

NONSOLVABLE FINITE GROUPS ALL OF WHOSE LOCAL SUBGROUPS ARE SOLVABLE, II

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In this second paper, the bulk of the work is devoted to characterizing $E_2(3)$ and $S_4(3)$. These two groups are "almost" N -groups and it is relevant to treat them separately. The actual characterizations (Theorems 8.1 and 9.1) are very technical but the hypotheses deal with the structure and embedding in a simple group of certain $\{2, 3\}$ -subgroups.

This paper is a continuation of an earlier paper.¹ The bibliographical references are to I.

7. Groups in which 1 is the only p -signalizer.

DEFINITION 7.1. $\mathcal{U}^*(p) = \{\mathfrak{B} \mid (\text{i}) \mathfrak{B} \text{ is a subgroup of } \mathfrak{G} \text{ of type } (p, p), (\text{ii}) N(\mathfrak{B}) \text{ contains a } S_p\text{-subgroup of } \mathfrak{G}\}$.

HYPOTHESIS 7.1. (i) p is a prime and if $\mathfrak{B} \in \mathcal{U}^*(p)$, then no S_p -subgroup of $C(\mathfrak{B})$ normalizes any nonidentity p' -subgroup of \mathfrak{G} .

(ii) The centralizer of every nonidentity p -subgroup of \mathfrak{G} is p -solvable.

Lemmas 7.1, 7.2, 7.3 are proved under Hypothesis 7.1.

LEMMA 7.1. (i) $\mathcal{U}(p) \subseteq \mathcal{E}(p)$. (See Definitions 2.8 and 2.10 of I).
 (ii) If $p \geq 5$, then $\mathcal{U}^*(p) \subseteq \mathcal{E}(p)$.
 (iii) If $p = 3$ and if no element of $\mathcal{U}(3)$ centralizes a quaternion subgroup of \mathfrak{G} , then $\mathcal{U}^*(3) \subseteq \mathcal{E}(3)$.

Proof. If p is odd, choose $\mathfrak{B} \in \mathcal{U}^*(p)$, while if $p = 2$, choose $\mathfrak{B} \in \mathcal{U}(2)$. We must show that either \mathfrak{B} centralizes every element of $\mathcal{N}(\mathfrak{B}; p')$ or $p = 3$, $\mathfrak{B} \in \mathcal{U}^*(3) - \mathcal{U}(3)$ and some element of $\mathcal{U}(3)$ centralizes a quaternion subgroup of \mathfrak{G} .

Let \mathfrak{F} be a S_p -subgroup of $N(\mathfrak{B})$, so that \mathfrak{F} is a S_p -subgroup of \mathfrak{G} . Proceeding by way of contradiction, let \mathfrak{Q} be an element of $\mathcal{N}(\mathfrak{B}; p')$ minimal subject to $[\mathfrak{Q}, \mathfrak{B}] \neq 1$. Then \mathfrak{Q} is a q -group for some prime $q \neq p$, $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{B}]$, and $\mathfrak{B}_0 = C_{\mathfrak{F}}(\mathfrak{Q})$ has order p . Let $\mathfrak{C} = C(\mathfrak{B}_0)$, $\mathfrak{C}_1 = C_{\mathfrak{F}}(\mathfrak{B}_0)$, and let \mathfrak{F}^* be a S_p -subgroup of \mathfrak{C} containing \mathfrak{C}_1 . Hypothesis 7.1 implies that $O_p(\mathfrak{C}) = 1$. Let $\mathfrak{F}_0 = O_p(\mathfrak{C})$. If $[\mathfrak{F}_0, \mathfrak{B}] \subseteq \mathfrak{B}$, then

¹ *Non-solvable finite groups all of whose local subgroups are solvable*, I, Bull. Amer. Math. Soc. **74** (1968), 383-437, which will be referred to as I.

Lemma 5.16 is violated. Hence, we have $|\mathfrak{P}^* : \mathbb{C}_1| = |\mathfrak{P}_0 : \mathfrak{P}_0 \cap \mathbb{C}_1| = p$ and $[\mathfrak{P}_0, \mathfrak{B}] \not\subseteq \mathfrak{B}$.

Suppose $\mathfrak{B} \subseteq \mathfrak{P}_0$. Then $\Omega = [\Omega, \mathfrak{B}] \subseteq \mathfrak{P}_0$, so $\Omega = 1$. This is not the case, so $\mathfrak{B} \not\subseteq \mathfrak{P}_0$. By Lemma 6.1, it follows that $\mathfrak{B} \notin \mathcal{Z}(p)$. Hence, by construction, p is odd. Since $\mathfrak{B} \not\subseteq \mathfrak{A}_0$, (\mathfrak{B}) implies that $p \leq 3$. Thus, $p = 3$ and $\mathfrak{B} \in \mathcal{Z}^*(3) - \mathcal{Z}(3)$. By definition of $\mathcal{Z}(3)$ and $\mathcal{Z}^*(3)$, it follows that $Z(\mathfrak{B})$ is non cyclic and \mathfrak{B} is not contained in the center of any S_3 -subgroup of \mathfrak{G} .

Since $[\mathfrak{P}_0, \mathfrak{B}, \mathfrak{B}] = 1$ and since $\mathfrak{B} \not\subseteq \mathfrak{P}_0$, it follows that $\mathfrak{B} \not\subseteq O_3(\mathbb{C}_{2,3})$, where $\mathbb{C}_{2,3}$ is a $S_{2,3}$ -subgroup of \mathbb{C} containing \mathfrak{P}^* .

Since $\mathfrak{B}_0 \not\subseteq Z(\mathfrak{B})$, we have $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$, where $\mathfrak{B}_1 \subseteq Z(\mathfrak{B})$. Since $\mathfrak{B}_0 \subseteq Z(\mathbb{C})$, we have $\mathfrak{B}_0 \subseteq O_3(\mathbb{C}_{2,3})$, and so $\mathfrak{B}_1 \not\subseteq O_3(\mathbb{C}_{2,3})$.

Let \mathfrak{H} be a subgroup of $\mathbb{C}_{2,3}$ such that

- (a) $\mathfrak{P}^* \subseteq \mathfrak{H}$.
- (b) $\mathfrak{B}_1 \not\subseteq O_3(\mathfrak{H})$.
- (c) \mathfrak{H} is minimal subject to (a) and (b).

Let $\mathfrak{H}_1 = O_3(\mathfrak{H})$. Since the fixed point subspace of \mathfrak{B}_1 on $\mathfrak{H}_1/D(\mathfrak{H}_1)$ is of codimension 1, Lemma 5.30 implies that $\mathfrak{H} = \mathfrak{P}^* \Omega^*$, where Ω^* is a quaternion group and $|\mathfrak{P}^* : \mathfrak{H}_1| = 3$, so that $\mathfrak{H}_1 \mathfrak{B}_1 = \mathfrak{P}^*$. Since \mathfrak{B}_1 centralizes $D(\mathfrak{H}_1)$, so does Ω^* . Let $C_{\mathfrak{H}_1}(\Omega^*) = \mathfrak{P}^*$. Thus, \mathfrak{P}_1 is a normal subgroup of \mathfrak{H} and $|\mathfrak{H}_1 : \mathfrak{P}_1^*| = 9$.

Let $\mathfrak{P}_2^* = [\mathfrak{H}_1, \Omega^*]$. Then \mathfrak{P}_2^* is generated by 2 elements and $\mathfrak{P}_2^* \cap \mathfrak{P}_1^*$ is of index 9 in \mathfrak{P}_2^* . Hence, \mathfrak{P}_2^* is either elementary of order 9 or is a nonabelian group of exponent 3 and order 27. Furthermore, $\mathfrak{H}_1 = \mathfrak{P}_1^* \mathfrak{P}_2^*$, $\mathfrak{P}_1^* \cap \mathfrak{P}_2^* = D(\mathfrak{P}_2^*)$, and $[\mathfrak{P}_1^*, \mathfrak{P}_2^*] = 1$.

Since $\mathfrak{B}_1 \not\subseteq \mathfrak{H}_1$, it follows that $\mathbb{C}_1 = \mathfrak{B}_1 \times (\mathbb{C}_1 \cap \mathfrak{H}_1)$. Hence, $D(\mathbb{C}_1) = D(\mathbb{C}_1 \cap \mathfrak{H}_1) \subseteq \mathfrak{P}_1^*$. We will show that $D(\mathbb{C}_1) = 1$. Suppose false. Let $\mathbb{C}^* = C(D(\mathbb{C}_1))$, so that \mathbb{C}^* is 3-solvable. Since $\mathbb{C}^* \triangleleft N(D(\mathbb{C}_1))$, it follows that $\mathbb{C}^* \mathfrak{B}$ is 3-solvable. Since $\mathfrak{B}_1 \subseteq Z(\mathfrak{B})$, we have $\mathfrak{B}_1 \subseteq O_3(\mathbb{C}^* \mathfrak{B})$. Since Ω^* centralizes $D(\mathfrak{H}_1)$, it follows that $\langle \mathfrak{B}_1, \Omega^* \rangle \subseteq \mathbb{C}^*$. Thus, $\langle \mathfrak{B}_1, \Omega^* \rangle$ is 3-closed. This is impossible, since $\langle \mathfrak{B}_1, \Omega^* \rangle$ covers $\mathfrak{H}/\mathfrak{H}_1$.

If \mathfrak{P}_2^* is nonabelian, then Ω^* centralizes $Z(\mathfrak{P}_2^*)$. Since Ω^* is a quaternion group, we are done in this case.

We may now assume that \mathfrak{P}_2^* is abelian, so elementary of order 9. Thus, $\mathfrak{H}_1 = \mathfrak{P}_1^* \times \mathfrak{P}_2^*$, \mathfrak{P}_1^* and \mathbb{C}_1 are elementary and $Z(\mathfrak{P}^*) = \mathfrak{P}_1^* \times \mathfrak{B}$, where $\mathfrak{B} = \mathfrak{P}_2^* \cap Z(\mathfrak{P}^*)$. Notice that $\mathfrak{B}_0 \subseteq \mathfrak{P}_1^*$. If $\mathfrak{P}_1^* \supset \mathfrak{B}_0$, then since every subgroup of \mathfrak{P}_1^* of type (3,3) is in $\mathcal{Z}(3)$ and since the quaternion group Ω^* centralizes \mathfrak{P}_1^* , we are done. We may therefore assume that $\mathfrak{P}_1^* = \mathfrak{B}_0$. Hence, $Z(\mathfrak{B})$ has order 9, $|\mathfrak{P}^*| = 3^4$, $|\mathbb{C}_1| = 3^3$. Also, $Z(\mathfrak{P}^*) = \mathfrak{B}_0 \times \mathfrak{B}$. Let B be a generator for \mathfrak{B}_0 and let I be the involution of Ω^* . Then I inverts \mathfrak{B} and centralizes B .

Let $\mathfrak{N} = \langle \mathfrak{B}, \mathfrak{P}^* \rangle \subseteq N(\mathbb{C}_1)$. Since $SL(3, 3)$ is a minimal simple group, it follows that $N(\mathbb{C}_1)$ is solvable. As $O_3(\mathfrak{N}) = 1$, we have $\mathbb{C}_1 = C(\mathbb{C}_1)$. Since $[\mathbb{C}_1, \mathfrak{P}^*, \mathfrak{P}^*] = 1$, it follows that \mathfrak{N} contains a normal subgroup

\mathfrak{B}^* of order 3 and that S_2 -subgroups of \mathfrak{N} are quaternion. Now $\mathfrak{B}^* \not\subseteq \mathfrak{B}$, since \mathfrak{B} does not normalize \mathfrak{B}_0 and \mathfrak{B}^* centralizes no subgroup of \mathfrak{B} other than 1 and \mathfrak{B}_0 . Suppose $\mathfrak{B}^* = \mathfrak{B}$. Since $\mathfrak{B} = \mathfrak{B}^{*'}$, it follows that S_3 -subgroups of $\mathfrak{N}/\mathfrak{B}^*$ are abelian. Thus \mathfrak{N} is 3-closed. But this is impossible since $\mathfrak{B} \neq \mathfrak{B}^*$. Hence, \mathfrak{B}^* is a subgroup of $Z(\mathfrak{B}^*)$ of order 3 which is different from \mathfrak{B} and from \mathfrak{B}_0 . Since $Z(\mathfrak{B}^*) = \mathfrak{B}_0 \times \mathfrak{B}$, there is a generator for \mathfrak{B}^* of the shape BV , where V is a generator for \mathfrak{B} .

Let J be any involution of \mathfrak{N} . Then $JVJ = V^{-1}$ and $JBVJ = BV$. Now I and J both normalize \mathfrak{B}^* and $\mathfrak{B}^{*'} = \mathfrak{B}$. Hence, $\langle I, J \rangle$ maps onto an abelian subgroup of $A_{\mathfrak{G}}(Z(\mathfrak{B}^*))$, which implies that J normalizes $Z(\mathfrak{B}^*) \cap C(I) = \mathfrak{B}_0$. Hence, $JBJ = B^f$ for some integer f , and the previous equations yield $V^2 = 1$, which is not the case. The proof is complete.

HYPOTHESIS 7.2. (i) If $\mathfrak{A} \in \mathcal{L}_{\text{inv}}(p)$, then $\mathfrak{N}(\mathfrak{A})$ contains only 1.

(ii) If \mathfrak{B} is of order p and is in the center of some S_p -subgroup of \mathfrak{G} , then $O_p(\mathfrak{M})$ is of symplectic type and width w , where $\mathfrak{M} = N(\mathfrak{B})$.

LEMMA 7.2. *Suppose Hypothesis 7.2 is satisfied and that if $p = 2$, then $w \geq 3$, while if $p = 3$, then $w \geq 2$. Let \mathfrak{B} be a subgroup of $O_p(\mathfrak{M})$ of type (p, p) which contains \mathfrak{B} . Then $\mathfrak{B} \in \mathcal{E}(p)$.*

Proof. Let \mathcal{V} be the set of subgroups of $O_p(\mathfrak{M})$ which violate the lemma. Let \mathcal{V}_0 be the subset of those \mathfrak{B} in \mathcal{V} which centralize at least one element \mathfrak{B} of $\mathcal{U}(p)$ which $\mathfrak{B} \subset \mathfrak{B} \subset O_p(\mathfrak{M})$. If $\mathcal{V}_0 \neq \emptyset$, choose $\mathfrak{B} \in \mathcal{V}_0$, while if $\mathcal{V}_0 = \emptyset$, choose \mathfrak{B} in \mathcal{V} .

Let $\mathfrak{G} = O_p(\mathfrak{M})$, $\mathfrak{G}_0 = C_{\mathfrak{G}}(\mathfrak{B})$. We first argue that $\mathfrak{N}(\mathfrak{G}_0; p')$ is trivial. Namely, \mathfrak{M} is p -solvable with $O_{p'}(\mathfrak{M}) = 1$, so $C_{\mathfrak{M}}(\mathfrak{G}) = \mathfrak{Z}(\mathfrak{G})$. This implies that $C_{\mathfrak{M}}(\mathfrak{G}_0)$ is a p -group. Hence, $\mathfrak{N}_{\mathfrak{M}}(\mathfrak{G}_0; p')$ is trivial. Suppose $\mathfrak{R} \in \mathfrak{N}(\mathfrak{G}_0; p')$. It suffices to show that $\mathfrak{R} \subseteq \mathfrak{M}$. If \mathfrak{G}_0 contains an element \mathfrak{B} of $\mathcal{U}(p)$ with $\mathfrak{B} \subset \mathfrak{B}$, then by Lemma 7.1 we get that \mathfrak{B} centralizes \mathfrak{R} . Hence, $\mathfrak{R} \subseteq C(\mathfrak{B}) \subseteq \mathfrak{M}$. If no such elements of $\mathcal{U}(p)$ are available, then by construction, $\mathcal{V}_0 = \emptyset$. But $\mathfrak{G} \triangleleft \mathfrak{M}$, so if \mathfrak{B} is a S_p -subgroup of \mathfrak{M} , then \mathfrak{G} contains an element \mathfrak{B} of $\mathcal{U}(\mathfrak{B})$. Let $\mathfrak{G}_1 = C_{\mathfrak{G}}(\mathfrak{B})$ so that $|\mathfrak{G} : \mathfrak{G}_1| = p$. If $\mathfrak{G}_0 \cap \mathfrak{G}_1$ contains more than one subgroup of order p , then there is a subgroup \mathfrak{B}^* of $\mathfrak{G}_0 \cap \mathfrak{G}_1$ of type (p, p) which contains \mathfrak{B} . Since $\mathcal{V}_0 = \emptyset$, $\mathfrak{B}^* \in \mathcal{E}(p)$, so $\mathfrak{R} \subseteq C(\mathfrak{B}^*) \subseteq C(\mathfrak{B}) \subseteq \mathfrak{M}$. Suppose $\mathfrak{G}_0 \cap \mathfrak{G}_1$ contains only one subgroup of order p . Then by hypothesis, we have $p \geq 5$, and so \mathfrak{G} is of width 1 and is a S_p -subgroup of \mathfrak{G} . Hypothesis 7.1 guarantees in this case that $\mathfrak{N}(\mathfrak{G}_0; p')$ is trivial, so $\mathfrak{R} = 1$. We have thus shown that $\mathfrak{N}(\mathfrak{G}_0; p')$ is trivial.

Choose \mathfrak{Q} in $\mathfrak{N}(\mathfrak{B}; p')$ minimal subject to $[\mathfrak{B}, \mathfrak{Q}] \neq 1$. Then $\mathfrak{Q} = [\mathfrak{B}, \mathfrak{Q}]$ and $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{Q})$ is of order p . Clearly, $\mathfrak{B}_0 \neq \mathfrak{B}$. Let $\mathfrak{M}_1 = N(\mathfrak{B}_0)$

so that \mathfrak{M}_1 is p -solvable. By the preceding argument, $O_p(\mathfrak{M}_1) = 1$. Hence, $\mathfrak{Z} \not\subseteq O_p(\mathfrak{M}_1)$, so that \mathfrak{H}_0 contains an extra special subgroup \mathfrak{H}^* of width $w - 1$ with $\mathfrak{H}^* \cap O_p(\mathfrak{M}_1) = 1$.

