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A CHARACTERIZATION OF PERFECT RINGS

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J. P. Jans has shown that if a ring R is right perfect, then a certain torsion in the category $\operatorname{Mod} R$ of left R-modules is closed under taking direct products. Extending his method, J. S. Alin and E. P. Armendariz showed later that this is true for every (hereditary) torsion in $\operatorname{Mod} R$. Here, we offer a very simple proof of this result. However, the main purpose of this paper is to present a characterization of perfect rings along these lines: A ring R is right perfect if and only if every (hereditary) torsion in $\operatorname{Mod} R$ is fundamental (i.e., derived from "prime" torsions) and closed under taking direct products; in fact, then there is a finite number of torsions, namely 2^n for a natural number n. Finally, examples of rings illustrating that the above characterization cannot be strengthened are provided. Thus, an example of a ring R_1 is given which is not perfect, although there are only fundamental torsions in $\operatorname{Mod} R_1$, and only $4 = 2^2$ of these. Furthermore, an example of a ring R_{2*} is given which is not perfect and which, at the same time, has the property that there is only a finite number (namely, 3) of (hereditary) torsions in $\operatorname{Mod} R_{2*}$ all of which are closed under taking direct products. Moreover, the ideals of R_{2*} form a chain (under inclusion) and $\operatorname{Rad} R_{2*}$ is a nil idempotent ideal; all the other proper ideals are nilpotent and R_{2*} can be chosen to have a (unique) minimal ideal.

In what follows, R stands always for a ring with unity, $\mathscr{L}$ for the set of all left ideals of R and $\operatorname{Mod} R$ for the category of all (unital) left R-modules and R-homomorphisms. Given $L \in \mathscr{L}$ and $\rho \in R$, $L: \rho$ denotes the (right) ideal-quotient of L by ρ, i.e., the left ideal of all $\chi \in R$ such that $\chi\rho \in L$. We shall call a subset $\mathscr{K}$ of $\mathscr{L}$ a *Q-set* if it is closed with respect to this operation, i.e., if $K \in \mathscr{K}$ and $\rho \in R$ implies $K: \rho \in \mathscr{K}$; obviously, $\mathscr{L}$ and $\{R\}$ are the greatest and the least Q-sets, respectively. Thus, a topologizing idempotent filter (briefly, a *T-set*) of left ideals of P. Gabriel [4] is a Q-set $\mathscr{K}$ satisfying, in addition to the filter properties, also the following "radical" condition: If L is a left ideal of R such that $L: \kappa \in \mathscr{K}$ for every element κ of $K \in \mathscr{K}$, then $L \in \mathscr{K}$, as well.

By a *torsion* T *in* $\operatorname{Mod} R$ we shall always understand a hereditary torsion; thus, a torsion T in $\operatorname{Mod} R$ is a full subcategory of $\operatorname{Mod} R$ such that

(a) T is closed under taking submodules,

(b) for every $M \in \operatorname{Mod} R$, there is the greatest submodule (the T-torsion part) $T(M)$ of M belonging to T and

(c) $T(M/T(M)) = 0$ for every $M \in \operatorname{Mod} R$.
As a consequence, every torsion in $\operatorname{Mod} R$ is closed under taking quotients, direct sums and inductive limits. There is a one-to-one correspondence between the torsions in $\operatorname{Mod} R$ and the T-sets of left ideals of R:

If $\mathscr{K}$ is a T-set, then the class $T(\mathscr{K})$ of all R-modules whose elements have orders from $\mathscr{K}$ is a torsion in $\operatorname{Mod} R$; on the other hand, if T is a torsion in $\operatorname{Mod} R$, then the T-set $\mathscr{K}(T) = \{L \mid L \in \mathscr{L}$ and $R \bmod L \in T\}$ satisfies $T = T[\mathscr{K}(T)]$. Given an R-module M, let us always denote the T-torsion part of it by $T(M)$.

