM 3. *If*

(iii) $$\Psi(t) = O(t)$$

as $t \rightarrow +0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{B_n(x_0)}{n^{\alpha}}$$

is summable $|C, 1|$ *for every* $\alpha > 0$.

THEOREM 4. *If*

$$\text{(iv)} \qquad \Psi(t) = O\left\{\frac{t}{\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}}\right\}$$

as $t \to +0$, *then the series*

$$\sum_{n=n_0}^{\infty} \frac{B_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} \qquad (\log^k n_0 > 0)$$

is summable $|C, 1|$ *for every* $\varepsilon > 0$.

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