Let $\mathfrak{X} = R_p(\mathfrak{M}_1)$ (see Definition 2.2), $\mathfrak{Y} = C_{\mathfrak{M}_1}(\mathfrak{X})$. Suppose $\mathfrak{Z} \subseteq \mathfrak{Y}$. Since $\Omega = [\Omega, \mathfrak{Z}]$ and $\mathfrak{Y} \triangleleft \mathfrak{M}_1$, we have $\Omega \subseteq \mathfrak{Y}$. Let \mathfrak{X}_p be a S_p -subgroup of \mathfrak{M}_1 which contains \mathfrak{H}^* . Then \mathfrak{H}^* centralizes $Z(\mathfrak{X}_p)$, so $\mathfrak{Y}\mathfrak{H}^*$ centralizes $Z(\mathfrak{X}_p)$. Let \mathfrak{F} be a S_p -subgroup of \mathfrak{G} containing \mathfrak{X}_p . Then $Z(\mathfrak{F}) \subseteq Z(\mathfrak{X}_p)$, so \mathfrak{H}^* is contained in a conjugate $\tilde{\mathfrak{M}}$ of \mathfrak{M} , $\tilde{\mathfrak{M}} = N(\Omega_1(Z(\mathfrak{F})))$. Furthermore, since $\mathfrak{Z}\Omega \subseteq \mathfrak{Y} \subseteq \tilde{\mathfrak{M}}$, \mathfrak{H}^* is faithfully represented on $\Omega_p^1(\tilde{\mathfrak{M}})$. Let $\tilde{\mathfrak{H}} = O_p(\tilde{\mathfrak{M}})$, and let $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}_0 \supset \tilde{\mathfrak{H}}_1 \supset \dots \supset \tilde{\mathfrak{H}}_k = 1$ be part of a chief series for $\tilde{\mathfrak{M}}$. Then since $\mathfrak{Z} \not\subseteq O_p(\tilde{\mathfrak{M}})$, it follows that \mathfrak{Z} does not centralize $\tilde{\mathfrak{H}}_n/\tilde{\mathfrak{H}}_{n+1}$ for at least one value of n , $0 \leq n < k$. Hence, $|\tilde{\mathfrak{H}}_n : \tilde{\mathfrak{H}}_{n+1}| = p^{a_n}$ where $a_n \geq r_p(\mathfrak{Z}; \tilde{\mathfrak{M}})$. Then by Lemma 5.4, $r_p(\mathfrak{Z}; \tilde{\mathfrak{M}}) \geq r_c(\mathfrak{Z}; \mathfrak{F})$. Clearly,

$$r_c(\mathfrak{Z}; \mathfrak{F}) \geq r_c(\mathfrak{Z}; \mathfrak{H}^*) = p^{w-1}.$$

On the other hand, $\tilde{\mathfrak{H}}_n$ is a subgroup of $\tilde{\mathfrak{H}}$, so $2w \geq a_n$. Hence $2w \geq p^{w-1}$. If $p \geq 5$, then $w = 1$ is forced, so every p -solvable subgroup of \mathfrak{G} has p -length at most 1. This is absurd, so $p \leq 3$. If $p = 3$, then $w = 2$, since $w \geq 2$ by hypothesis. It is clear that this is impossible since \mathfrak{H}^* is faithfully represented on $\Omega_3^1(\tilde{\mathfrak{M}})$. If $p = 2$, then $w = 3$ or $w = 4$, since by hypothesis $w \geq 3$. This is also impossible by Lemma 5.13. We have shown that $\mathfrak{Z} \not\subseteq \mathfrak{Y}$.

Since \mathfrak{X} is a p -group, $\mathfrak{X} \subseteq \mathfrak{M}^G$ for some G in \mathfrak{G} . Then $\mathfrak{X} \cap \mathfrak{H}^G$ is an abelian subgroup of \mathfrak{H}^G , so $m(\mathfrak{X} \cap \mathfrak{H}^G) \leq w + 1 + e$, where $e = 0$ if p is odd and $e = 1$ if $p = 2$.

If p is odd, then $\mathfrak{X}/\mathfrak{X} \cap \mathfrak{H}^G$ is faithfully represented on the Frattini quotient group of $\Omega_1(\mathfrak{H}^G)$, and this latter group is generated by $2w$ elements. If $p = 2$, write $O_{2,2}(\mathfrak{M}^G) = \mathfrak{H}^G \cdot \mathfrak{R}$ where $|\mathfrak{R}|$ is odd. Then $[\mathfrak{R}, \mathfrak{G}^G]$ is generated by $2w$ elements and $\mathfrak{X}/\mathfrak{X} \cap \mathfrak{H}^G$ is faithfully represented on the Frattini quotient group of $[\mathfrak{R}, \mathfrak{H}^G]$. Thus, by a result of Schur [32], we have $m(\mathfrak{X}/\mathfrak{X} \cap \mathfrak{M}^G) \leq w^2$, that is,

$$(7.1) \quad m(\mathfrak{X}) \leq w^2 + w + 1 + e.$$

If $w = 1$, then $p \geq 5$ implies that $O_p(\mathfrak{M})$ is a S_p -subgroup of \mathfrak{G} . So every p -solvable subgroup of \mathfrak{G} has p -length at most 1, a contradiction. Hence, $w \geq 2$.

There is an elementary subgroup \mathfrak{X}_1 of \mathfrak{X} such that $A_{\mathfrak{G}}(\mathfrak{X}_1)$ contains a subgroup $\tilde{\mathfrak{H}}^*\tilde{\Omega}$, where $\tilde{\Omega} \triangleleft \tilde{\mathfrak{H}}^*\tilde{\Omega}$ is special and $\tilde{\mathfrak{H}}^* \cong \mathfrak{H}^*$ operates faithfully and irreducibly on $\tilde{\Omega}/D(\tilde{\Omega})$. Also, $\tilde{\mathfrak{H}}^*\tilde{\Omega}$ acts irreducibly on \mathfrak{X}_1 .

Assume that p is odd.

Since $\tilde{\mathfrak{H}}^*$ is extra special of width $w - 1$, it follows that $m(\tilde{\Omega}) \geq p^{w-1}a$, where $a = |F_q(\zeta) : F_q|$. Here, $\tilde{\Omega}$ is a q -group and ζ is a primitive

p^{th} root of 1 in an extension field of the prime field F_q . By Lemma 5.3, $m(\mathfrak{X}) \geq m(\tilde{\mathfrak{Q}})b$, where $b = 2/3$ if $q = 2$ and $b = |F_p(\tau):F_p|$ if q is odd. Here τ is a primitive q^{th} root of 1 in an extension field of the prime field F_p . Together with (7.1), we get $abp^{w-1} \leq w^2 + w + 1$. Clearly, $ab > 1$. Suppose $w \geq 4$. Then $3^{w-1} \leq p^{w-1} < w^2 + w + 1$, a contradiction. Suppose $w = 3$. If $p \geq 5$, then $5^2 = 5^{w-1} \leq p^{w-1} < 3^2 + 3 + 1$, a contradiction. Thus, $p = 3$ and $q = 2$. Since $p = 3, w = 3$, it follows that $\mathfrak{M}^G/\mathfrak{G}^G$ is isomorphic to a 3-solvable subgroup of the 6 by 6 symplectic group over F_3 . It follows readily that $\mathfrak{M}^G/\mathfrak{G}^G$ has no elementary subgroup of order 3^4 . Thus, in this case, $m(\mathfrak{X}) \leq 4 + m(\mathfrak{X} \cap \mathfrak{G}^G) \leq 8$, against $(4/3) \cdot 3^3 = 12 = abp^{w-1} \leq m(\mathfrak{X})$. Hence $w = 2$. We now get $abp^{w-1} < 7$, so $p \leq 5$. Suppose $p = 5$ and q is odd. Then $ab \geq 2$, so that $10 < 7$. Suppose $p = 5$ and $q = 2$. Then $a = 4$, so $ab = 8/3$. We get $(8/3) \cdot 5 < 7$. Hence, $p = 3$. Suppose q is odd. Since a is characterized as the smallest positive integer n with $3^n \equiv 1 \pmod{q}$, it follows that $a \geq 3$, so $ab \geq 3$. This gives $9 \leq abp^{w-1} < 7$. Hence, $q = 2$. Since $p = 3, w = 2$, it follows that $\mathfrak{M}^G/\mathfrak{G}^G$ has no elementary subgroup of order 27. Hence, $m(\mathfrak{X}) \leq 3 + 2 = 5$. In particular, $m(\mathfrak{X}_1) \leq 5$. Suppose first that $\tilde{\mathfrak{Q}}$ is abelian. Since $Z(\tilde{\mathfrak{G}}^*) = \tilde{\mathfrak{Z}}$ acts without fixed points on $\tilde{\mathfrak{Q}}$, it follows that $C_{\tilde{\mathfrak{G}}^*}(\lambda) \cap \tilde{\mathfrak{Z}} = 1$ for every non trivial character λ of $\tilde{\mathfrak{Q}}$. So $|\tilde{\mathfrak{G}}^*:C_{\tilde{\mathfrak{G}}^*}(\lambda)| \geq 9$ for all $\lambda \neq 1$. Hence, $m(\mathfrak{X}_1) \geq 9$, a contradiction. Suppose $\tilde{\mathfrak{Q}}$ is nonabelian. Let \mathfrak{X}_2 be a subgroup of \mathfrak{X}_1 on which $\tilde{\mathfrak{Q}}$ acts irreducibly. Thus $m(\mathfrak{X}_2) \geq 2$, since $\tilde{\mathfrak{Q}}$ does not centralize \mathfrak{X}_2 . Since $m(\mathfrak{X}_1) \leq 5$, and $p = 3$, it follows that $\mathfrak{X}_2 = \mathfrak{X}_1$ is an irreducible $\tilde{\mathfrak{Q}}$ -group. Thus, $\tilde{\mathfrak{Q}}$ is extra special. But $m(\tilde{\mathfrak{Q}}) = 6$, since $q = 2$ and $|\tilde{\mathfrak{G}}^*| = 27$. This yields $m(\mathfrak{X}_1) \geq 2^3$. All possibilities have led to contradictions. So $p = 2$.

Since $\tilde{\mathfrak{G}}^*$ is extra special of width $w - 1$, we get that $m(\tilde{\mathfrak{Q}}) \geq 2^{w-1}$. Now Lemma 5.3(a) applied with $\tilde{\mathfrak{Q}}$ in the role of $\mathfrak{B}, \mathfrak{X}_1$ in the role of V , yields $m(\mathfrak{X}_1) \geq 2^w$. On the other hand, \mathfrak{B}_0 is a normal subgroup of \mathfrak{M}_1 of order 2, so $m(\mathfrak{X}) \geq 1 + 2^w$.

Let \mathfrak{C} be an elementary subgroup of \mathfrak{X} with $m(\mathfrak{C}) = 2^w + 1$, let $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{G}^G$, and let \mathfrak{C}_1 be a complement to \mathfrak{C}_0 in \mathfrak{C} . Since $m(\mathfrak{E}_0) \leq w + 2$, we get $m(\mathfrak{C}_1) = a \geq 2^w - 1 - w$. Since \mathfrak{C}_1 acts faithfully on $O_{2,2}(\mathfrak{M}^G)/\mathfrak{G}^G$, Lemma 5.34 implies that $O_{2,2}(\mathfrak{M}^G)/\mathfrak{G}^G$ has a subgroup $\hat{\mathfrak{D}}/\mathfrak{G}^G$ which admits \mathfrak{C}_1 and such that $\mathfrak{C}_1\hat{\mathfrak{D}}/\mathfrak{G}^G$ is the direct product of a dihedral groups of order twice an odd prime. Let \mathfrak{R} be a S_2 -subgroup of $\hat{\mathfrak{D}}$. By Lemma 5.12, $[\mathfrak{G}^G, \mathfrak{R}] = \mathfrak{R}$ is extra special of width $w_1 \leq w$. Since \mathfrak{G}^G is the central product of \mathfrak{R} and $C_{\mathfrak{G}^G}(\mathfrak{R})$, and since $\mathfrak{C}_1\hat{\mathfrak{D}} = \mathfrak{R} \cdot N_{\mathfrak{C}_1\hat{\mathfrak{D}}}(\mathfrak{R})$, it follows that if $M = \mathfrak{R}/D(\mathfrak{R})$, then $A_{\mathfrak{C}_1\hat{\mathfrak{D}}}(M)$ has a subgroup which is the direct product of a dihedral groups of order twice an odd prime. Let $m(M) = m$. Then $m = 2w_1 \leq 2w$. Since $w \geq 3$ by hypothesis, we get $w < 2^w - 1 - w \leq a$, and so $2w < 2a$, whence $m < 2a$.

This violates Lemma 5.8, and completes the proof.

HYPOTHESIS 7.3. (i) p is odd.

(ii) \mathfrak{B} is a S_p -subgroup of \mathfrak{G} , \mathfrak{A} is a normal elementary subgroup of \mathfrak{B} with $m(\mathfrak{A}) \geq 3$, $Z(\mathfrak{B})$ is cyclic, and $A_{\mathfrak{G}}(\mathcal{E}) = A(\mathcal{E})$, where $\mathcal{E}: \mathfrak{A} \supset \mathfrak{B} \cap Z(\mathfrak{B}) \supset 1$. Also, $\mathfrak{A} \triangleleft N(Z(\mathfrak{B}) \cap \mathfrak{A})$.

LEMMA 7.3. *Suppose Hypothesis 7.3 is satisfied. Let $\mathfrak{Z} = Z(\mathfrak{B}) \cap \mathfrak{A}$. Then each subgroup of \mathfrak{A} of type (p, p) which contains \mathfrak{Z} is in $\mathcal{E}(p)$.*

Proof. The lemma is an immediate consequence of Lemma 5.5, together with Hypothesis 7.1 (i).

HYPOTHESIS 7.4. (i) \mathfrak{G} is simple.

(ii) $\{2, 3\} \subseteq \pi_4(\mathfrak{G})$.

(iii) The centralizer of every involution of \mathfrak{G} is solvable.

(iv) The normalizer of every nonidentity 3-subgroup of \mathfrak{G} is solvable.

(v) If $\mathfrak{A} \in \mathcal{S}_{\text{Sol}_3}(2) \cup \mathcal{S}_{\text{Sol}_3}(3)$, then $\mathfrak{N}(\mathfrak{A})$ contains only 1.

All remaining lemmas in this section are proved under Hypothesis 7.4.

DEFINITION 7.2.

- $$\mathcal{N} = \{(\mathfrak{A}, \mathfrak{B}) \mid \begin{array}{l} 1. \ \mathfrak{A} \text{ is a 2-subgroup of } \mathfrak{G}. \\ 2. \ \mathfrak{B} \text{ is a 3-subgroup of } \mathfrak{G}. \\ 3. \ \langle \mathfrak{A}, \mathfrak{B} \rangle \text{ is not solvable.} \end{array}\}$$

We remark that in the following lemmas, Lemma 7.1 may be invoked, since Hypothesis 7.4 implies that Hypothesis 7.1 is satisfied for $p = 2$ and for $p = 3$.

LEMMA 7.4. *If \mathfrak{A} is a four-subgroup of \mathfrak{G} which centralizes every element of $\mathfrak{N}(\mathfrak{A}; 3)$ and \mathfrak{B} is a subgroup of \mathfrak{G} of type $(3, 3)$ which centralizes every element of $\mathfrak{N}(\mathfrak{B}; 2)$, then $(\mathfrak{A}, \mathfrak{B}) \in \mathcal{N}$.*

Proof. Notice that if $G, H \in \mathfrak{G}$, then the pair $(\mathfrak{A}^G, \mathfrak{B}^H)$ satisfies the hypothesis of the lemma.

Suppose the lemma is false and $\mathfrak{A}, \mathfrak{B}$ are chosen so that $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is minimal. It follows as in Lemma 0.10.2 that $\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathfrak{A} \times \mathfrak{B}$. We may then choose A in \mathfrak{A}^* such that $E(A)$ contains an element \mathfrak{A}_1 of $\mathcal{Z}(2)$. Hence, $\langle \mathfrak{A}_1, \mathfrak{B} \rangle$ is solvable. Thus, we may assume that $\mathfrak{A} \in \mathcal{Z}(2)$. Let $\mathfrak{N} = N(\mathfrak{A})$. Since $2 \in \pi_4$, we have $O_2(\mathfrak{N}) = 1$. This is absurd since

\mathfrak{B} centralizes $O_2(\mathfrak{N})$ and \mathfrak{N} is solvable. The proof is complete.

We set

$$\begin{aligned} \mathfrak{G}_1 &= \{G \mid G \in \mathfrak{G}, C(G) \text{ is solvable.}\} \\ \mathfrak{G}_p &= \{G \mid G \in \mathfrak{G}, C(G) \text{ contains an elementary subgroup } \mathfrak{E} \text{ of} \\ &\quad \text{order } p^2 \text{ which centralizes every element of } \mathfrak{N}(\mathfrak{E}; q)\}, \\ &\quad p = 2, 3, q = 2, 3, p \neq q. \end{aligned}$$

We conclude from Lemma 7.4 that

$$(7.2) \quad \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3 = \emptyset.$$

There are some subtle consequence of (7.2).

DEFINITION 7.3.

- $\mathcal{D} = \{\mathfrak{B} \mid$
 1. \mathfrak{B} is a noncyclic elementary 3-subgroup of \mathfrak{G} .
 2. Every element of \mathfrak{B} centralizes an element of $\mathcal{U}(3)$.
 3. \mathfrak{B} centralizes every abelian subgroup in $\mathfrak{N}(\mathfrak{B}; 2)$.\}

LEMMA 7.5. *Suppose $\mathfrak{A} \in \mathcal{U}(2)$, $\mathfrak{B} \in \mathcal{D}$ and \mathfrak{X} is a 2, 3-subgroup of \mathfrak{G} which contains $\langle \mathfrak{A}, \mathfrak{B} \rangle$. Let \mathfrak{X}_2 be a S_2 -subgroup of \mathfrak{X} . Then $\mathcal{MS}(\mathfrak{G})$ (see Definition 2.7) contains an element \mathfrak{M} such that*

- (a) $O_2(\mathfrak{M}) = 1$.
- (b) $O_2(\mathfrak{M})$ is the central product of $[O_2(\mathfrak{M}), \mathfrak{B}]$, which is extra special of width $w = 2, 3$ or 4, and of $C_{O_2(\mathfrak{M})}(\mathfrak{B})$, which is either cyclic or of maximal class ≥ 3 .
- (c) $[O_2(\mathfrak{M}), \mathfrak{B}]$ is the central product of w \mathfrak{B} -invariant quaternion groups $\mathfrak{Q}_1, \dots, \mathfrak{Q}_w$ whose centralizers in \mathfrak{B} are w distinct subgroups of order 3. In particular, no element of $\mathfrak{B}^\#$ centralizes any four-subgroup of $[O_2(\mathfrak{M}), \mathfrak{B}]$. If $w > 2$, then $C_{O_2(\mathfrak{M})}(\mathfrak{B}) = [O_2(\mathfrak{M}), \mathfrak{B}]'$ is the center of $O_2(\mathfrak{M})$.
- (d) $\mathfrak{X}_2 \subset \mathfrak{M}$.
- (e) $\mathfrak{B} \subset \mathfrak{M}$, and if \mathfrak{Q} is a quaternion subgroup of \mathfrak{X}_2 which is normalized by \mathfrak{B} but is not centralized by \mathfrak{B} , then $\mathfrak{Q} \subset O_2(\mathfrak{M})$.
- (f) If J is an involution of $\mathfrak{M} \cap C(\mathfrak{B})$, then $J \in O_2(\mathfrak{M})$. If \mathfrak{M} contains a S_2 -subgroup of $C(J)$ (e.g., if $C(J) = \mathfrak{X}$), then $C(J) \subseteq \mathfrak{M}$.
- (g) \mathfrak{M} contains a S_2 -subgroup of \mathfrak{G} .