Thus, given a torsion T, we can define the following two-sided ideals I_T and $J_T \supseteqq I_T$ of R:

$$I_T = \bigcap_{L \in \mathscr{K}(T)} L$$

and

$$J_T/I_T = T(R/I_T) .$$

Using this notation, we can prove easily

PROPOSITION 1. *The following four statements are equivalent*:
(i) *A torsion T in* $\operatorname{Mod} R$ *is closed under taking direct products.*
(ii) *For every subset $\mathscr{S}$ of $\mathscr{K}(T)$,*

$$\bigcap_{L \in \mathscr{S}} L \in \mathscr{K}(T) .$$

(iii) $$I_T \in \mathscr{K}(T)$$

(iv) $$J_R = R .$$

Proof. The equivalence of (ii), (iii), and (iv) is trivial. Also the implication (ii) $\longrightarrow$ (i) follows easily; for, the order of an element of a direct product is evidently the intersection of the orders of its components. Finally, in order to show that (i) $\longrightarrow$ (iv), we consider the monogenic submodule of the direct product

$$\prod_{L \in \mathscr{K}(T)} R \bmod L$$

generated by the element whose components are generators of $R \bmod L$; it is obviously R-isomorphic to R/I_T.

PROPOSITION 2. *Let every proper (i.e., $\neq R$) two-sided ideal J of R satisfy the following condition: There is $\kappa \notin J$ such that, for every $\rho \in R$ with $\rho\kappa \notin J$, there exists $\sigma \in R$ with $\sigma\rho\kappa = \kappa$. Then every torsion in* $\operatorname{Mod} R$ *is closed under taking direct products.*

Proof. Let T be a torsion and J_T the two-sided ideal defined above. Assume that $J_T \neq R$. Thus, there exists $\kappa \notin J_T$ with the properties stated in our assumption. Since

$$\bigcap_{L \in \mathscr{K}(T)} L = I_T \subseteqq J_T ,$$

there is $L_0 \in \mathscr{K}(T)$ such that $\kappa \notin L_0$. Hence

$$L_0 \colon \kappa = (R\kappa \cap L_0) \colon \kappa \subseteqq J_T \colon \kappa ,$$

and therefore $J_T \colon \kappa \in \mathscr{K}(T)$, in contradiction to the fact that R/J_T has no nonzero element of order belonging to $\mathscr{K}(T)$. Consequently, $J_T = R$ and Proposition 2 follows in view of Proposition 1.

THEOREM A. *If a ring R satisfies the minimum condition on principal left ideals, i.e., if R is right perfect (cf. H. Bass [2]), then every torsion in* Mod R *is closed undertaking direct products.*

Proof. Given an ideal $J \neq R$, consider the (nonempty) set of all principal left ideals which are not contained in J; take a minimal element K of this set, $\kappa \in K \backslash J$ and apply Proposition 2.

REMARK 1. We can see easily that if R satisfies the minimum condition on principal left ideals, then every R-module M has a nonzero socle; the latter property is, in turn, obviously equivalent to either of the following two statements:

(i) Every monogenic R-module has a nonzero socle.

(ii) For every proper left ideal L of R, there is $\rho \in R \backslash L$ such that $L \colon \rho \neq R$ is maximal in R.

Before we proceed to establish the characterization of perfect rings, left us introduced the following convenient notation and terminology. Denote by $\mathscr{W} \subseteqq \mathscr{L}$ the Q-set of all maximal left ideals of R (R itself including). Obviously, for every $W \in \mathscr{W}$, $W \neq R$, the subset

$$\{W \colon \rho \mid \rho \in R\}$$

is a minimal Q-set contained in $\mathscr{W}$. Denoting by $\mathscr{W}_\omega$, $\omega \in \Omega$, all such (distinct) minimal Q-sets, it is easy to see that $\{\mathscr{W}_\omega \mid \omega \in \Omega\}$ is a covering of $\mathscr{W}$, i.e.,

$$\mathscr{W} = \bigcup_{\omega \in \Omega} \mathscr{W}_\omega \quad \text{and} \quad \mathscr{W}_{\omega_1} \cap \mathscr{W}_{\omega_2} = \{R\} \quad \text{for} \quad \omega_1 \neq \omega_2 .$$