Proof. Let \mathcal{S} be the set of 2, 3-subgroups of \mathfrak{G} which contain $\langle \mathfrak{B}, \mathfrak{X}_2 \rangle$. Choose \mathfrak{S} in \mathcal{S} so that $|\mathfrak{S}|_2$ is maximal. Let \mathfrak{S}_p be a S_p -subgroup of \mathfrak{S} , $p = 2, 3$, chosen so that $\mathfrak{X}_2 \subseteq \mathfrak{S}_2, \mathfrak{B} \subseteq \mathfrak{S}_3$. Let $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ be all the elements of $\mathcal{U}(2)$ in \mathfrak{S}_2 . By Lemma 7.1, each \mathfrak{A}_i centralizes $O_3(\mathfrak{S})$, so by Lemma 7.4, $|O_3(\mathfrak{S})| \leq 3$. In particular, \mathfrak{B} is not contained in $O_2(\mathfrak{S})$ and \mathfrak{B} centralizes $O_3(\mathfrak{S})$. Since $\mathfrak{B} \not\subseteq F(\mathfrak{S}), \mathfrak{B}$

does not centralize $O_2(\mathfrak{C})$. Hence, $O_2(\mathfrak{C})$ is nonabelian since $\mathfrak{B} \in \mathcal{S}$. Let $\mathfrak{R} = O_2(\mathfrak{C})' \cap \mathfrak{Z}(O_2(\mathfrak{C}))$, so that $\mathfrak{R} \neq 1$. Let $\mathfrak{N} = N(\mathfrak{R})$, $\mathfrak{C} = C(\mathfrak{R})$, and observe that $\mathfrak{B} \subseteq \mathfrak{C}$. Since the centralizer of every involution is solvable, \mathfrak{C} is solvable. Let \mathfrak{C}^* be a S_2 -subgroup of \mathfrak{N} which contains \mathfrak{C}_2 . Then $\mathfrak{C}\mathfrak{C}_2^*$ is solvable. By $D_{2,3}$ in $\mathfrak{C}\mathfrak{C}_2^*$ and maximality of $|\mathfrak{C}|_2$, it follows that $\mathfrak{C}_2^* = \mathfrak{C}_2$. Hence, if \mathfrak{C}_2^{**} is a S_2 -subgroup of \mathfrak{G} containing \mathfrak{C}_2 , then \mathfrak{C}_2 contains every element of $\mathcal{Z}(\mathfrak{C}_2^{**})$. We may therefore assume that $\mathfrak{A}_1 \in \mathcal{Z}(\mathfrak{C}_2^{**})$.

Since \mathfrak{A}_1 centralizes $O_3(\mathfrak{C})$, it follows that $\mathfrak{A}_1 \cap \mathfrak{Z}(\mathfrak{C}_2^{**}) \subseteq \mathfrak{Z}(O_2(\mathfrak{C}))$. Since \mathfrak{B} centralizes $\mathfrak{Z}(O_2(\mathfrak{C}))$, maximality of $|\mathfrak{C}|_2$ guarantees that $\mathfrak{C}_2 = \mathfrak{C}_2^{**}$ is a S_2 -subgroup of \mathfrak{G} .

Let $\mathfrak{C} \subseteq \mathfrak{M} \in \mathcal{MS}(\mathfrak{G})$. Thus, (g) holds, as does (d). Since \mathfrak{M} contains a S_2 -subgroup of \mathfrak{G} , and since 1 is the only 2-signalizer of \mathfrak{G} , it follows that $O_2(\mathfrak{M}) = 1$, and (a) holds. Let $\mathfrak{H} = O_2(\mathfrak{M})$. Suppose \mathfrak{H} contains a noncyclic characteristic abelian subgroup \mathfrak{H}_0 . Then \mathfrak{B} centralizes $\mathfrak{H}_0\mathfrak{Z}(\mathfrak{H})$ and $\mathfrak{H}_0\mathfrak{Z}(\mathfrak{H})$ contains an element of $\mathcal{Z}(\mathfrak{C}_2)$. This violates Lemma 7.4.

Clearly, \mathfrak{H} is noncyclic, since $\mathfrak{H} = F(\mathfrak{M})$ and \mathfrak{M} is solvable. Thus, \mathfrak{H} is of symplectic type. The width w of \mathfrak{H} is at least 2, since \mathfrak{B} is faithfully represented on \mathfrak{H} .

Suppose $w \geq 3$ and $B \in \mathfrak{B}^*$ centralizes a four-subgroup \mathfrak{B} of \mathfrak{H} with $\Omega_1(\mathfrak{Z}(\mathfrak{H})) \subset \mathfrak{B}$. By Lemma 7.2, \mathfrak{B} centralizes every element of $\mathfrak{N}(\mathfrak{B}; 2')$. Since $C(B)$ contains an element of $\mathcal{Z}(3)$, (7.2) is violated. Thus, if $w \geq 3$, then no element of \mathfrak{B}^* centralizes any four-subgroup of \mathfrak{H} . This immediately implies that \mathfrak{H} is extra special and $w \leq 4$. Now (b) and (c) follow from Lemma 5.12.

We next prove the first assertion of (f). Let J be an involution of $\mathfrak{M} \cap C(\mathfrak{B})$. If $w = 2$, then \mathfrak{B} is a S_2 -subgroup of \mathfrak{M} and (f) is clear. Suppose $w \geq 3$. In this case, \mathfrak{H} is extra special, so $\mathfrak{H} \cap C(\mathfrak{B}) = \mathfrak{H}'$ is of order 2. Let \mathfrak{B}_0 be any subgroup of \mathfrak{B} of order 3. Since J centralizes \mathfrak{B}_0 , it follows that J normalizes $C_{\mathfrak{H}}(\mathfrak{B}_0)$. We will show that J centralizes $C_{\mathfrak{H}}(\mathfrak{B}_0)$. This is clear if $C_{\mathfrak{H}}(\mathfrak{B}_0) = \mathfrak{H}'$, so suppose $C_{\mathfrak{H}}(\mathfrak{B}_0) \supset \mathfrak{H}'$. Since \mathfrak{H} is extra special, so is $C_{\mathfrak{H}}(\mathfrak{B}_0)$, so $C_{\mathfrak{H}}(\mathfrak{B}_0)$ is a quaternion group on which $\mathfrak{B}/\mathfrak{B}_0$ is faithfully represented. Since a quaternion group has no automorphism of order 6, J necessarily centralizes $C_{\mathfrak{H}}(\mathfrak{B}_0)$. Hence, J centralizes $\langle C_{\mathfrak{H}}(\mathfrak{B}_0) \mid \mathfrak{B}_0 \subset \mathfrak{B}, |\mathfrak{B}_0| = 3 \rangle = \mathfrak{H}$, so $J \in \mathfrak{H}$. This proves the first assertion of (f).

Now for the second assertion of (f). If $w > 2$, then $\langle J \rangle = \mathfrak{H}'$, by what we have just shown, together with (c). So suppose $w = 2$ and $\langle J \rangle \triangleleft \mathfrak{M}$. Let $\mathfrak{H}_0 = [\mathfrak{H}, \mathfrak{B}]$, $\mathfrak{H}_1 = C_{\mathfrak{H}}(\mathfrak{B})$. Thus, $J \in \mathfrak{H}_1$, $J \neq Z$, where Z is central involution of \mathfrak{H}_1 . Since $w = 2$, \mathfrak{B} is a S_2 -subgroup of \mathfrak{M} . Let \mathfrak{X}_0 be a S_2 -subgroup of $C(J)$ which is contained in \mathfrak{M} . Thus, $C(J) \cong \mathfrak{X}_0\mathfrak{B}$, $\mathfrak{X}_0 \cong \mathfrak{H}_0 \times \langle J \rangle$, and \mathfrak{H}_0 is the central product of 2 quaternion groups.

Since $C(J)$ contains an element of $\mathcal{Z}(2)$, it follows that $O_2(C(J)) = 1$. Since Z centralizes \mathfrak{X}_0 , a S_2 -subgroup of $C(J)$, it follows that $Z \in O_2(C(J))$, and so $Z \in Z(O_2(C(J)))$. Since $\mathfrak{B} \subseteq C(J)$, it follows that $O_2(C(J)) \in \mathcal{N}_{\mathfrak{M}}(\mathfrak{B}; 2)$. Hence, $O_2(C(J)) \subseteq O_2(\mathfrak{M}) = \mathfrak{G}$, since \mathfrak{B} is a S_2 -subgroup of \mathfrak{M} . Hence, $O_2(C(J)) \subseteq \mathfrak{G} \cap \mathfrak{X}_0 = \mathfrak{G}_0 \times \langle J \rangle$. If $O_2(C(J))$ is not elementary, then $\langle Z \rangle \text{ char } O_2(C(J))$, and so $C(J) \subseteq \mathfrak{M}$. Suppose $O_2(C(J))$ is elementary. Since \mathfrak{B} is faithfully represented on $O_2(C(J))$, it follows that $|O_2(C(J))| \geq 2^4$. However, $O_2(\mathfrak{M})$ contains no elementary subgroup of order 2^4 on which \mathfrak{B} is faithfully represented. This completes the proof of the second assertion of (f).

We turn to the proof of (e). Let \mathfrak{Q} be a quaternion subgroup of \mathfrak{M} normalized but not centralized by \mathfrak{B} . Let $\mathfrak{B}_0 = \mathfrak{B} \cap C(\mathfrak{Q})$, so that $\mathfrak{B}\mathfrak{Q} = \mathfrak{B}_0 \times \mathfrak{B}_1\mathfrak{Q}$, where $|\mathfrak{B}_0| = 3$ and \mathfrak{B}_1 is faithfully represented on \mathfrak{Q} . Let $\mathfrak{Q}_0 = \mathfrak{Q} \cap \mathfrak{G}$. By (f), $\mathfrak{Q}_0 \cong \mathfrak{Q}'$. Since $\mathfrak{Q}/\mathfrak{Q}'$ is an irreducible \mathfrak{B} -group, we may assume by way of contradiction that $\mathfrak{Q}_0 = \mathfrak{Q}'$.

Let \mathfrak{R} be a $\mathfrak{B}\mathfrak{Q}$ -invariant subgroup of $Q_2(\mathfrak{M})$ minimal subject to $[\mathfrak{R}, \mathfrak{Q}] \neq 1$. Thus, \mathfrak{R} may be viewed as a $\mathfrak{B}\mathfrak{Q}/\mathfrak{Q}'$ -group; as such $\mathfrak{B}_1\mathfrak{Q}/\mathfrak{Q}'$ acts faithfully. Since $w \leq 4$, it follows that \mathfrak{R} is elementary of order 3^3 and is centralized by \mathfrak{B}_0 . Thus, $w = 4$ and S_3 -subgroups of \mathfrak{M} are of order 3^5 . Also, \mathfrak{R} is incident with an elementary subgroup \mathfrak{R}_0 of \mathfrak{M} such that $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$ is elementary of order 3^4 .

Let \mathfrak{F} be a S_3 -subgroup of \mathfrak{G} containing $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$ and choose \mathfrak{U} in $\mathcal{Z}(\mathfrak{F})$. Then $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$ contains an elementary subgroup \mathfrak{E} of order 3^3 which centralizes \mathfrak{U} . Since $\mathfrak{E} \subseteq \mathfrak{M}$, there is an element E of $\mathfrak{E}^\#$ such that $\mathfrak{G} \cap C(E)$ contains a four-group. But then $E \in \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3$, against (7.2). This contradiction completes the proof of (e) and the lemma.

Throughout the remainder of this section, \mathfrak{F} denotes a S_3 -subgroup of \mathfrak{G} .

LEMMA 7.6. *Suppose $|\mathfrak{F}| > 3^4$.*

(a) *If \mathfrak{F}_0 is a subgroup of \mathfrak{F} of index at most 9 and \mathfrak{F}_0 contains an element of $\mathcal{Z}^*(\mathfrak{F})$, then $\mathcal{N}(\mathfrak{F}_0; 2)$ contains only 1.*

(b) *If \mathfrak{A} is a subgroup of \mathfrak{F} of type (3, 3) and $|\mathfrak{F}: C_{\mathfrak{F}}(\mathfrak{A})| \leq 3$, then \mathfrak{A} centralizes every element of $\mathcal{N}(\mathfrak{A}; 2)$.*

(c) *If \mathfrak{A} is a subgroup of \mathfrak{F} of type (3, 3), if $|\mathfrak{F}: C_{\mathfrak{F}}(\mathfrak{A})| \leq 9$, and if $C_{\mathfrak{F}}(\mathfrak{A})$ contains an element of $\mathcal{Z}^*(\mathfrak{F})$, then $\mathfrak{A} \in \mathcal{D}$.*

(d) *If \mathfrak{E} is a normal elementary subgroup of \mathfrak{F} of order 27 and $|\mathfrak{F}: C_{\mathfrak{F}}(\mathfrak{E})| = 3$, then $\mathfrak{A} \in \mathcal{E}(3)$ for each subgroup \mathfrak{A} of index 3 in \mathfrak{E} .*

Proof. (a) Let \mathfrak{B} be an element of $\mathcal{Z}^*(\mathfrak{F})$ with $\mathfrak{B} \subseteq \mathfrak{F}_0$. We will show that $\mathcal{N}(\mathfrak{B}; 2)$ contains only 1. To do this, we first show that if $\mathfrak{X} \in \mathcal{Z}(3)$, then $|C(\mathfrak{X})|$ is odd. Suppose J is an involution of $C(\mathfrak{X})$. By Lemma 5.38, $C(J)$ contains an element \mathfrak{Y} of $\mathcal{Z}(2)$. Hence,

by Lemmas 7.1 and 7.4, $\langle \mathfrak{X}, \mathfrak{Y} \rangle$ is nonsolvable, against $\langle \mathfrak{X}, \mathfrak{Y} \rangle \subseteq C(J)$. In particular, \mathfrak{X} does not centralize any quaternion subgroup of \mathfrak{G} . By Lemma 7.1(iii), it follows that \mathfrak{B} centralizes every element of $\mathcal{N}(\mathfrak{B}; 2)$. Suppose K is an involution of $C(\mathfrak{B})$. Then by Lemma 5.38, $C(\mathfrak{B})$ contains an element \mathfrak{Z} of $\mathcal{U}(2)$, so by Lemmas 7.1 and 7.4, $\langle \mathfrak{B}, \mathfrak{Z} \rangle$ is nonsolvable, against $\langle \mathfrak{B}, \mathfrak{Z} \rangle \subseteq C(K)$. We conclude that $|C(\mathfrak{B})|$ is odd, and so $\mathcal{N}(\mathfrak{B}; 2)$ contains only 1. Since $\mathcal{N}(\mathfrak{B}_0; 2) \subseteq \mathcal{N}(\mathfrak{B}; 2)$, (a) follows.

Suppose (b) is false. Let \mathfrak{Q} be a 2-group normalized by \mathfrak{A} and minimal subject to $[\mathfrak{Q}, \mathfrak{A}] \neq 1$. Then $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}]$ is either a quaternion group or a four-group, and $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$ where $|\mathfrak{A}_1| = 3$ and $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{Q})$.

Let $\mathfrak{C} = C(\mathfrak{A}_0) \cong \langle C_{\mathfrak{B}}(\mathfrak{A}), \mathfrak{Q} \rangle$. Since $C_{\mathfrak{B}}(\mathfrak{A})$ is of index at most 3 in \mathfrak{B} , it follows that $C_{\mathfrak{B}}(\mathfrak{A})$ contains an element \mathfrak{U} of $\mathcal{U}(\mathfrak{B})$. We argue that $C_{\mathfrak{B}}(\mathfrak{A})$ contains an element of $\mathcal{S}_{m_3}(\mathfrak{B})$. Namely, let

$$\mathfrak{Z} = \Omega_1(Z(C_{\mathfrak{B}}(\mathfrak{A}))) .$$

If $m(\mathfrak{Z}) \geq 3$, let \mathfrak{B} be an element of $\mathcal{S}_{m_3}(\mathfrak{B})$ which contains \mathfrak{Z} . Since $\mathfrak{A} \subseteq \mathfrak{B}$, we get $\mathfrak{B} \subseteq C_{\mathfrak{B}}(\mathfrak{A})$. Suppose $m(\mathfrak{Z}) \leq 2$. Then $\mathfrak{Z} = \mathfrak{A} \triangleleft \mathfrak{B}$, so by Lemma 0.8.9, \mathfrak{A} is contained in some element of $\mathcal{S}_{m_3}(\mathfrak{B})$. So $C_{\mathfrak{B}}(\mathfrak{A})$ contains an element of $\mathcal{S}_{m_3}(\mathfrak{B})$. By Hypothesis 7.4(v), $O_3(\mathfrak{C}) = 1$.

Let \mathfrak{P}^* be a S_3 -subgroup of C which contains $C_{\mathfrak{B}}(\mathfrak{A})$. Since \mathfrak{A}_1 does not centralize $O_3(\mathfrak{C}) = \mathfrak{H}$, it follows that $\mathfrak{P}^* = \mathfrak{H}C_{\mathfrak{B}}(\mathfrak{A})$ is a S_3 -subgroup of \mathfrak{G} . Also, since $\mathfrak{A}_1\mathfrak{H}/\mathfrak{H} \subseteq Z(\mathfrak{P}^*/\mathfrak{H})$, it follows that $\mathfrak{A}_1 \subseteq O_{3,3',3}(\mathfrak{C})$. Hence, $\mathfrak{Q} \subseteq O_{3,3',3}(\mathfrak{C})$. Since $C_{\mathfrak{Q}}(\mathfrak{A}_1)$ is of index 3 in \mathfrak{H} , it follows that $[Q_3(\mathfrak{C}), \mathfrak{A}_1]$ is a quaternion group. Hence, $\tilde{\mathfrak{C}} = \mathfrak{Q}\mathfrak{P}^*$ is a group. Let $\tilde{\mathfrak{H}} = O_3(\tilde{\mathfrak{C}})$. Thus, $\mathfrak{P}^* = \tilde{\mathfrak{H}}\mathfrak{A}_1$ and $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q})$ is of index 9 in \mathfrak{H} , while $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q}) \triangleleft \mathfrak{P}^*$. Since \mathfrak{Q}' centralizes no element of $\mathcal{U}^*(\mathfrak{P}^*)$, it follows that $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q})$ is cyclic. Since $|\mathfrak{B}| > 3^4$, so also $|\mathfrak{P}^*| > 3^4$, so \mathfrak{A}_0 is a proper subgroup of $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q}) = \tilde{\mathfrak{A}}_0$.

Let $\tilde{\mathfrak{G}}_0 = [\mathfrak{Q}, \tilde{\mathfrak{H}}]$. By the three subgroups lemma, $\tilde{\mathfrak{G}}_0$ and $\tilde{\mathfrak{A}}_0$ commute elementwise. Furthermore, either $\tilde{\mathfrak{G}}_0$ is elementary of order 9 and $\tilde{\mathfrak{H}} = \tilde{\mathfrak{G}}_0 \times \tilde{\mathfrak{A}}_0$ or $\tilde{\mathfrak{G}}_0$ is a non abelian group of order 27 and exponent 3 and $\tilde{\mathfrak{H}}$ is the central product of $\tilde{\mathfrak{G}}_0$ and $\tilde{\mathfrak{A}}_0$.

Since $C(\mathfrak{A}_1) \cap \tilde{\mathfrak{H}}$ is of index 3 in $\tilde{\mathfrak{H}}$, it follows that \mathfrak{A}_1 centralizes $\tilde{\mathfrak{A}}_0$. Set $\mathfrak{B} = \langle \mathfrak{A}_1, \tilde{\mathfrak{A}}_0 \rangle = \mathfrak{A}_1 \times \tilde{\mathfrak{A}}_0$, and let I be the involution of \mathfrak{Q} . Thus, $\mathfrak{B} \subseteq C(I)$ and $C(I)$ contains an element of $\mathcal{U}(2)$. Thus, $C(I)$ contains no element of $\mathcal{U}^*(3)$. Since \mathfrak{B} is of index 9 in \mathfrak{P}^* , it follows that \mathfrak{B} is a S_3 -subgroup of $C(I)$. Let \mathfrak{B} be a $S_{2,3}$ -subgroup of $C(I)$ which contains $\mathfrak{B}\mathfrak{Q}$. Then $O_3(\mathfrak{B}) = 1$, so \mathfrak{B} is faithfully represented on $O_2(\mathfrak{B})$. We can thus choose a subgroup \mathfrak{B}_0 of order 3 in \mathfrak{B} such that $\tilde{\mathfrak{A}}_0$ is faithfully represented on $O_2(\mathfrak{B}) \cap C(\mathfrak{B}_0)$. Let $\mathfrak{X} = C(\mathfrak{B}_0)$. Then $O_{3'}(\mathfrak{X})$ is of odd order by (a). Thus, $O_{3',3}(\mathfrak{X}) \cap \tilde{\mathfrak{A}}_0 = 1$, so that $|O_{3',3}(\mathfrak{X})|_3 \leq 27$. But $\tilde{\mathfrak{A}}_0$ is faithfully represented on the Frattini quotient group \mathfrak{B} of $O_{3',3}(\mathfrak{X})/O_3(\mathfrak{X})$. Since $|\mathfrak{B}| \leq 27$ and $\tilde{\mathfrak{A}}_0$ is cyclic of order ≥ 9 , we have a contradiction. The proof of (b) is complete.

Suppose (c) is false. Let \mathfrak{B} be a four group in $\mathfrak{N}(\mathfrak{A})$ which is not centralized by \mathfrak{A} . Then $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$ where $|\mathfrak{A}_i| = 3$ and $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{B})$.

Set $\mathfrak{C} = C(\mathfrak{A}_0)$ and let \mathfrak{P}^* be a S_3 -subgroup of \mathfrak{C} which contains $C_{\mathfrak{P}}(\mathfrak{A})$. By (a), $O_3(\mathfrak{C})$ is of odd order, so $\mathfrak{A}_1\mathfrak{B}$ is faithfully represented on $O_{3',3}(\mathfrak{C})/O_3(\mathfrak{C})$. Set $\mathfrak{G} = \mathfrak{P}^* \cap O_{3',3}(\mathfrak{C})$. By (B), $|\mathfrak{P}^*: C_{\mathfrak{G}}(\mathfrak{A}_1)| \geq 9$. Thus, $\mathfrak{P}^* = \mathfrak{G}C_{\mathfrak{P}}(\mathfrak{A})$ is a S_3 -subgroup of \mathfrak{G} , so that $O_3(\mathfrak{C}) = 1$.

We may now apply Lemma 5.42 with $\mathfrak{C}/\mathfrak{G}$ in the role of \mathfrak{S} , $\mathfrak{G}/D(\mathfrak{G})$ in the role of \mathfrak{C} , and \mathfrak{A}_1 in the role of \mathfrak{Z} . Let \mathfrak{C}_1 be the inverse image in \mathfrak{C} of $[Q_3(\mathfrak{C}), \mathfrak{A}_1]$. Thus, $\mathfrak{C}_1 = \mathfrak{G}\Omega$ where Ω is either a four-group or is the central product of 2 quaternion groups. Since $C_{\mathfrak{P}}(\mathfrak{A}_1)$ covers $\mathfrak{P}^*/\mathfrak{G}$, it follows that \mathfrak{C}_1 is a minimal subgroup of the group $\mathfrak{K} = \mathfrak{B}\mathfrak{P}^*$. Let $\mathfrak{L} = O_3(\mathfrak{K})$, so that $\mathfrak{P}^*/\mathfrak{L}$ is elementary of order 3 or 9. Since $\mathfrak{B} \subset \mathfrak{C}_1$, we assume without loss of generality that $\mathfrak{B} \subseteq \Omega$.

Since \mathfrak{A}_1 centralizes $D(\mathfrak{B})$, so does Ω . Thus, $C(\Omega) \cap \mathfrak{L} \triangleleft \mathfrak{L}$. Since $N(\Omega) \cap \mathfrak{K}$ normalizes $C(\Omega) \cap \mathfrak{L}$, it follows that $C(\Omega) \cap \mathfrak{L} \triangleleft \mathfrak{P}^*$. Since Ω centralizes no element of $\mathcal{Z}^*(3)$, it follows that $C(\Omega) \cap \mathfrak{L}$ is cyclic. Naturally, $\mathfrak{A}_0 \subseteq C(\Omega) \cap \mathfrak{L}$.

Case 1. $\mathfrak{P}^* = \mathfrak{L}\mathfrak{A}_1$.

Since \mathfrak{A}_1 normalizes \mathfrak{B} , it follows that $\mathfrak{K}_1 = \mathfrak{P}^*\mathfrak{B}$ is a group and that $\mathfrak{L} = O_3(\mathfrak{K}_1)$. Let $\mathfrak{L}_1 = C_{\mathfrak{L}}(\mathfrak{B}) \cong D(\mathfrak{B})$. Thus, $\mathfrak{L}_1 \triangleleft \mathfrak{K}_1$ and $\mathfrak{L}/\mathfrak{L}_1$ is elementary of order 27. Also, \mathfrak{L}_1 is cyclic, since no element of $\mathcal{Z}^*(3)$ is centralized by \mathfrak{B} . Since $\mathfrak{L}/\mathfrak{L}_1$ is a chief factor of \mathfrak{K}_1 or order 27, it follows that $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$, where $\mathfrak{L}_2 = [\mathfrak{L}, \mathfrak{B}]$ is elementary of order 27, $\mathfrak{L}_2 \triangleleft \mathfrak{K}_1$. Let V be an involution of \mathfrak{B} . Thus, $\mathfrak{B} = \langle C(V) \cap \mathfrak{L}_2, \mathfrak{A}_0 \rangle$ is elementary of order 9 and $|\mathfrak{P}^*: C_{\mathfrak{P}^*}(\mathfrak{B})| = 3$.

By (b), \mathfrak{B} centralizes every element of $\mathfrak{N}(\mathfrak{B}; 2)$. Since $C(V)$ contains an element of $\mathcal{Z}(2)$, (7.2) is violated.

Case 2. $\mathfrak{P}^* \supset \mathfrak{L}\mathfrak{A}_1$.

In this case, $\mathfrak{P}^*/\mathfrak{L}$ is elementary of order 9, so Ω is the central product of 2 quaternion groups.

Suppose \mathfrak{L} is abelian. Then $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$ where $\mathfrak{L}_1 = [\mathfrak{L}, \Omega]$ is elementary of order 3^4 and $\mathfrak{L}_2 = C_{\mathfrak{L}}(\Omega)$ is cyclic. Notice that $\mathfrak{A}_0 \subseteq \mathfrak{L}_2$. Since $C_{\mathfrak{L}}(\mathfrak{B})$ is of index 27 in \mathfrak{L} by (B), it follows that $\mathfrak{L} \cap C(\mathfrak{A}_1) \cap C(\mathfrak{B})$ contains a subgroup \mathfrak{B} of type (3, 3). But then $|\mathfrak{P}^*: C_{\mathfrak{P}^*}(\mathfrak{B})| \leq 3$, so by (b), \mathfrak{B} centralizes every element of $\mathfrak{N}(\mathfrak{B}; 2)$. Hence, $\mathfrak{B}^\# \subseteq \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3$, against (7.2). We conclude that \mathfrak{L} is non abelian.

Since \mathfrak{A}_1 centralizes $D(\mathfrak{B})$, so does Ω , so $\mathfrak{L}_2 = C_{\mathfrak{L}}(\Omega) \triangleleft \mathfrak{L}$. Hence $\mathfrak{L}_2 \triangleleft \mathfrak{P}^*$, and \mathfrak{L}_2 is cyclic. Let $\mathfrak{L}_1 = [\mathfrak{L}, \Omega]$. Then $\mathfrak{L}_1/D(\mathfrak{L}_1)$ is elementary of order 3^4 , $\mathfrak{L}'_1 = D(\mathfrak{L}_1)$ and $\mathfrak{L}_1/D(\mathfrak{L}_1)$ is a chief factor of K . Being a chief factor, $\mathfrak{L}_1/D(\mathfrak{L}_1)$ is centralized by \mathfrak{L} . Hence, $[\mathfrak{L}_2, \mathfrak{L}_1] \subseteq D(\mathfrak{L}_1) \subseteq \mathfrak{L}_2$, so $[\mathfrak{L}_2, \mathfrak{L}_1, \Omega] = 1$. Since $[\Omega, \mathfrak{L}_2] = 1$, so also $[\Omega, \mathfrak{L}_2, \mathfrak{L}_1] = 1$. By the three subgroups lemma, $[\mathfrak{L}_1, \Omega, \mathfrak{L}_2] = 1$, that is, $[\mathfrak{L}_1, \mathfrak{L}_2] = 1$. Hence,

$D(\mathfrak{L}_1) \subseteq \mathbf{Z}(\mathfrak{L}_1)$. Since \mathfrak{L} is nonabelian, so is \mathfrak{L}_1 . Since $\mathfrak{L}_1/D(\mathfrak{L}_1)$ is a chief factor of \mathfrak{R} , $D(\mathfrak{L}_1) = \mathbf{Z}(\mathfrak{L}_1) = \mathfrak{L}'_1 \subseteq \mathfrak{L}_2$, so \mathfrak{L}_1 is extra special of order 3^5 .

Now $\mathfrak{A}_1\mathfrak{B}$ is faithfully represented on \mathfrak{L}_1 . Also, $|\mathfrak{L}:\mathfrak{L} \cap C(\mathfrak{B})| = |\mathfrak{L}_1:\mathfrak{L}_1 \cap C(\mathfrak{B})| = 3^3$, by (B). Hence, $\mathfrak{L}_1 \cap C(\mathfrak{B}) = 3^2$. This is not the case, since $\mathfrak{L}_1 \cap C(\mathfrak{B})$ is either extra special or is \mathfrak{L}'_1 . The proof of (c) is complete.

Suppose (d) is false. Let \mathfrak{Q} be an element of $\mathfrak{N}(\mathfrak{A}; 3')$ minimal subject to $[\mathfrak{A}, \mathfrak{Q}] \neq 1$. Then \mathfrak{Q} is a q -group for some prime q , $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}]$, and $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$, where $|\mathfrak{A}_i| = 3$ and $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{Q})$. By (b), $q \neq 2$. Let $\mathfrak{C} = C(\mathfrak{A}_0)$. Since $C_{\mathfrak{B}}(\mathfrak{A})$ contains an element of $\mathcal{SBN}_3(\mathfrak{B})$, it follows from Hypothesis 7.4(v) that $O_3(\mathfrak{C}) = 1$. Let \mathfrak{B}^* be a S_3 -subgroup of \mathfrak{C} which contains $C_{\mathfrak{B}}(\mathfrak{A}_0)$. Since $|\mathfrak{B}:C_{\mathfrak{B}}(\mathfrak{A})| \leq 3$, so also $|\mathfrak{B}^*:C_{\mathfrak{B}}(\mathfrak{A}_0)| \leq 3$, and so $[\mathfrak{B}^*, \mathfrak{A}, \mathfrak{A}] = 1$.

Let \mathfrak{H} be a $S_{3,q}$ -subgroup of \mathfrak{C} which contains \mathfrak{B}^* . Since q is odd, (B) implies that $\mathfrak{A} \subseteq O_3(\mathfrak{H})$. Let \mathfrak{H}^* be a $S_{3,q}$ -subgroup of \mathfrak{C} which contains $\mathfrak{A}\mathfrak{Q}$. By Lemma 0.7.5, we get $\mathfrak{A} \subseteq O_3(\mathfrak{H}^*)$, so $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}] \subseteq O_3(\mathfrak{H}^*)$. This contradiction completes the proof of (d) and the lemma.

LEMMA 7.7. *Assume the following:*

- (a) \mathfrak{W} is a normal elementary subgroup of \mathfrak{B} , $\mathfrak{A} = A_{\mathfrak{B}}(\mathfrak{W})$.
- (b) $\bar{\mathfrak{B}}$ is the image of \mathfrak{B} in \mathfrak{A} and $\bar{\mathfrak{B}}$ is faithfully represented on \mathfrak{Q} , \mathfrak{Q} being a non abelian special 2-subgroup of \mathfrak{A} .
- (c) $\bar{\mathfrak{B}}$ contains a subgroup $\bar{\mathfrak{B}}_0$ of order 3 which centralizes a hyperplane of \mathfrak{W} .

Then $\bar{\mathfrak{B}}$ centralizes \mathfrak{Q}' .

Proof. Let $\mathfrak{Q}_0 = [\bar{\mathfrak{B}}_0, \mathfrak{Q}]$. Thus, \mathfrak{Q}_0 is a quaternion group, and $\mathfrak{W} = \mathfrak{W}_0 \times \mathfrak{W}_1$, where $\mathfrak{W}_0 = [\mathfrak{Q}_0, \mathfrak{W}]$ is of order 9 and $\mathfrak{W}_1 = C_{\mathfrak{W}}(\mathfrak{Q}_0)$. Since $|C(\mathfrak{W})|$ is odd, some involution I of $N(\mathfrak{W})$ maps to the involution of \mathfrak{Q}_0 . Let $\bar{\mathfrak{B}}_1$ be the normal closure of $\bar{\mathfrak{B}}_0$ in $\bar{\mathfrak{B}}$. Thus, $\bar{\mathfrak{B}}_1$ centralizes \mathfrak{Q}' . Let $\mathfrak{W}_2 = C_{\mathfrak{W}}(\bar{\mathfrak{B}}_1)$ so that \mathfrak{Q}' is faithfully represented on \mathfrak{W}_2 . Suppose $\bar{\mathfrak{B}}$ does not centralize \mathfrak{Q}' . Then by Lemma 4.4 of [17], there is an elementary subgroup \mathfrak{W}^* of \mathfrak{W}_2 which is of order 27, normal in \mathfrak{B} and with $|\mathfrak{B}:C_{\mathfrak{B}}(\mathfrak{W}^*)| = 3$. Since $\bar{\mathfrak{B}}_0$ centralizes \mathfrak{W}^* , it follows that $\mathfrak{W}^* \cap \mathfrak{W}_1$ is noncyclic. Let \mathfrak{B} be a subgroup of $\mathfrak{W}^* \cap \mathfrak{W}_1$ of order 9. With \mathfrak{W}^* in the role of \mathfrak{C} in Lemma 7.6(d), we conclude that $\mathfrak{B} \in \mathcal{E}(3)$. But now $C(I)$ contains an element of $\mathcal{Z}(2)$ and also contains \mathfrak{B} , against Lemma 7.4. The proof is complete.

LEMMA 7.8. *Suppose that \mathfrak{B} is of exponent 3, order 81 and that $|\mathbf{Z}(\mathfrak{B})| = 9$. Then $N(\mathfrak{B})$ is the unique element of $\mathcal{MS}(\mathfrak{G})$ which contains \mathfrak{B} .*

Proof. Suppose false. Let \mathfrak{S} be a solvable subgroup of \mathfrak{G} which

contains \mathfrak{P} and is minimal subject to $\mathfrak{P} \triangleleft \mathfrak{G}$. Let $\mathfrak{P}_0 = O_3(\mathfrak{G})$. Since $Z(\mathfrak{P}) \subset \mathfrak{P}_0 \subset \mathfrak{P}$, it follows that \mathfrak{P}_0 is abelian of order 27. Since \mathfrak{G} is not 3-closed, it follows that $\mathfrak{G} = \mathfrak{P}\Omega$ where Ω is a quaternion group.