Furthermore, for every $\Omega_1 \subseteqq \Omega$, put

$$\mathscr{W}_{\Omega_1} = \bigcap_{\omega \in \Omega_1} \mathscr{W}_\omega ;$$

of course, $\mathscr{W} = \mathscr{W}_\Omega$ and $\mathscr{W}_\omega = \mathscr{W}_{\{\Omega\}}$ for each $\omega \in \Omega$. Now, for every $\Omega_1 \subseteqq \Omega$, denote the smallest T-set containing $\mathscr{W}_{\Omega_1}$ by $\mathscr{W}_{\Omega_1}^*$. It can be easily shown (cf. [3]) that $\mathscr{W}_{\Omega_1}^*$ is the unique T-set $\sim$-equivalent to $\mathscr{W}_{\Omega_1}$ in the sense that, for every proper left ideal $L \in \mathscr{W}_{\Omega_1}^*$,

$$\{L:\rho \mid \rho \in R\} \cap \mathscr{W}_{\Omega_1} \neq \{R\} .$$

As a consequence,

$$\mathscr{W}_{\Omega_1}^* \cap \mathscr{W} = \mathscr{W}_{\Omega_1} .$$

Let us call the torsions $T(\mathscr{W}_\omega^*)$, $\omega \in \Omega$, the *prime torsions* in Mod R and, more generally, torsions $T(\mathscr{W}_{\Omega_1}^*)$ corresponding to the subsets Ω_1 of Ω, the *fundamental torsions* (i.e., derived from prime ones) in Mod R.

On the basis of the above characterization of the T-sets $\mathscr{W}_{\Omega_1}^*$, one can derive very easily the following well-known

PROPOSITION 3. *For any ring R, all the fundamental torsions $T(\mathscr{W}_{\Omega_1}^*)$ in* Mod R *are distinct and form a lattice ideal of the complete lattice of all torsions in* Mod R, *which is isomorphic to the lattice 2^Ω of all subsets of Ω.*

Proof. In order to complete the proof we need only to show that every torsion T in Mod R contained in $T(\mathscr{W}^*)$ is fundamental. But this follows from the fact that the T-set $\mathscr{K}(T) \subseteqq \mathscr{W}^*$ is evidently $\sim$-equivalent to $\mathscr{K}(T) \cap \mathscr{W}$ and since $\mathscr{K}(T) \cap \mathscr{W} = \mathscr{W}_{\Omega_0}$ for a suitable $\Omega_0 \subseteqq \Omega$, we have, in view of the fact that there is unique T-set $\sim$-equivalent to $\mathscr{W}_{\Omega_0}$,

$$\mathscr{K}(T) = \mathscr{W}_{\Omega_0}^* ,$$

as required.

REMARK 2. We can see easily that the assertion that every torsion in Mod R is fundamental is equivalent to the assertion that $\mathscr{W}^* = \mathscr{L}$, which in turn is equivalent to any of the statements of the previous Remark 1 (for, $\mathscr{W}^* \sim \mathscr{W}$).

Now, let us formulate the following

THEOREM B. *Let R be a ring such that every fundamental torsion in* Mod R *is closed under taking direct products. Then $R/\text{Rad}\, R$ is semisimple (i.e., artinian); in particular, Ω is finite.*

Proof. For each $\omega \in \Omega$, put

$$W_\omega^0 = \bigcap_{W \in \mathscr{W}} W$$

and notice that the intersection

$$\operatorname{Rad} R = \bigcap_{\omega \in \Omega} W_\omega^0$$

is, according to Proposition 3, irredundant. For, $\mathscr{W}_\omega^*$ (for each $\omega \in \Omega$) and $\mathscr{W}^*$ are the smallest T-sets containing W_ω^0 and Rad R, respectively.