Let $\mathfrak{P}_0 = \mathfrak{P}_1 \times \mathfrak{P}_2$ where $\mathfrak{P}_1 = C_{\mathfrak{P}_0}(\Omega)$, $\mathfrak{P}_2 = [\mathfrak{P}_0, \Omega]$. Thus, $|\mathfrak{P}_i| = 3^i, i = 1, 2$. Let $\tilde{\mathfrak{P}}_1 = \mathfrak{P}_2 \cap Z(\mathfrak{P})$. Thus, $Z(\mathfrak{P}) = \mathfrak{P}_1 \times \tilde{\mathfrak{P}}_1$ and $\mathfrak{P}' = \tilde{\mathfrak{P}}_1$. Let $\Omega' = \langle I \rangle \subseteq N(\mathfrak{P})$. Write $N(\mathfrak{P}) = \mathfrak{P}\mathfrak{R}$ where \mathfrak{R} is a complement to \mathfrak{P} in $N(\mathfrak{P})$ which contains I . Since \mathfrak{R} normalizes $\tilde{\mathfrak{P}}_1$, it follows that $A_{\mathfrak{G}}(Z(\mathfrak{P}))$ is abelian. Hence, $Z(\mathfrak{P}) \cap C(I) \triangleleft N(\mathfrak{P})$. Since $Z(\mathfrak{P}) \cap C(I) = \tilde{\mathfrak{P}}_1$, we get that $N(\mathfrak{P}) \subseteq N(\mathfrak{P}_1)$. Since $I \in N(\mathfrak{P})$, it follows that \mathfrak{P}_1 may be characterized as the only subgroup of $Z(\mathfrak{P})$ of order 3 which is normal in $N(\mathfrak{P})$ and is not contained in \mathfrak{P}' .

Let \mathfrak{R} be any solvable subgroup of \mathfrak{G} which contains \mathfrak{P} . We will show that $\mathfrak{R} \subseteq N(\mathfrak{P}_1)$. We may assume that $\mathfrak{P} \triangleleft \mathfrak{R}$. Let $\tilde{\mathfrak{P}}_0 = O_3(\mathfrak{R}) \supset Z(\mathfrak{P})$. Thus, $\tilde{\mathfrak{P}}_0$ is abelian of order 27. By our characterization of \mathfrak{P}_1 , it follows that $\mathfrak{P}_1 \triangleleft \mathfrak{R}$, that is, $\mathfrak{R} \subseteq N(\mathfrak{P}_1)$.

Set $\mathfrak{M} = N(\mathfrak{P}_1)$, so that \mathfrak{M} is the unique element of $\mathcal{MS}(\mathfrak{G})$ which contains \mathfrak{P} . Let \mathfrak{A} be any elementary subgroup of \mathfrak{P} of order 27. Then $\mathfrak{P} \subseteq N(\mathfrak{A})$, so $N(\mathfrak{A}) \subseteq \mathfrak{M}$. Now let A be any element of \mathfrak{P}^* . We will show that $C(A) \subseteq \mathfrak{M}$. This is clear if $A \in Z(\mathfrak{P})$. Suppose $A \notin Z(\mathfrak{P})$. Then $C_{\mathfrak{P}}(A) = \mathfrak{A}$ is of order 27 and is abelian. Hence, $N(\mathfrak{A}) \subseteq \mathfrak{M}$. This implies that some S_3 -subgroup of $C(A)$ is contained in \mathfrak{M} . If $C(A)$ contains a S_3 -subgroup of \mathfrak{G} , then $C(A) \subseteq \mathfrak{M}$, by uniqueness of \mathfrak{M} . So suppose that \mathfrak{A} is a S_3 -subgroup of $C(A)$. Then since $\mathfrak{N}(\mathfrak{A})$ is trivial, we get that $\mathfrak{A} \triangleleft C(A)$, so in any case, $C(A) \subseteq \mathfrak{M}$.

Let \mathfrak{G} be any non identity subgroup of \mathfrak{P} . We will show that $N(\mathfrak{G}) \subseteq \mathfrak{M}$. If $|\mathfrak{G}'| = 3$, it suffices to show that $N(\mathfrak{G}') \subseteq \mathfrak{M}$. If $|\mathfrak{G}'| \neq 3$, then \mathfrak{G} is abelian, since $|\mathfrak{P}'| = 3$. This, we may assume that \mathfrak{G} is abelian. By the preceding paragraph, $C(\mathfrak{G}) \subseteq \mathfrak{M}$. Let \mathfrak{G}^* be a S_3 -subgroup of $C(\mathfrak{G})$. Then $N(\mathfrak{G}) = C(\mathfrak{G}) \cdot (N(\mathfrak{G}) \cap N(\mathfrak{G}^*))$, so it suffices to show that $N(\mathfrak{G}^*) \subseteq \mathfrak{M}$. But $|\mathfrak{G}^*| \geq 27$, so $N(\mathfrak{G}^*) \subseteq \mathfrak{M}$.

It is a consequence of the preceding results, that if \mathfrak{H} is a solvable subgroup of \mathfrak{G} such that $\mathfrak{H} \cap \mathfrak{P}$ is noncyclic, then $\mathfrak{H} \subseteq \mathfrak{M}$.

Let $\mathfrak{B} = \mathfrak{P} \cap N(\Omega')$ so that \mathfrak{B} is noncyclic. Hence, $N(\Omega') \subseteq \mathfrak{M}$. This is not the case since $N(\Omega')$ contains an element of $\mathcal{Z}(2)$, while \mathfrak{M} contains an element of $\mathcal{Z}(3)$.

8. A characterization of $E_2(3)$.

THEOREM 8.1. $E_2(3)$ is the only simple group \mathfrak{G} with the following properties:

- (i) 1 is the only 3-signalizer of \mathfrak{G} .
- (ii) The center of a S_3 -subgroup of \mathfrak{G} is noncyclic.
- (iii) The normalizer of every nonidentity 3-subgroup of \mathfrak{G} is solvable.

- (iv) *The centralizer of every involution of \mathfrak{G} is solvable.*
- (v) *S_2 -subgroup of \mathfrak{G} contain normal elementary subgroups of order 8.*
- (vi) *If \mathfrak{X} is a S_2 -subgroup of \mathfrak{G} and $\mathfrak{A} \in \mathcal{L}_{\mathfrak{G}_3}(\mathfrak{X})$, then $\mathfrak{N}(\mathfrak{A})$ is trivial.*
- (vii) *$2 \sim 3$.*

The proof of Theorem 8.1 is elaborate. I am indebted to J. Tits for helpful discussion.

We first derive some properties of $E_2(q)$. We use the notation and calculations of Ree [30]. In addition, we let $\mathfrak{B} = \mathfrak{U}\mathfrak{S}$, $\mathfrak{N} = \langle \mathfrak{S}, \omega_a, \omega_b \rangle$. F_q is the field of $q = p^n$ elements, and if $x \in F_q$, then $tr(x) = tr_{F_q/F_p}(x) = \sum x^\sigma$, σ ranging over all the automorphisms of F_q . If $r \in \Sigma$, then $\mathfrak{X}_r = \langle x_r(t) \mid t \in F_q \rangle$.

We need the usual sort of omnibus lemma.

LEMMA 8.1. *Let $\mathfrak{U}, \mathfrak{B}, \mathfrak{S}, \mathfrak{N}$ denote the subgroups of $E_2(q)$ given above.*

(i) *$\mathfrak{B}_0 = \langle \omega_b^2 \omega_a, \omega_a^2 \omega_b \rangle$ is a dihedral group of order 12 and is a complement to \mathfrak{S} in \mathfrak{N} .*

(ii) *\mathfrak{S} is the direct product of two cyclic groups of order $q - 1$, with generators $H_1 = h(\chi_{a,z}), H_2 = h(\chi_{b,z})$. Here z is a generator for F_q^* . If $W_1 = \omega_b^2 \omega_a, W_2 = \omega_a^2 \omega_b$, then*

$$\begin{aligned} W_1^{-1}H_1W_1 &= H_1^{-1}, & W_1^{-1}H_2W_1 &= H_1H_2, \\ W_2^{-1}H_1W_2 &= H_1H_2^3, & W_2^{-1}H_2W_2 &= H_2^{-1}, \end{aligned}$$

(iii) *If q is a power of 3 and ν is a nonsquare in F_q , then*

$$\{x_{3a+2b}(1), x_{2a+b}(1), x_{3a+2b}(1)x_{2a+b}(1), x_{a+b}(1)x_{3a+b}(1), x_{a+b}(1)x_{3a+b}(\nu)\}$$

is a set of representatives for the conjugacy classes of $E_2(q)$ of order 3. If $c \in F_q$ satisfies $tr(c) = 1$, then $\{x_a(1)x_b(1)x_{3a+b}(ec), e = 0, 1, -1\}$ is a set of representatives for the conjugacy classes of elements of $E_2(q)$ of order 9. \mathfrak{U} is of exponent 9.

(iv) *Assume that q is odd.*

(a) *Let $\mathfrak{B} = C_{\mathfrak{B}}(\omega_a^2), \mathfrak{N} = C_{\mathfrak{N}}(\omega_a^2)$. Let $\mathfrak{C} = C_{E_2(q)}(\omega_a^2)$. Then $\mathfrak{C} = \mathfrak{B}\mathfrak{N}\mathfrak{B}$. \mathfrak{C} contains a subgroup $\mathfrak{C}_0 = \mathfrak{C}_1\mathfrak{C}_2$, where $\mathfrak{C}_i \cong SL(2, q)$, $i = 1, 2, \mathfrak{C}_1 \cap \mathfrak{C}_2 = \mathbf{Z}(\mathfrak{C}_i) = \langle \omega_a^2 \rangle, \mathfrak{C}_1$ and \mathfrak{C}_2 commute elementwise and $|\mathfrak{C} : \mathfrak{C}_0| = 2$. Furthermore, $\mathfrak{C}_i \triangleleft \mathfrak{C}, i = 1, 2$.*

(b) *For $i = 1, 2$, let α_i be the isomorphism from \mathfrak{C}_i to $SL(2, q)$ induced by $x_{r_i}(t) \rightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, z_{-r_i}(t) \rightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, where $r_1 = a, r_2 = 3a + 2b$. Each element X in $\mathfrak{C} - \mathfrak{C}_0$ induces an automorphism $\varphi_X^{(i)}$ of \mathfrak{C}_i such that $\alpha_i \varphi_X^{(i)} \alpha_i^{-1}$ coincides with the automorphism of $SL(2, q)$*

induced by an element of $GL(2, q)$ whose determinant is a nonsquare.

(c) There are involutions in $\mathfrak{C} - \mathfrak{C}_0$. If X is an involution in $\mathfrak{C} - \mathfrak{C}_0$, and $q \equiv \varepsilon \pmod{4}$, $\varepsilon = \pm 1$, then $C(X) \cap \mathfrak{C}_0$ has order $2(q + \varepsilon)^2$ and

$$\begin{aligned} C(X) \cap \mathfrak{C}_0 &\cong gp\langle w, x, y, z \mid x^{(q+\varepsilon)/2} \\ &= y^{(q+\varepsilon)/2} = w, w^2 = 1, xy = yx, z^{-1}xz \\ &= x^{-1}, z^{-1}yz = y^{-1}, z^2 = 1 \rangle. \end{aligned}$$

(v) If q is odd, then $i(E_2(q)) = 1$.

Proof. The Weyl group of G_2 is dihedral of order 12, so $w_a w_b$ is of order 6. By (1.8) of [30], $(\omega_a \omega_b)^6 = h(\chi)$, for some $\chi \in X$. We show that $\chi = 1$. It suffices to show that $\chi(a) = \chi(b) = 1$, that is, $\eta_a = \eta_b = 1$. This follows readily from table (3.4) of [30]. Since $\omega_a^{-1} \omega_b^2 \omega_a = \omega_b^2 \omega_a^2$, and $\omega_b^{-1} \omega_a^2 \omega_b = \omega_a^2 \omega_b^2$, the elements $\omega_b^2 \omega_a$ and $\omega_a^2 \omega_b$ are involutions. We have $(\omega_b^2 \omega_a)(\omega_a^2 \omega_b) = \omega_b^2 \omega_a^{-1} \omega_b \sim \omega_b^{-1} \omega_a^{-1} = (\omega_a \omega_b)^{-1}$, proving (i).

It is convenient for calculations to use the following character table:

	a	b
$\chi_{a,z}$	z^2	z^{-3}
$\chi_{b,z}$	z^{-1}	z^2

To determine this character table, we need to compute the values $u(r)$, u , $r \in \Sigma$ (see [30], p. 433). The relevant values of $u(r)$ are given as follows:

r	a	b
u	2	-1
a	-3	2

Using this table, we compute the values $w_r(s)$, as follows:

s	a	b
r	$-a$	$3a + b$
a	$a + b$	$-b$

(Using the geometric interpretation of w_r , we can read these results directly from Figure 1 of [30].)

We next compute $w_r(\chi)$ for $r = a, b$ and $\chi = \chi_{a,z}, \chi_{b,z}$. For example,

$[w_a(\chi_{a,z})](a) = \chi_{a,z}(w_a(a)) = \chi_{a,z}(-a) = \chi_{a,z}(a)^{-1} = z^{-2}$. Continuing in this fashion, we get the following table of values:

	a	b
$w_a(\chi_{a,z})$	z^{-2}	z^3
$w_a(\chi_{b,z})$	z	z^{-1}
$w_b(\chi_{a,z})$	z^{-1}	z^3
$w_b(\chi_{b,z})$	z	z^{-2}

Referring back to the character table, we have

$$w_a(\chi_{a,z}) = \chi_{a,z}^{-1}, w_a(\chi_{b,z}) = \chi_{a,z}\chi_{b,z},$$

$$w_b(\chi_{a,z}) = \chi_{a,z}\chi_{b,z}^3, w_b(\chi_{b,z}) = \chi_{b,z}^{-1}.$$

The map from X to \mathfrak{S} induced by $\chi_{r,z} \rightarrow h(\chi_{r,z})$ is an isomorphism, since X and \mathfrak{S} have order $(q - 1)^2$. The previous information, together with (1.7) of [30] implies that (ii) holds.

Let $\mathfrak{U}_1 = \mathfrak{U} \cap \mathfrak{U}^{w_a}, \mathfrak{U}_2 = \mathfrak{U} \cap \mathfrak{U}^{w_b}$. By using (3.10) of [30] it is straightforward to verify that $\mathfrak{U}_1 \cup \mathfrak{U}_2$ is the set of elements of \mathfrak{U} of order 1 or 3. This then implies easily that every element of $E_2(q)$ of order 3 is conjugate to an element of $\mathfrak{U}_1 \cap \mathfrak{U}_2 = \langle \mathfrak{x}_{a+b}, \mathfrak{x}_{2a+b}, \mathfrak{x}_{3a+b}, \mathfrak{x}_{3a+2b} \rangle$. Since $\mathfrak{B} = N(\mathfrak{U})$, it follows from Lemma 14.3.1 of [21] that elements of $Z(\mathfrak{U})$ are conjugate in $E_2(q)$ only if they are conjugate in \mathfrak{B} . Since the action of \mathfrak{S} on $Z(\mathfrak{U}) = \langle \mathfrak{x}_{2a+b}, \mathfrak{x}_{3a+2b} \rangle$ is determined by (1.5) of [30] and our character table, it follows that any element of $E_2(q)^*$ which is conjugate to an element of $Z(\mathfrak{U})$ is conjugate to exactly one of $x_{2a+b}(1), x_{3a+2b}(1), x_{2a+b}(1)x_{3a+2b}(1)$. Furthermore, since the Weyl group permutes transitively the roots of a given length, and since $2a + b$ and $3a + 2b$ have different lengths, it follows that every element of the shape $x_r(t), r \in \Sigma$, is conjugate to an element of $Z(\mathfrak{U})$. Suppose $x \in \mathfrak{U}_1 \cap \mathfrak{U}_2, x = x_{a+b}(t_1)x_{2a+b}(t_2)x_{3a+b}(t_3)x_{3a+2b}(t_4)$, and that x is conjugate to no element of $Z(\mathfrak{U})$. Hence, either $t_1 \neq 0$ or $t_3 \neq 0$. Suppose $t_3 = 0$. Conjugation by $x_a(-t_1^{-1}t_2/2)$ enables us to assume that $t_2 = 0$. Conjugation by ω_a then yields that x is conjugate to an element of $Z(\mathfrak{U})$. Hence, $t_3 \neq 0$. Suppose $t_1 = 0$. Conjugation by $x_b(t_3^{-1}t_4)$ enables us to assume that $t_4 = 0$. Conjugation by ω_b yields that x is conjugate to an element of $Z(\mathfrak{U})$. Hence, $t_1 t_3 \neq 0$. Conjugation by $x_a(-t_1^{-1}t_2/2)x_b(t_3^{-1}t_4)$ enables us to assume that $t_2 = t_4 = 0$. Since $h(\chi_{a,z})x_{a+b}(t_1)h(\chi_{a,z})^{-1} = x_{a+b}(z^{-1}t_1)$, we may assume that $t_1 = 1$. Since $h(\chi_{a,z}\chi_{b,z})$ centralizes $x_{a+b}(1)$ and since $h(\chi_{a,z}\chi_{b,z})x_{3a+b}(t_3)h(\chi_{a,z}\chi_{b,z})^{-1} = x_{3a+b}(t_3z^2)$, we may assume that $t_3 = 1$ or ν . A direct calculation shows that the centralizer of $x_{a+b}(1)x_{3a+b}(u)$ does not contain a S_3 -subgroup of $E_2(q)$ for any u in F_q^* , and a further calculation shows that $x_{a+b}(1)x_{3a+b}(1)$ is not conjugate to $x_{a+b}(1)x_{3a+b}(\nu)$,

completing the proof of the first part of (iii).

If $tu \neq 0$, it is easy to verify that $x_a(t)x_b(u)x$ has order 9 for all x in $\mathfrak{U}_1 \cap \mathfrak{U}_2$ and that $(x_a(t)x_b(u))^3 = (x_a(t)x_b(u)x)^3$. A calculation shows that \mathfrak{S} permutes transitively the elements $x_a(t)x_b(u)$, $tu \in F_q^*$, so every element of $E_2(q)$ of order 9 is conjugate to an element of the shape $x_a(1)x_b(1)x$, with x in $\mathfrak{U}_1 \cap \mathfrak{U}_2$. Let $x = x_{a+b}(t_1)x_{2a+b}(t_2)x_{3a+b}(t_3)x_{3a+2b}(t_4)$. Conjugation by $x_a(u)$ enables us to assume that $t_3 = 0$. A further conjugation by $x_{a+b}(u_1)x_{3a+b}(u_2)$ enables us to assume that $t_2 = t_4 = 0$. Thus, it suffices to show that

$$x_a(1)x_b(1)x_{a+b}(u) \text{ is conjugate to } x_a(1)x_b(1)x_{a+b}(v)$$

if and only if $tr(u) = tr(v)$. If g conjugates the first element into the second then g centralizes $(x_a(1)x_b(1))^3$. A calculation shows that the centralizer of $(x_a(1)x_b(1))^3$ is \mathfrak{U} , and a further calculation completes the proof of (iii).

By a direct calculation, $\tilde{\mathfrak{B}} = \langle \mathfrak{X}_a, \mathfrak{X}_{3a+2b}, \mathfrak{S} \rangle$, $\tilde{\mathfrak{N}} = \langle \mathfrak{S}, \omega_a, (\omega_a \omega_b)^3 \rangle$. Suppose ω_a^2 centralizes $xh\omega x'$, $x \in \mathfrak{U}$, $h \in \mathfrak{S}$, $\omega \in \mathfrak{B}_0$, $x' \in \mathfrak{U}_w$, w being the image of ω in the Weyl group. Then the normal form implies that $x, x', h, \omega \in C(\omega_a^2)$, so the first assertion of (iv) is proved.

Let $\mathfrak{C}_1 = \langle \mathfrak{X}_a, \mathfrak{X}_{-a} \rangle$, $\mathfrak{C}_2 = \langle \mathfrak{X}_{3a+2b}, \mathfrak{X}_{-(3a+2b)} \rangle$, so that $\mathfrak{C}_1 \cong \mathfrak{C}_2 \cong SL(2, q)$. Clearly, \mathfrak{C}_1 and \mathfrak{C}_2 commute elementwise. Since $\chi_{3a+2b, -1} = \chi_{a, -1}$, it follows that $\mathfrak{C}_1 \cap \mathfrak{C}_2 = \langle \omega_a^2 \rangle$, so that $\mathfrak{C}_0 = \mathfrak{C}_1 \mathfrak{C}_2$ is the central product of \mathfrak{C}_1 and \mathfrak{C}_2 . Setting $\tilde{\mathfrak{U}} = \mathfrak{U} \cap \tilde{\mathfrak{B}}$, we have

$$|\tilde{\mathfrak{U}} \cap \tilde{\mathfrak{U}}^{\omega_a}| = |\tilde{\mathfrak{U}} \cap \tilde{\mathfrak{U}}^{\omega_a(\omega_a \omega_b)^3}| = q,$$

and $\tilde{\mathfrak{U}} \cap \tilde{\mathfrak{U}}^{(\omega_a \omega_b)^3} = 1$, it follows that $|\mathfrak{C}| = q^2(q-1)^2(1+2q+q^2)$. Hence,

$$(*) \quad |\mathfrak{C}: \mathfrak{C}_0| = |\mathfrak{S}: \mathfrak{S} \cap \mathfrak{C}_0| = 2.$$

Since \mathfrak{S} normalizes \mathfrak{X}_r for all r in Σ , (iv) (a) is proved.

We observe that by (1.5) of [30],

$$\begin{aligned} h(\chi_{b,z})x_a(t)h(\chi_{b,z})^{-1} &= x_a(z^{-1}t), \\ h(\chi_{b,z})x_{-a}(t)h(\chi_{b,z})^{-1} &= x_{-a}(zt). \end{aligned}$$

Hence, if $\eta = \varphi_{h(\chi_{b,z})}^{(1)}$ denotes the automorphism of \mathfrak{C}_1 induced by $h(\chi_{b,z})^{-1}$, then $\alpha_1 \eta \alpha_1^{-1}$ is the automorphism of $SL(2, q)$ induced by the map

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & z^{-1}t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ zt & 1 \end{pmatrix}.$$

This automorphism therefore coincides with the automorphism induced by $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$. A similar argument applies to \mathfrak{C}_2 . Since $\mathfrak{C} - \mathfrak{C}_0$ coincides with the coset $\mathfrak{C}_0 h(\chi_{b,z})$ whenever z is not a square of F_q^* , the proof of (iv)(b) is complete.

Let $K = (W_1W_2)^3$. By (i), K is an involution, and by (ii), K inverts \mathfrak{S} . Thus, $\mathfrak{S}K$ is a set of involutions in \mathfrak{C} . If $\mathfrak{S}K \subseteq \mathfrak{C}_0$, we get $\mathfrak{S} \subseteq \langle \mathfrak{S}K \rangle \subseteq \mathfrak{C}_0$, against (*). So $\mathfrak{C} - \mathfrak{C}_0$ contains an involution.

We will use (iv)(b) in the proof of (iv)(c). First, suppose $\varepsilon = -1$. In this case, -1 is not a square in F_q . Since \mathfrak{C}_1 and \mathfrak{C}_2 commute elementwise, we assume without loss of generality that for $i = 1, 2$, $\alpha_i\varphi_X^{(i)}\alpha_i^{-1}$ is the automorphism of $SL(2, q)$ induced by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, $\mathfrak{C}_i \cap \mathcal{C}(X)$ is cyclic of order $q - 1$. Since the commutator of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is $-I$, and since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ inverts $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ for all $x \in F_q^*$, (iv)(c) follows in this case.

Now suppose $\varepsilon = 1$. In this case, -1 is a square in F_q . Choose $a, b \in F_q$ such that $a^2 + b^2 = c$ is a nonsquare. We may assume that for $i = 1, 2$, $\alpha_i\varphi_X^{(i)}\alpha_i^{-1}$ is the automorphism of $SL(2, q)$ induced by $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. A short calculation shows that (iv)(c) holds.

Since $|\mathcal{C}(\omega_a^2)| = (q(q^2 - 1))^2$, it follows that $\mathcal{C}(\omega_a^2)$ contains a S_2 -subgroup of $E_2(q)$. Thus, to prove (v), it suffices to show that each involution X of \mathfrak{C} is conjugate to ω_a^2 in $E_2(q)$. Since $E_2(q)$ is simple, Lemma 5.38 (a)(i) implies that X is conjugate in $E_2(q)$ to an element of \mathfrak{C}_0 .

Thus, it suffices to show that all involutions of \mathfrak{C}_0 are conjugate in $E_2(q)$. Since \mathfrak{C}_i has just involution for $i = 1, 2$, it follows that every involution of \mathfrak{C}_0 different from ω_a^2 is of the shape I_1I_2 where $I_i \in \mathfrak{C}_i$ and $I_i^2 = \omega_a^2$. Since \mathfrak{C}_i has just 1 conjugacy class of elements of order 4, it follows that \mathfrak{C}_0 has two conjugacy classes of involutions.

Case 1. Every involution of \mathfrak{S} is in \mathfrak{C}_0 .

By (ii), all involutions of \mathfrak{S} are fused in \mathfrak{N} . By the preceding paragraph, (v) follows.

Case 2. J is an involution of $(\mathfrak{C} - \mathfrak{C}_0) \cap \mathfrak{S}$.

Set $I = \omega_a^2, K = (W_1W_2)^3$, so that K inverts \mathfrak{S} and so centralizes I and J . Let $\mathfrak{A} = \langle I, J, K \rangle$. By (ii), the involutions of \mathfrak{A} are fused in \mathfrak{N} as follows:

$$I \sim J \sim IJ, IK \sim JK \sim IJK.$$

It is clear that in \mathfrak{C} all the involutions of $\mathfrak{C} - \mathfrak{C}_0$ are conjugate. Let $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{C}_0$. Thus, \mathfrak{A}_0 is one of $\langle I, K \rangle, \langle I, JK \rangle$. Suppose $\mathfrak{A}_0 = \langle I, K \rangle$. Then, J, JI, JK, JIK are the involutions of $\mathfrak{A} - \mathfrak{A}_0$, so are all conjugate in \mathfrak{C} . Since $K \in \mathfrak{C}_0$, K and KI are conjugate in \mathfrak{C}_0 . Thus, all involutions of \mathfrak{A} are conjugate in $E_2(q)$, so (v) follows. Suppose $\mathfrak{A}_0 = \langle I, JK \rangle$. Then, J, JI, K, KI are the involutions of $\mathfrak{A} - \mathfrak{A}_0$, so are all conjugate in \mathfrak{C} . Again all involutions of \mathfrak{A} are conjugate in $E_2(q)$. The proof of (v) is complete.

LEMMA 8.2. (a) Suppose \mathfrak{G} is a finite group and \mathfrak{B} is a four-subgroup of \mathfrak{G} . Suppose also that whenever I and J are distinct involutions of \mathfrak{B} , I and IJ are conjugate in $C(J)$. Then $A(\mathfrak{B}) = \text{Aut}(\mathfrak{B})$.

(b) $A_{E_2(3)}(\mathfrak{B}) = \text{Aut}(\mathfrak{B})$ for every four-subgroup \mathfrak{B} of $E_2(3)$.

Proof. Choose X in $C(J)$ such that $X^{-1}IX = IJ$. Thus,

$$X \in N(V) \cap C(J).$$

Replacing the pair I, J by the pair J, I we choose Y in $C(I)$ with $Y^{-1}JY = IJ$. Then $\mathfrak{X} = \langle XY \rangle$ permutes I, J, IJ cyclically. Thus, $\langle X, Y \rangle$ maps onto $\text{Aut}(\mathfrak{B})$, proving (a).

Let \mathfrak{B} be a four-subgroup of $E_2(3)$ and let I, J be distinct involutions of \mathfrak{B} . We will produce X in $C(J)$ such that $X^{-1}IX = IJ$. We may assume that $J = \omega_a^2$, since $i(E_2(3)) = 1$. Since $O_2(C(\omega_a^2))$ is extra special, we are done in case $I \in O_2(C(\omega_a^2))$. If $I \notin O_2(C(\omega_a^2))$, then I induces an outer automorphism of both quaternion subgroups of $O_2(C(\omega_a^2))$, so again X is available. Now (b) follows from (a).

We omit the proof that $\mathfrak{C} = C_{E_2(3)}(\omega_a^2)$ has exactly $19 + 72$ involutions; namely, $O_2(\mathfrak{C})$ has exactly 19 involutions, while all involutions of $\mathfrak{C} - O_2(\mathfrak{C})$ are conjugate in \mathfrak{C} . Furthermore, it is straightforward to verify that \mathfrak{C} has exactly 3 conjugacy classes of elementary subgroups of order 8. Representatives $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ for these classes may be chosen so that if \mathfrak{X} denotes a fixed S_2 -subgroup of \mathfrak{C} , then $\mathfrak{C}_i \triangleleft \mathfrak{X}$, and $\mathfrak{C}_i \cong O_2(\mathfrak{C})$, $i = 1, 2$.

We argue that \mathfrak{C}_1 and \mathfrak{C}_2 are not conjugate in $E_2(3)$. Suppose $\mathfrak{C}_1^g = \mathfrak{C}_2$. Then \mathfrak{X}^g normalizes \mathfrak{C}_2 , as does \mathfrak{X} . Then $\mathfrak{X}^g = \mathfrak{X}^N$ for some N in $N(\mathfrak{C}_2)$. Hence, $GN^{-1} \in N(\mathfrak{X}) = \mathfrak{X}$, so $G \in \mathfrak{X}N \cong N(\mathfrak{C}_2)$. Since $\mathfrak{C}_1^g = \mathfrak{C}_2$, we get $\mathfrak{C}_1 = \mathfrak{C}_2$, a contradiction.

Set $\mathfrak{B} = \mathfrak{C}_1 \cap \mathfrak{C}_2$ so that \mathfrak{B} is a four-subgroup of \mathfrak{X} and $O_2(\mathfrak{X}) \cap C(\mathfrak{B}) = \mathfrak{C}_1\mathfrak{C}_2$ is the direct product of a group of order 2 and a dihedral group of order 8. Let $\mathfrak{D} = C_{E_2(3)}(\mathfrak{B}) = C_{\mathfrak{X}}(\mathfrak{B})$, a group of order 32. We omit the proof that \mathfrak{D} has exactly 4 elementary subgroups of order 8, among which are \mathfrak{C}_1 and \mathfrak{C}_2 . By Lemma 8.2 (b), $N(\mathfrak{B})$ has an element A of order 3 which permutes transitively the involutions of \mathfrak{B} . If A normalizes both \mathfrak{C}_1 and \mathfrak{C}_2 , then A normalizes the derived group of $\mathfrak{C}_1\mathfrak{C}_2$, that is, A normalizes $\langle \omega_a^2 \rangle$. Since this is not the case, we can choose i in $\{1, 2\}$ so that the orbit of \mathfrak{C}_i under $\langle A \rangle$ has 3 elements. Since \mathfrak{C}_1 and \mathfrak{C}_2 are in different orbits under $\langle A \rangle$, it follows that A normalizes \mathfrak{C}_j , where $\{i, j\} = \{1, 2\}$.

We omit the proof that $N(\mathfrak{C}_j) \cap \mathfrak{C}$ permutes transitively the involutions of $\mathfrak{C}_j - \langle \omega_a^2 \rangle$. Since A does not centralize ω_a^2 , it follows that $N(\mathfrak{C}_j)$ permutes transitively the involutions of \mathfrak{C}_j . Thus, $|N(\mathfrak{C}_j)| = 7 \cdot |N(\mathfrak{C}_j) \cap \mathfrak{C}| = 8 \cdot 24 \cdot 8$. Hence,

$$\mathfrak{A}_{E_2(3)}(\mathfrak{G}_j) = \text{Aut}(\mathfrak{G}_j).$$

We have proved (a) of the next lemma.

- LEMMA 8.3. (a) $E_2(3)$ is not an N -group.
 (b) $E_2(3)$ satisfies the hypotheses of Theorem 8.1.

Proof. It suffices to verify (b).

By Lemma 8.1 (iv), hypothesis (iv) of Theorem 8.1 is satisfied. By definition of \sim , so is hypothesis (vii), $C_{E_2(3)}(\omega_a^2)$ being the relevant solvable group. Hypothesis (ii) is clearly satisfied, since

$$Z(\mathfrak{U}) = \langle \mathfrak{x}_{2a+b}, \mathfrak{x}_{3a+2b} \rangle.$$

Clearly, 1 is the only 2-signalizer of $C(\omega_a^2)$, so if \mathfrak{X} is a S_2 -subgroup of $C(\omega_a^2)$ and \mathfrak{Y} is a nonidentity 2'-subgroup of $E_2(3)$ normalized by \mathfrak{X} , then ω_a^2 inverts \mathfrak{Y} , so \mathfrak{Y} is abelian. Furthermore, \mathfrak{Y} is a 3-group, as every $\{2, 3\}'$ -subgroup of $E_2(3)$ is cyclic. Since \mathfrak{Y} is a faithful \mathfrak{X} -module, $|\mathfrak{Y}| \geq 3^4$. Since \mathfrak{U} has no abelian subgroup of order 3^5 , it follows that \mathfrak{Y} is elementary of order 3^4 . It is straightforward to verify that every elementary subgroup of \mathfrak{U} of order 3^4 is conjugate to

$$\langle \mathfrak{x}_{2a+b}, \mathfrak{x}_{3a+b}, \mathfrak{x}_{a+b}, \mathfrak{x}_{3a+2b} \rangle;$$

the normalizer of this last group is \mathfrak{B} , so does not contain a S_2 -subgroup of $E_2(3)$. Thus, 1 is the only 2-signalizer of $E_2(3)$. It is trivial to verify that 1 is the only 3-signalizer of $E_2(3)$, so hypothesis (i) is verified.

Since $E_2(3)$ is of order $2^6 \cdot 3^6 \cdot 7 \cdot 13$, and since the centralizer of every nonidentity 3-element of $E_2(3)$ is a 2, 3-group, it is easy to check that hypothesis (iii) is satisfied. Since S_2 -subgroups of $E_2(3)$ are of order 64, and since (***) holds, hypothesis (v) is satisfied.

Suppose that $\mathfrak{A} \in \mathcal{S}\mathcal{C}\mathcal{N}_3(\mathfrak{B})$ for a S_2 -subgroup \mathfrak{B} of $E_2(3)$, and \mathfrak{B} is minimal nontrivial element of $\mathfrak{N}(\mathfrak{A})$. Then $\mathfrak{A}\mathfrak{B}$ is contained in the centralizer of an involution; say $\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{C} = C(\omega_a^2)$. But, by Lemma 8.1 (iv), \mathfrak{C} contains no nontrivial 2'-subgroup \mathfrak{B} for which $N_{\mathfrak{C}}(\mathfrak{B})$ contains an elementary subgroup of order 2^3 . This contradiction proves that $\mathfrak{N}(\mathfrak{A}) = \{1\}$, which is hypothesis (vi). The proof is complete.

The remaining results in this section are proved under the hypothesis that \mathfrak{G} satisfies the hypothesis of Theorem 8.1.

- LEMMA 8.4. (i) \mathfrak{G} satisfies Hypothesis 7.4.
 (ii) \mathfrak{G} satisfies Hypothesis 7.1 for $p = 2$ and for $p = 3$.

Proof. We first show that $\mathcal{S}\mathcal{C}\mathcal{N}_3(3) \neq \emptyset$. Suppose false. Let \mathfrak{B} be a S_3 -subgroup of \mathfrak{G} . Since $\mathcal{S}\mathcal{C}\mathcal{N}_3(\mathfrak{B}) = \emptyset$, it follows that

$\Omega_1(\mathfrak{B}) = \Omega_1(\mathbf{Z}(\mathfrak{B}))$ is of type (3, 3). This implies that every 3-solvable subgroup of \mathfrak{G} has 3-length at most 1. Since 1 is the only 3-signalizer of \mathfrak{G} , it follows that $\mathfrak{B} = C(\Omega_1(\mathbf{Z}(\mathfrak{B})))$. Hence, 1 is the only element in $\mathfrak{N}(\Omega_1(\mathbf{Z}(\mathfrak{B})); 3')$. Thus, if \mathfrak{R} is a 3-solvable subgroup of \mathfrak{G} and S_3 -subgroups of \mathfrak{R} are noncyclic, then \mathfrak{R} is 3-closed. This implies by definition of \sim that $A_{\mathfrak{G}}(\Omega_1(\mathbf{Z}(\mathfrak{B})))$ contains an abelian subgroup of type (4, 2) or an elementary subgroup of order 8. Neither of these possibilities holds in $\text{Aut}(\Omega_1(\mathbf{Z}(\mathfrak{B})))$. Hence, $\mathcal{S}_{\text{inv}_3}(3) \neq \emptyset$. We have shown that (i), (ii), (iii), (iv) of Hypothesis 7.4 hold. If $\mathfrak{A} \in \mathcal{S}_{\text{inv}_3}(2)$, then $\mathfrak{N}(\mathfrak{A})$ contains only 1 by Hypothesis (vi) of Theorem 8.1. Suppose $\mathfrak{A} \in \mathcal{S}_{\text{inv}_3}(3)$, and $\Omega \in \mathfrak{N}(\mathfrak{A})$, $\Omega \neq 1$, Ω minimal with these properties. Let \mathfrak{B} be a S_3 -subgroup of $N(\mathfrak{A})$. Since $\mathbf{Z}(\mathfrak{B})$ is noncyclic, we may choose Z in $C(\Omega) \cap \mathbf{Z}(\mathfrak{B})^\#$. It follows that $\Omega \subseteq O_3(C(Z))$ against Hypothesis (i) of Theorem 8.1. (i) is proved.