In order to prove our theorem, it is sufficient to show that the socle of $R/\operatorname{Rad} R$ is the whole quotient ring $R/\operatorname{Rad} R$; for, $R/\operatorname{Rad} R$ is a ring with unity. First, observe that, in view of the fact that $\operatorname{Rad} R \in \mathscr{W}^*$, the socle of $R/\operatorname{Rad} R$ is essential in $R/\operatorname{Rad} R$ in the sense that it intersect every nonzero left ideal of $R/\operatorname{Rad} R$ nontrivially. Write

$$S/\operatorname{Rad} R = \operatorname{Socle}(R/\operatorname{Rad} R)$$

with the two-sided ideal $S \supseteqq \operatorname{Rad} R$ of R and assume

$$S \neq R .$$

Then, there is a (proper) maximal left ideal W of R such that

$$S \subseteqq W \subset R ;$$

and, $W \in \mathscr{W}_{\omega_1}$ for a suitable $\omega_1 \in \Omega$. Moreover, clearly

$$S \subseteqq W_{\omega_1}^0 .$$

Hence, since $\bigcap_{\omega \in \Omega} W_\omega^0$ is irredundant,

$$\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_\omega^0 \neq \Big(\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_{\omega_1}^0\Big) \cap W_{\omega_1}^0 = \operatorname{Rad} R ;$$

on the other hand, since $\operatorname{Rad} R \subseteqq S \subseteqq W_{\omega_1}^0$,

$$\Big(\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_\omega^0\Big) \cap S = \operatorname{Rad} R ,$$

and thus

$$\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_\omega^0 = \operatorname{Rad} R ,$$

a contradiction.

The proof of the theorem is completed.

Now, the main result of the present paper, namely the characterization of perfect rings, follows straight forward from Theorem A, Remarks 1 and 2, Theorem B and the fact that a (right) perfect rings can be characterized as a ring R with unity such that every (left) R-module has a nonzero socle and that $R/\operatorname{Rad} R$ is artinian (H. Bass [2]):

COROLLARY. *A ring R is right perfect if and only if all torsions in* Mod R *are fundamental and are closed under taking direct products.*

In conclusion, let us remark that the above characterization cannot be strengthened, even if we take into account the additional condition that there is a finite number of fundamental torsions in Mod R (the fact which is a consequence of our characterization). To show this, we present the following two examples of rings (which can easily be generalized):

EXAMPLE 1. Let N be the set of all natural numbers, F a field. Denote by $R_1 = R_1(\aleph_0, F)$ the ring of all countable "bounded" matrices over F, i.e., the ring of all functions $f: N \times N \rightarrow F$ satisfying the condition that there is a natural number n_f such that

$$f(i, j) = 0 \qquad \text{for } i \neq j,\ i > n_f \text{ or } j > n_f$$

and

$$f(i, i) = f(n_f + 1, n_f + 1) \qquad \text{for all } i > n_f\ ,$$

with matrix addition and multiplication. It is easy to verify that, for every $n \in N$,

$$C_n = \{f \mid f \in R_1 \text{ and } f(i, j) = 0 \text{ for } j \neq n\}$$

are minimal left ideals in R_1 and that the socle

$$S = \bigoplus_{n \in N} C_n$$

of R_1 is a (two-sided) maximal ideal in R_1; obviously, $R_1/S \cong F$. Furthermore, $\mathscr{W}_1' = \{S, R_1\}$ is a minimal Q-set of left ideals of R_1. Also, for every $n \in N$, the left ideals

$$W_n = \{f \mid f \in R_1 \text{ and } f(i, n) = 0\}$$

are maximal in R_1 and belong to the same minimal Q-set $\mathscr{W}_2'$. It is easy to see that the set of all maximal left ideals of R_1

$$\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$$

and that there are 4 torsions in Mod R, all of them fundamental, namely

$$0 = T(\{R\}),\ T(\mathscr{W}_1{}^*),\ T(\mathscr{W}_2{}^*) \text{ and } \operatorname{Mod} R = T(\mathscr{W}^*)\ .$$

Only $T(\mathscr{W}_2{}^*)$ is not closed under taking direct products. Of course, R_1 is not perfect.