Hypothesis 7.1 follows from Hypothesis 7.4 since if $p = 2$ or 3 and $\mathfrak{B} \in \mathcal{Z}^*(p)$, then $C(\mathfrak{B})$ contains an element of $\mathcal{S}_{\text{inv}_3}(p)$.

In the remainder of this section, \mathfrak{B} denotes a S_3 -subgroup of \mathfrak{G} , and $\mathfrak{B} \in \mathcal{Z}(\mathfrak{B})$.

Let $\mathfrak{B}_i, 1 \leq i \leq 4$, be the subgroups of \mathfrak{B} of order 3. Let $\mathfrak{N}_i = N(\mathfrak{B}_i)$, let $\mathfrak{D}_i = \mathfrak{B}^{\mathfrak{N}_i}$ and let $\mathfrak{C}_i = C_{\mathfrak{N}_i}(\mathfrak{D}_i)$. Since $3 \in \pi_4$ and $\mathfrak{B} \subseteq \mathfrak{N}_i$, we have $O_3(\mathfrak{N}_i) = 1$. Hence, by Lemma 5.10, \mathfrak{D}_i is 3-reducible in \mathfrak{N}_i . Finally, let $\mathfrak{E}_i = \mathfrak{N}_i/\mathfrak{C}_i$. Thus, \mathfrak{E}_i may be identified with a subgroup of $\text{Aut}(\mathfrak{D}_i)$, $\mathfrak{E}_i \cong A_{\mathfrak{N}_i}(\mathfrak{D}_i)$, and as such \mathfrak{E}_i is a 3-solvable group with no nontrivial normal 3-subgroups. We let $\mathfrak{R}_i = O^3(\mathfrak{E}_i)$, so that \mathfrak{R}_i is that subgroup of \mathfrak{E}_i generated by the 3-elements of \mathfrak{E}_i .

The following lemma is important.

LEMMA 8.5. *Suppose for some $i, 1 \leq i \leq 4$, \mathfrak{R}_i contains an element of order 3 which centralizes a subgroup of \mathfrak{D}_i of index 3. Then*

- (a) $|\mathfrak{D}_i| = 27$.
- (b) $\mathfrak{R}_i \cong SL(2, 3)$.
- (c) $\mathfrak{D}_i = \mathfrak{B}_i \times \mathfrak{C}_i$, where $\mathfrak{C}_i \triangleleft \mathfrak{N}_i$.
- (d) \mathfrak{R}_i is faithfully and irreducibly represented on \mathfrak{C}_i .

Proof. Let \mathfrak{U} be the set of 3-elements of \mathfrak{N}_i which centralize some subgroup of index 3 in \mathfrak{D}_i . Since $\mathfrak{D}_i \triangleleft \mathfrak{N}_i$, \mathfrak{U} is an invariant subset of \mathfrak{N}_i . By hypothesis, $\mathfrak{U} - \mathfrak{C}_i \neq \emptyset$.

Let $\mathfrak{U}^* = \mathfrak{U} \cap \mathfrak{B}$, and let $\mathfrak{U}_1 = \langle \mathfrak{U} \mid \mathfrak{U} \in \mathfrak{U}^* \rangle$. For any subset \mathfrak{H} of \mathfrak{N}_i , let $\bar{\mathfrak{H}} = \mathfrak{H}\mathfrak{C}_i/\mathfrak{C}_i$. Since \mathfrak{E}_i is 3-reduced, so is \mathfrak{R}_i . Furthermore, if $U \in \mathfrak{U} - \mathfrak{C}_i$, then \bar{U} is an exceptional element in the sense of Hall-Higman [26, p. 10], or as we might say, an exceptional element, being the identity on a hyperplane of \mathfrak{D}_i . (In a perhaps more frequently used terminology, \bar{U} is a transvection.)

Let $\mathfrak{H} = O_3(\mathfrak{N}_i \text{ mod } \mathfrak{C}_i)$, so that $\bar{\mathfrak{U}}_1$ is faithfully represented on $\bar{\mathfrak{H}}$.

By (B), \bar{U}_1 centralizes some S_2 -subgroup of $\bar{\mathfrak{G}}$. Let $\bar{\mathfrak{K}} = [\bar{\mathfrak{G}}, \bar{U}_1]$. Since $\bar{\mathfrak{K}}$ is solvable, it follows that $\bar{\mathfrak{K}}$ is a $\bar{\mathfrak{G}}$ -invariant 2-group on which U_1 is faithfully represented, and that $\bar{\mathfrak{K}} = [\bar{\mathfrak{K}}, \bar{U}_1]$. By Lemma 5.17, $\bar{\mathfrak{K}}$ is special, since by (B), \bar{U}_1 centralizes every abelian \bar{U}_1 -invariant subgroup of $\bar{\mathfrak{K}}$.

It may now be verified that if $U \in U^* - \mathfrak{C}_i$ and $\mathfrak{B} = \langle U \rangle$, then $[\bar{\mathfrak{K}}, \mathfrak{B}]$ is a quaternion group and $[\bar{\mathfrak{K}}, \mathfrak{B}]$ centralizes a subgroup \mathfrak{F} of index 9 in \mathfrak{G}_i . Furthermore, $\mathfrak{D}_i = \mathfrak{C}_i \times \mathfrak{F}$, where $\mathfrak{C}_i = [\mathfrak{K}, \mathfrak{B}, \mathfrak{D}_i]$ is of order 9, and $\mathfrak{C}_{i0} = [\mathfrak{C}_i, \mathfrak{B}]$ is of order 3. Since \mathfrak{C}_i and \mathfrak{F} are U -invariant, and since U centralizes some hyperplane of \mathfrak{D}_i , it follows that $C_{\mathfrak{D}_i}(U) = \mathfrak{C}_{i0} \times \mathfrak{F}$.

Let $\bar{\mathfrak{P}}_1 = \bar{\mathfrak{P}} \cap C(\bar{\mathfrak{K}})$ and let $\bar{\mathfrak{X}} = \bar{\mathfrak{P}}\bar{\mathfrak{K}}$. Since $\bar{\mathfrak{X}}$ is faithfully represented on \mathfrak{D}_i , so is its subgroup $\bar{\mathfrak{P}}_1\bar{\mathfrak{K}} = \bar{\mathfrak{P}}_1 \times \bar{\mathfrak{K}}$. Hence, by Lemma 3.7 of [20], $\bar{\mathfrak{K}}$ is faithfully represented on $\bar{\mathfrak{D}}_i = C_{\mathfrak{D}_i}(\bar{\mathfrak{P}}_1)$. Since $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$ is faithfully represented on $\bar{\mathfrak{K}}$, it follows that $\bar{\mathfrak{X}}/\bar{\mathfrak{P}}_1$ is faithfully represented on $\bar{\mathfrak{D}}_i$. By Lemma 7.7, $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$ centralizes $\bar{\mathfrak{K}}'\bar{\mathfrak{P}}_1/\bar{\mathfrak{P}}_1$. Since $\bar{\mathfrak{K}}\bar{\mathfrak{P}}_1 = \bar{\mathfrak{K}} \times \bar{\mathfrak{P}}_1$, it follows that $\bar{\mathfrak{P}}$ centralizes $\bar{\mathfrak{K}}'$.

Since \bar{U}_1 is faithfully represented on $\bar{\mathfrak{K}}/\bar{\mathfrak{K}}'$, and since each element \bar{U}^* centralizes a subgroup of $\bar{\mathfrak{K}}/\bar{\mathfrak{K}}'$ of index 4, it is straightforward to verify that \bar{U}_1 is elementary. It then follows that every element of \bar{U}_1^* is exceptional (though we don't contend that every element of \bar{U}_1 centralizes a hyperplane of \mathfrak{D}_i).

The preceding paragraph, together with $[\bar{\mathfrak{K}}', \bar{\mathfrak{P}}] = 1$ and Corollary 2 of § 2.6 of [24] imply that $\bar{U}_1 \subseteq Z(\bar{\mathfrak{P}})$. Returning to \mathfrak{B} , we see that $[\bar{\mathfrak{K}}, \mathfrak{B}]$ is $\bar{\mathfrak{P}}$ -invariant. This in turn implies that \mathfrak{F} is \mathfrak{P} -invariant. If $|\mathfrak{F}| \geq 9$, then \mathfrak{F} contains an element of $\mathcal{Z}^*(3)$ and Lemma 7.4 is violated. Hence, $|\mathfrak{F}| < 9$. Since $\mathfrak{B}_i \subseteq \mathfrak{F}$, we see that (a) and (c) hold. By construction, (b) and (d) follow. The proof is complete.

J denotes the subset of $\{1, 2, 3, 4\}$ whose elements satisfy the hypothesis of Lemma 8.5.

LEMMA 8.6. *Let $i \in J$ and let \mathfrak{A} be a subgroup of \mathfrak{C}_i of order 3. Let $\mathfrak{N} = N(\mathfrak{A})$, let $\bar{\mathfrak{N}}$ be the normal closure of \mathfrak{B}_i in \mathfrak{N} and $\bar{\mathfrak{N}}_1$ be the normal closure of \mathfrak{D}_i in \mathfrak{N} . Then*

- (a) $[\bar{\mathfrak{N}}, \bar{\mathfrak{N}}_1] = 1$.
- (b) $[\bar{\mathfrak{N}}_1, \bar{\mathfrak{N}}_1] \subseteq \mathfrak{A}$.

Proof. Let $\mathfrak{A}^* = \mathfrak{A} \times \mathfrak{B}_i$. Since the subgroups of \mathfrak{C}_i of order 3 are permuted transitively in \mathfrak{N}_i , it follows that $C_{\mathfrak{N}_i}(\mathfrak{A}^*)$ contains a S_3 -subgroup \mathfrak{P}^* of \mathfrak{G} . Thus, \mathfrak{B}_i is contained in the center of a S_3 -subgroup of \mathfrak{N} , namely \mathfrak{P}^* . By Lemma 5.10, $\bar{\mathfrak{N}}$ is 3-reducible in \mathfrak{N} . Since $\mathfrak{P}^* \subseteq C_{\mathfrak{G}}(\mathfrak{A})$ and $3 \in \pi_4$, we have $O_3(C_{\mathfrak{G}}(\mathfrak{A})) = 1$, which implies that $O_3(C_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}) = 1$. Since $\mathfrak{D}_i/\mathfrak{A} \subseteq Z(\mathfrak{P}^*/\mathfrak{A})$, we conclude that

$$\mathfrak{D}_i/\mathfrak{A} \cong O_3(C_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}) \cong O_3(\mathfrak{N}/\mathfrak{A}),$$

and so $\mathfrak{D}_i \subseteq C_{\mathfrak{N}}(\tilde{\mathfrak{N}})$, by 3-reducibility of $\tilde{\mathfrak{N}}$ in \mathfrak{N} . Since $C_{\mathfrak{N}}(\tilde{\mathfrak{N}}) \triangleleft \mathfrak{N}$, we have $[\tilde{\mathfrak{N}}, \tilde{\mathfrak{N}}_i] = 1$. Since $\mathfrak{D}_i/\mathfrak{A} \subseteq Z(O_3(\mathfrak{N}/\mathfrak{A}))$, we also have $[\tilde{\mathfrak{N}}_i, \tilde{\mathfrak{N}}_i] \subseteq \mathfrak{A}$.

We now set $\mathfrak{D} = \langle \mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4 \rangle$. Since $\mathfrak{B} \subseteq Z(\mathfrak{B})$, it is clear that $\mathfrak{B} \subseteq Z(\mathfrak{D})$ and that $\mathfrak{D} \triangleleft \mathfrak{B}$.

HYPOTHESIS 8.1. (i)

$$\begin{aligned} \mathfrak{B} &\subseteq \mathfrak{M} \in \mathcal{MS}(\mathfrak{G}), \mathfrak{B} = \Omega_1(Z(\mathfrak{B}))^{\mathfrak{M}}, \\ \mathfrak{C} &= C_{\mathfrak{M}}(\mathfrak{B}), \mathfrak{B}^* = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{B}). \end{aligned}$$

(ii) $\mathfrak{B}^* \not\subseteq \mathfrak{C}$.

The long argument to follow is carried out under Hypothesis 8.1.

Choose G in \mathfrak{G} so that $\mathfrak{D}^G \subseteq \mathfrak{B}$ but $\mathfrak{D}^G \not\subseteq \mathfrak{C}$. The element G plays a passive but important role. If \mathfrak{H} is any subset of \mathfrak{G} , we set $\mathfrak{H}^{\circ} = \mathfrak{H}^G$, while if \mathfrak{H} is any subset of \mathfrak{M} , we set $\tilde{\mathfrak{H}} = \mathfrak{H}\mathfrak{C}/\mathfrak{C}$.

Let \mathfrak{R} be any subgroup of $O_3(\overline{\mathfrak{M}})$ which admits \mathfrak{D}° and is minimal subject to $[\overline{\mathfrak{D}}^{\circ}, \mathfrak{R}] \neq 1$. (Notice that \mathfrak{R} is available.) Let $N = N(\mathfrak{R}) = \{i \mid 1 \leq i \leq 4, \overline{\mathfrak{D}}_i \text{ does not centralize } C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)\}$. We argue that $N \neq \emptyset$. This is clear if $\overline{\mathfrak{B}}^{\circ}$ centralizes \mathfrak{R} , so we may assume that $[\overline{\mathfrak{B}}^{\circ}, \mathfrak{R}] \neq 1$. Since \mathfrak{B}° is noncyclic, it follows that $\mathfrak{R} = \langle \mathfrak{R} \cap C(\mathfrak{B}_i) \mid 1 \leq i \leq 4 \rangle$, so we can choose i such that $\overline{\mathfrak{D}}^{\circ}$ does not centralize $C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)$. Minimality of \mathfrak{R} guarantees that $\mathfrak{R} = C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)$. Thus, \mathfrak{B}° does not centralize $C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)$. Since $\mathfrak{B}^{\circ} \subseteq \mathfrak{D}_i^{\circ}$, we have $i \in N(\mathfrak{R})$. In the following discussion, \mathfrak{R} is a fixed subgroup of $O_3(\overline{\mathfrak{M}})$ which admits \mathfrak{D}° and is minimal subject to $[\overline{\mathfrak{D}}^{\circ}, \mathfrak{R}] \neq 1$, and j is a fixed element of $N(\mathfrak{R})$. As already observed, $\overline{\mathfrak{B}}_j$ centralizes \mathfrak{R} .

Let $\tilde{\mathfrak{Q}}$ be a \mathfrak{D}_j -subgroup of \mathfrak{R} minimal subject to $[\overline{\mathfrak{D}}_j, \tilde{\mathfrak{Q}}] \neq 1$. Let $\mathfrak{D}_0 = \ker(\mathfrak{D}_j \rightarrow \text{Aut}(\tilde{\mathfrak{Q}}))$, so that $|\mathfrak{D}_j : \mathfrak{D}_0| = 3$ and $\mathfrak{B}_j \subseteq \mathfrak{D}_0$.

Since $\tilde{\mathfrak{Q}}$ is faithfully represented on \mathfrak{B} , Lemma 3.7 of [18] implies that $\tilde{\mathfrak{Q}}$ faithfully represented on $C_{\mathfrak{B}}(\mathfrak{D}_0)$. Since \mathfrak{D}_j does not centralize $C_{\mathfrak{B}}(\mathfrak{D}_0)$, we may choose V in $\mathfrak{D}_{\mathfrak{B}}(\mathfrak{D}_0) - C(\mathfrak{D}_j)$. Then

$$V \in C(\mathfrak{B}_j) \subseteq N(\mathfrak{B}_j) = \mathfrak{R}_j.$$

Thus, GVG^{-1} is a 3-element of $\mathfrak{R}_j - \mathfrak{C}_j$ which centralizes a subgroup of \mathfrak{D}_j of index 3. By Lemma 8.5,

$$(8.1) \quad j \in J, |\mathfrak{D}_j| = 27, \dots$$

This implies that $|C_{\mathfrak{B}}(\mathfrak{D}_0) : C_{\mathfrak{B}}(\mathfrak{D}_j)| = 3$, which in turn implies that $\tilde{\mathfrak{Q}}$ is a quaternion group.

Since $\tilde{\mathfrak{Q}}$ is a quaternion group, the following assertions hold:

(a) \mathfrak{D} centralizes a S_2 -subgroup of $O_3(\overline{\mathfrak{M}})$.

(b) \mathfrak{D} centralizes every abelian subgroup of $O_3(\overline{\mathfrak{M}})$ which \mathfrak{D} normalizes.

(c) If $\overline{\mathfrak{F}}$ is the normal closure in \mathfrak{P} of \mathfrak{D} , then $[\overline{\mathfrak{F}}, O_3(\overline{\mathfrak{M}})] = \overline{\mathfrak{E}}$ is a special 2-group whose derived group is centralized by $\overline{\mathfrak{F}}$. Namely, if either (a) or (b) were false, we could find \mathfrak{R} such that \mathfrak{R} contains no quaternion group. Since this is not the case, (a) and (b) hold. Now (c) follows from Lemma 5.17, together with the solvability of $O_3(\overline{\mathfrak{M}})$. We retain the previous notation and continue the argument.

Let $\mathfrak{A} = [C_{\mathfrak{B}}(\mathfrak{D}_i), \mathfrak{D}_i]$. Thus, \mathfrak{A} is a subgroup of \mathfrak{E}_i of order 3.