EXAMPLE 2. Denote by Q^+ the set of all nonnegative rational numbers endowed with the usual order $\leqq$. Let F be a field. Denote by $R_2 = R(Q^+, F)$ the ring of all functions $f: Q^+ \to F$ such that the support

$$\operatorname{Sup} f = \{r \mid r \in Q^+ \text{ and } f(r) \neq 0\}$$

is contained in a well-ordered (with respect to $\leqq$) subset of Q^+ which has no finite limit point, with the addition and multiplication defined by

$$(f_1 + f_2)(r) = f_1(r) + f_2(r)$$

and

$$(f_1 * f_2)(r) = \sum_{\substack{t \in Q^+ \\ t \leqq r}} f_1(t) \cdot f_2(r - t) ,$$

respectively.

It is a matter of routine to verify that R_2 is a (commutative) ring. Now, for every $f \in R_2$, denote by r_f the least nonzero rational number such that $f(r_f) \neq 0$. Moreover, for every $t \in Q^+$, denote by $f^{(t)}$ the function of R_2 defined by

$$f^{(t)}(r) = \begin{cases} 1 & \text{for } r = t , \\ 0 & \text{otherwise} . \end{cases}$$

Now, we can see easily that, for every $f \in R_2$,

$$f = f^{(r_f)} * \bar{f} ,$$

where $\bar{f}(r) = f(r + r_f)$ for $r \in Q^+$ (and thus, $r_{\bar{f}} = 0$). First, we are going to prove the following

LEMMA. *If $\bar{f} \in R_2$ such that $r_{\bar{f}} = 0$, then there is $\bar{g} \in R_2$ satisfying*

$$\bar{f} * \bar{g} = f^{(0)} \ (= \textit{unity of } R_2) .$$

Proof. In order to ease the technical difficulties of the proof, observe first that having a well-ordered subset S of Q^+ with no finite limit point, we can consider the subsemigroup $\bar{S}$ of Q^+ generated by S: $\bar{S}$ is again well-ordered and has no finite limit point. Hence, we may consider, for a moment, that our function $\bar{f}$ is defined on a well-ordered subsemigroup $\bar{S}$ of Q^+ with no limit point and try to find $\bar{g}$ defined on the same set $\bar{S}$, i.e., with $\operatorname{Sup} \bar{g} \subseteqq \bar{S}$. Write

$$S = \{r_i\}_{i=0}^{\infty} \text{ with } 0 = r_0 < r_1 < r_2 < \cdots < r_n < \cdots .$$

Let us proceed by induction: Denoting by $\bar{g}_1$ the function defined by

$$\bar{g}_1(0) = [\bar{f}(0)]^{-1},\ \bar{g}_1(r_1) = -[\bar{f}(0)]^{-2}\cdot f(r_1) \text{ and } \bar{g}_1(r) = 0 \text{ otherwise,}$$

we can see easily that

$$\bar{f}*\bar{g}_1 = f^{(0)} + h_1 ,$$

where

$$\text{Sup}\ \bar{g}_1 \subseteqq \{r_i\}_{i=0}^{1} \text{ and } \text{Sup}\ h_1 \subseteqq \{r_i\}_{i=2}^{\infty} .$$

Assuming that, for a natural $n \geqq 1$, we have $\bar{g}_n \in R_2$ and $h_n \in R_2$ with

$$\text{Sup}\ \bar{g}_n \subseteqq \{r_i\}_{i=0}^{n} \text{ and } \text{Sup}\ h_n \subseteqq \{r_i\}_{i=n+1}^{\infty}$$

such that

$$\bar{f}*\bar{g}_n = f^{(0)} + h_n ,$$

let us define

$$\bar{g}_{n+1} = \bar{g}_n + g_{n+1} ,$$

where

$$g_{n+1}(r_{n+1}) = -[\bar{f}(0)]^{-1}h_n(r_{n+1}) \text{ and } g_{n+1}(r) = 0 \text{ otherwise} .$$

Then,

$$\bar{f}*\bar{g}_{n+1} = \bar{f}*\bar{g}_n + \bar{f}*g_{n+1} = f^{(0)} + h_n + \bar{f}*g_{n+1}$$

and, writing

$$h_{n+1} = h_n + \bar{f}*g_{n+1} ,$$

we can easily check that

$$\text{Sup}\ h_{n+1} \subseteqq \{r_i\}_{i=n+2}^{\infty} .$$

For,

$$h_{n+1}(r) = (\bar{f}*g_{n+1})(r) = \sum_{0\leqq t\leqq r} \bar{f}(t)g_{n+1}(r-t) = 0 \quad \text{for } r < r_{n+1}$$

and

$$\begin{aligned} h_{n+1}(r_{n+1}) &= h_n(r_{n+1}) + \sum_{0\leqq t\leqq r_{n+1}} \bar{f}(t)g_{n+1}(r_{n+1}-t) \\ &= h_n(r_{n+1}) + \bar{f}(0)g_{n+1}(r_{n+1}) = h_n(r_{n+1}) - h_n(r_{n+1}) = 0 , \end{aligned}$$

as required.

Finally, to complete the proof of our lemma, denote by $\bar{g}$ the function defined by

$$\bar{g}(r) = \begin{cases} g_i(r_i) & \text{for } r = r_i,\ i = 0, 1, \cdots \\ 0 & \text{elsewhere} . \end{cases}$$

Then,

$$\bar{f}*\bar{g} = f^{(0)} ;$$

for, if $i = 1, 2, \cdots$

$$\begin{aligned}(\bar{f}*\bar{g})(r_i) &= (\bar{f}*[\bar{g}_i + (\bar{g} - \bar{g}_i)])(r_i) \\ &= (\bar{f}*\bar{g}_i)(r_i) + [\bar{f}*(\bar{g} - \bar{g}_i)](r_i) \\ &= (f^{(0)} + h_i)(r_i) + [\bar{f}*(\bar{g} - \bar{g}_i)](r_i) \\ &= 0 + \sum_{0 \leqq t \leqq r_i} \bar{f}(t)(\bar{g} - \bar{g}_i)(r_i - t) \\ &= 0 .\end{aligned}$$

As a consequence, $f \in R_2$ is a unit in R_2 if and only if $r_f = 0$. Moreover, for every $r \in Q^+$, there exist two ideals

$$\bar{I}_r = \{f \mid f \in R_2 \text{ and } r_f \geqq r\}$$

and

$$I_r = \{f \mid f \in R_2 \text{ and } r_f > r\} ;$$

these are all ideals of R_2. Notice that,

$$I_r \subset \bar{I}_r$$

and that

$$r_1 < r_2 \quad \text{implies} \quad I_{r_1} \supset \bar{I}_{r_2} ;$$

in particular,

$$\bar{I}_0 = R_2 \quad \text{and} \quad I_0 = \operatorname{Rad} R_2 .$$

It is also easy to see that there are no divisors of zero in R_2 and that

$$(\operatorname{Rad} R_2)^2 = \operatorname{Rad} R_2 .$$

For, if $f \in \operatorname{Rad} R_2$, then $r_f > 0$ and obviously,

$$f = f^{((1/2)r_f)} * g ,$$

where

$$g(r) = f\left(r + \frac{1}{2} r_f\right) \qquad \text{for } r \in Q^+ ;$$

here, both $f^{((1/2)r_f)}$ and g evidently belong to $\operatorname{Rad} R_2$.

Finally, given a positive rational number q, define

$$R_{2q} = R_2/I_q$$

(similarly, we can consider $\bar{R}_{2q} = R_2/\bar{I}_q$). It is easy to see that

$$\text{Rad}\, R_{2q} \cong I_0/I_q$$

satisfies again

$$(\text{Rad}\, R_{2q})^2 = \text{Rad}\, R_{2q} \ ,$$

but that every other proper ideal (which is isomorphic to either I_r/I_q or I_r/I_q for $r \geqq q$) is nilpotent; besides,

$$\text{Socle}\,(R_{2q}) \cong \bar{I}_q/I_q \ .$$

Thus, there are only three torsions in Mod R_{2q}, namely

$$0 = T(\{R\}),\ T(\{R_{2q}, \text{Rad}\, R_{2q}\}) \quad \text{and} \quad \text{Mod}\, R_{2q} = T(\mathscr{L}^{R_{2q}}) \ .$$

All of them are evidently closed under taking direct products; but, only the first two are fundamental. And, R_{2q} is not perfect.

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