Let $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$, where $\mathfrak{B}_0 = C_{\mathfrak{B}}(\tilde{\mathfrak{Q}})$, $\mathfrak{B}_1 = [\mathfrak{B}, \tilde{\mathfrak{Q}}]$. Since $\mathfrak{A} \subseteq \mathfrak{B}$, we have $\mathfrak{B} \subseteq N(\mathfrak{A}) = \mathfrak{A}$, so that $[\mathfrak{B}, \mathfrak{B}_i] \subseteq \mathfrak{B}_i^{\mathfrak{A}}$. By Lemma 8.6, $[\mathfrak{B}, \mathfrak{B}_i, \mathfrak{D}_i] = 1$. This implies that $\tilde{\mathfrak{Q}}$ centralizes $[\mathfrak{B}, \mathfrak{B}_i]$, which in turn implies that $[\mathfrak{B}, \mathfrak{B}_i] \subseteq \mathfrak{B}_0$. Hence, \mathfrak{B}_i centralizes \mathfrak{B}_1 . Hence, \mathfrak{D}_i centralizes $[\mathfrak{B}_1, \mathfrak{D}_i]$. As $\tilde{\mathfrak{Q}}$ normalizes $[\mathfrak{B}_1, \mathfrak{D}_i]$, it follows that $\tilde{\mathfrak{Q}}$ centralizes $[\mathfrak{B}_1, \mathfrak{D}_i]$. By definition of \mathfrak{B}_1 , we get $[\mathfrak{B}_1, \mathfrak{D}_i] = 1$. Hence, $[\mathfrak{B}_1, \mathfrak{D}_i] = \mathfrak{A}$ and $|\mathfrak{B}_1| = 9$.

Suppose $\overline{\mathfrak{F}}$ centralizes $\overline{\mathfrak{E}}$. Since $\tilde{\mathfrak{Q}} \subseteq \overline{\mathfrak{E}}$, it follows that $\overline{\mathfrak{F}}$ centralizes $\tilde{\mathfrak{Q}}$, a group of order 2. Hence, $\overline{\mathfrak{F}}$ normalizes $\mathfrak{B}_0 = C_{\mathfrak{B}}(\tilde{\mathfrak{Q}})$. Since the inverse image of $\tilde{\mathfrak{Q}}$ in \mathfrak{M} contains an involution, it follows that \mathfrak{B}_0 contains no element of $\mathcal{U}^*(3)$. But $\mathfrak{B}_0 \triangleleft \mathfrak{P}$, so the only possibility is that \mathfrak{B}_0 is cyclic. Since $Z(\mathfrak{P})$ is non cyclic, we get $|\mathfrak{B}_0| = 3$.

Suppose $\overline{\mathfrak{F}}$ does not centralize $\overline{\mathfrak{E}}$. Let $\overline{\mathfrak{E}}_1 = [\overline{\mathfrak{E}}, \overline{\mathfrak{F}}]$, and let \mathfrak{B} be a subgroup of \mathfrak{B} which admits $\overline{\mathfrak{F}}\overline{\mathfrak{E}}$ and is minimal subject to $[\overline{\mathfrak{E}}_1, \mathfrak{B}] \neq 1$. Since $\overline{\mathfrak{F}}$ centralizes $\overline{\mathfrak{E}}$, it follows that $\overline{\mathfrak{F}}$ centralizes \mathfrak{B} ; so \mathfrak{D} centralizes \mathfrak{B} . Hence, $\mathfrak{B} \subseteq \mathfrak{B}_0 \times \mathfrak{A}$. By Lemma 4.4 of [19], \mathfrak{B} contains a subgroup \mathfrak{B}_0 of order 27 such that $\mathfrak{B}_0 \triangleleft \mathfrak{P}$, $|\mathfrak{P} : C_{\mathfrak{P}}(\mathfrak{B}_0)| = 3$. Since $|\mathfrak{A}| = 3$, it follows that $\mathfrak{B}_0 \cap \mathfrak{A}$ is noncyclic. Let \mathfrak{B}_1 be a subgroup of $\mathfrak{B}_0 \cap \mathfrak{A}$ of order 9. Since $|\mathfrak{P}|$ is clearly larger than 3^4 , we conclude from Lemma 7.6 (d) that $\mathfrak{B}_1 \in \mathcal{E}(3)$. Let I be an involution in the inverse image of $\tilde{\mathfrak{Q}}$ in \mathfrak{M} ; then I centralizes \mathfrak{B}_0 , so centralizes \mathfrak{B}_1 . Hence, by Lemma 7.4, $C(I)$ is nonsolvable. This contradiction shows that $[\overline{\mathfrak{F}}, \overline{\mathfrak{E}}] = 1$. Hence, $|\mathfrak{B}_0| = 3$, an important equality.

Since $\mathfrak{M} \in \mathcal{MS}(G)$, it follows that $\mathfrak{M} = N(\mathfrak{B}_0)$, so that $\mathfrak{M} = \mathfrak{N}_i$ for some i , $1 \leq i \leq 4$. Thus, $i \in J$, $\mathfrak{B}_0 = \mathfrak{B}_i$, $\mathfrak{B}_1 = \mathfrak{E}_i$, $\mathfrak{B} = \mathfrak{D}_i$, $\mathfrak{C} = \mathfrak{E}_i$.

Let $\mathfrak{P}_0 = \mathfrak{P} \cap \mathfrak{E}_i$, $\mathfrak{N}_0 = N_{\mathfrak{N}_i}(\mathfrak{P}_0)$, so that $\mathfrak{N}_0\mathfrak{E}_i = \mathfrak{N}_i$. Let \mathfrak{D}_0 be a S_2 -subgroup of \mathfrak{N}_0 permutable with \mathfrak{P} . Let $\mathfrak{C} = \mathfrak{P}\mathfrak{D}_0 \cap C(\mathfrak{B}_i)$, and set $\mathfrak{D} = \mathfrak{C} \cap \mathfrak{D}_0$. Then $\mathfrak{C} = \mathfrak{P}\mathfrak{D}$.

Since $i \in J$, Lemma 8.5 implies that a $S_{2,3}$ -subgroup of $\mathfrak{N}_i/\mathfrak{E}_i$ is not 3-closed. Since $\mathfrak{P}\mathfrak{D}_0$ is not 3-closed, neither is \mathfrak{C} , since $|\mathfrak{P}\mathfrak{D}_0 : \mathfrak{C}| \leq 2$. Let I be an involution of \mathfrak{D} . Suppose $\mathfrak{D}_i \cap C(I) \supset \mathfrak{B}_i$. Then $\mathfrak{D}_i \cap C(I)$ contains a subgroup $\tilde{\mathfrak{D}}$ of order 9 with $\tilde{\mathfrak{D}} \supset \mathfrak{B}_i$. Since \mathfrak{C} permutes transitively the subgroups of \mathfrak{E}_i of order 3, it follows that $\tilde{\mathfrak{D}}$ is central

in some S_3 -subgroup of \mathfrak{C} , that is, I centralizes an element of $\mathcal{U}(3)$. This is not the case, since $C(I)$ contains an element of $\mathcal{U}(2)$. This contradiction forces $\mathfrak{D}_i \cap C(I) = \mathfrak{B}_i$ for all involutions I of \mathfrak{Q} . Since $\mathfrak{B} \not\triangleleft \mathfrak{C}$, \mathfrak{Q} is not cyclic. Thus, \mathfrak{Q} is a quaternion group. Also, $\mathfrak{F}_0 = \mathcal{O}_3(\mathfrak{C}) = \mathfrak{C} \cap \mathfrak{C}_i$ and $\mathfrak{C}/\mathfrak{F}_0 \cong SL(2, 3)$. In particular, $\mathfrak{B}_j \subseteq \mathfrak{F}_0$, while $\mathfrak{D}_j \not\subseteq \mathfrak{F}_0$.

Since $j \in J$, it follows that $[C_{\mathfrak{F}_0}(\mathfrak{B}_j), \mathfrak{D}_j] \subseteq \mathfrak{C}_j$. Since $\mathfrak{B}_j \subseteq \mathfrak{F}_0$ and $\mathfrak{D}_j \not\subseteq \mathfrak{F}_0$, it follows that $\mathfrak{C}_j \not\subseteq \mathfrak{F}_0$. Hence, $\mathfrak{C}_j \cap \mathfrak{F}_0 = \mathfrak{U}$. This implies that

$$(8.2) \quad [C_{\mathfrak{F}_0}(\mathfrak{B}_j), \mathfrak{D}_j] = \mathfrak{U}.$$

For any subset \mathfrak{I} of \mathfrak{C} , let $\bar{\mathfrak{I}} = \mathfrak{I}\mathfrak{C}_i/\mathfrak{C}_i$. It is important to show that

$$(8.3) \quad C_{\bar{\mathfrak{F}}_0}(\bar{\mathfrak{B}}_j) = C_{\mathfrak{F}_0}(\mathfrak{B}_j)/\mathfrak{C}_i.$$

Namely, suppose P in \mathfrak{F}_0 satisfies $[\mathfrak{B}_j, P] \subseteq \mathfrak{C}_i$. Now $\mathfrak{C}_i = \mathfrak{U} \times \mathfrak{U}^*$, where \mathfrak{U} and \mathfrak{U}^* are of order 3 and $\mathfrak{U} \subseteq \mathfrak{C}_j$. We may apply Lemma 8.6 to \mathfrak{U} . Since $P \in \mathfrak{U}$, we get $[\mathfrak{B}_j, P, \mathfrak{D}_j] = 1$. Hence, $[\mathfrak{B}_j, P] \subseteq \mathfrak{C}_i \cap C(\mathfrak{D}_j) = \mathfrak{U}$. Consider the group $\mathfrak{B}_j \times \mathfrak{U}$, which is normalized by the 3-element P . Since \mathfrak{N}_j permutes transitively the subgroups of \mathfrak{C}_j of order 3, it follows that $\mathfrak{B}_j \times \mathfrak{U}$ is in the center of some S_3 -subgroup of \mathfrak{C} . Hence, $A_{\mathfrak{G}}(\mathfrak{B}_j \times \mathfrak{U})$ is a 3'-group, so P centralizes $\mathfrak{B}_j \times \mathfrak{U}$. We have proved (8.3).

Since $\mathfrak{U} \subseteq \mathfrak{C}_i$, so also $\mathfrak{F}_0 \subseteq \mathfrak{U}$. Hence

$$(8.4) \quad [\mathfrak{F}_0, \mathfrak{D}_j, \mathfrak{D}_j] \subseteq \mathfrak{U},$$

by Lemma 8.6 (b) applied to \mathfrak{U} . We will use this fact several times.

We next show that

$$(8.5) \quad C_{\mathfrak{F}_0}(\mathfrak{Q}') = C_{\mathfrak{F}_0}(\mathfrak{Q}).$$

Since $\mathfrak{Q}\mathfrak{C}_i$ is a Frobenius group, it suffices to show that $C_{\bar{\mathfrak{F}}_0}(\mathfrak{Q}') = C_{\bar{\mathfrak{F}}_0}(\mathfrak{Q})$. Let \mathcal{C} be part of a chief series of \mathfrak{C} from \mathfrak{F}_0 to 1, one of whose terms is \mathfrak{C}_i . If \mathfrak{F} is a chief factor of \mathcal{C} , it suffices to show that if \mathfrak{Q}' centralizes \mathfrak{F} , so does \mathfrak{Q} . If this were not the case, then elements of $\mathfrak{D}_j - \mathfrak{F}_0$ would have minimal polynomial $(x - 1)^3$ on \mathfrak{F} , against (8.4). Thus, (8.5) holds.

Suppose \mathfrak{Q}' centralizes $\bar{\mathfrak{F}}_0$. Let $\mathfrak{F}_1 = C_{\mathfrak{F}_0}(\mathfrak{Q}')$. We get $\mathfrak{F}_0 = \mathfrak{F}_1\mathfrak{C}_i$, $\mathfrak{F}_1 \cap \mathfrak{C}_i = 1$. Since \mathfrak{C}_i is an irreducible \mathfrak{Q} -module, we have $\mathfrak{F}_0 = \mathfrak{F}_1 \times \mathfrak{C}_i$. If \mathfrak{F}_1 is not cyclic, then \mathfrak{Q}' centralizes an element of $\mathcal{U}^*(3)$, which is not the case, since \mathfrak{Q}' centralizes an element of $\mathcal{U}(2)$. Thus, \mathfrak{F}_1 is cyclic. Clearly, $\mathfrak{F}_1 \neq 1$, since $\mathfrak{B}_j \subseteq \mathfrak{F}_1$. If

$$|\mathfrak{F}_1| > 3, \text{ then } \mathcal{O}^1(\mathfrak{F}_1) \triangleleft \langle \mathfrak{N}_i, \mathfrak{N}_j \rangle,$$

while it is trivial that $\langle \mathfrak{N}_i, \mathfrak{N}_j \rangle$ is non solvable. Hence, $\mathfrak{F}_1 = \mathfrak{B}_j$. But then Lemma 7.8 is violated.

Let $\mathfrak{P}_2 = [\mathfrak{P}_0, \Omega]$. By the preceding paragraph,

$$(8.6) \quad \overline{\mathfrak{P}}_2 \neq 1 .$$

We will show that

$$(8.7) \quad \overline{\mathfrak{P}}_2 \text{ is of exponent 3 and class at most 2 .}$$

Let $\mathfrak{R}_1 = [\mathfrak{P}_2, \mathfrak{D}_j]$. Since $\mathfrak{P}_2 \subseteq \mathfrak{R}^*$, Lemma 8.6 (b) implies that $[\mathfrak{R}_1, \mathfrak{R}_1] \subseteq \mathfrak{R}^*$, so that $\overline{\mathfrak{R}}_1$ is abelian. Since \mathfrak{D}_j is elementary so is $\overline{\mathfrak{R}}_1$. Thus $\overline{\mathfrak{R}}_1$ is a normal elementary subgroup of $\overline{\mathfrak{P}}_2$. Let Q be an element of Ω of order 4, and set $\mathfrak{R}_2 = \mathfrak{R}_1^Q$. We argue that $\overline{\mathfrak{R}}_1 \overline{\mathfrak{R}}_2 = \overline{\mathfrak{P}}_2$. To see this, observe that since Ω' inverts $\mathfrak{P}_2/D(\mathfrak{P}_2)$, and since the minimal polynomial of each element of \mathfrak{D}_j on $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$ is a divisor of $(x - 1)^2$, it follows that $\mathfrak{R}_1, \mathfrak{R}_2$ map onto subspaces of $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$ which generate $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$, so our assertion follows. Since $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2$ are normal elementary subgroups of $\overline{\mathfrak{P}}_2$, (8.7) holds. Since we now have $D(\overline{\mathfrak{P}}_2) = [\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2] \subseteq \overline{\mathfrak{R}}_1$, and since $\overline{\mathfrak{D}}_j$ centralizes $\overline{\mathfrak{R}}_1$, it follows that

$$(8.8) \quad \Omega \text{ centralizes } D(\overline{\mathfrak{P}}_2) .$$

Since Ω has no fixed points on $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$, it follows from (8.4) that

$$(8.9) \quad \begin{array}{l} \Omega \text{ operates on } \overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2) \text{ as a multiple } d \text{ of the} \\ \text{faithful irreducible } \Omega\text{-representation .} \end{array}$$

In particular,

$$(8.10) \quad |\overline{\mathfrak{P}}_2 : D(\overline{\mathfrak{P}}_2)| = 3^{2d} .$$

Let B a generator for \mathfrak{P}_j , and for any element S of \mathfrak{S} , let $S]$ be the mapping of \mathfrak{P}_2 into itself which sends P to $[P, S]$. We may view $B]$ in more than one way. Since Ω centralizes $\mathfrak{P}_0/\mathfrak{P}_2$, we have $B = CU$, where $C \in C_{\mathfrak{P}_0}(\Omega)$ and $U \in \mathfrak{P}_2$. Since $U \in \mathfrak{P}_2$, $B]$ and $C]$ induce the same mapping from $\mathfrak{P}_2/D(\mathfrak{P}_2)$ to itself. In particular, $[\mathfrak{P}_2, B]D(\mathfrak{P}_2)$ admits Ω . By Lemma 8.6 applied to \mathfrak{R}^* , we have $[\mathfrak{P}_2, B, \mathfrak{D}_j] = 1$. This implies that Ω centralizes $[\mathfrak{P}_2, B]D(\mathfrak{P}_2)/D(\mathfrak{P}_2)$, so by construction of \mathfrak{P}_2 , we have

$$(8.11) \quad [\mathfrak{P}_2, B] \subseteq D(\mathfrak{P}_2) .$$

Hence, $[\mathfrak{P}_2, C] \subseteq D(\mathfrak{P}_2)$. Since C centralizes Ω and Ω centralizes $D(\overline{\mathfrak{P}}_2)$, we conclude that C centralizes $\overline{\mathfrak{P}}_2$, by the three subgroups lemma. Hence, B and U induce the same automorphism of $\overline{\mathfrak{P}}_2$.

By Lemma 8.6 applied to \mathfrak{R}^* , B centralizes the normal closure of \mathfrak{D}_j in \mathfrak{R}^* . Hence, $C_{\mathfrak{P}_2}(B) \cong [\mathfrak{P}_2, \mathfrak{D}_j]\mathfrak{C}_i\mathfrak{P}'_2$. We will show that

$$(8.12) \quad C_{\mathfrak{P}_2}(B) = [\mathfrak{P}_2, \mathfrak{D}_j]\mathfrak{C}_i\mathfrak{P}'_2 ,$$

$$(8.13) \quad |\overline{C_{\mathfrak{P}_2}(B)} : D(\overline{\mathfrak{P}}_2)| = 3^d .$$

Let \mathfrak{W}_1 be the set of fixed points of \mathfrak{D}_j on $\overline{\mathfrak{F}}_2/D(\overline{\mathfrak{F}}_2)$, let

$$\mathfrak{W}_2 = [\overline{\mathfrak{D}}_j, \overline{\mathfrak{F}}_2]D(\overline{\mathfrak{F}}_2)/D(\overline{\mathfrak{F}}_2), \text{ and let } W_3 = \overline{C_{\mathfrak{F}_2}(B)}/D(\overline{\mathfrak{F}}_2).$$

From (8.2) and (8.3), we get that $\mathfrak{W}_3 \subseteq \mathfrak{W}_1$. By Lemma 8.6 (a) applied to \mathfrak{N} , we get $\mathfrak{W}_2 \subseteq \mathfrak{W}_3$. By (8.10) and Lemma 5.2, it follows that $|\mathfrak{W}_1| \leq 3^d$. Using (8.10) once again, we get $|\mathfrak{W}_2| \geq 3^d$. Since $\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_1$, it follows that $\mathfrak{W}_1 = \mathfrak{W}_2 = \mathfrak{W}_3$ is of order 3^d . This yields (8.12) and (8.13).

Let $\mathfrak{B}^* = \mathfrak{B}_j^{\mathfrak{F}}$. Since \mathfrak{F} centralizes \mathfrak{U} , we have $\mathfrak{B}^* \subseteq \mathfrak{B}_j^{\mathfrak{U}}$. By Lemma 8.6, \mathfrak{B}^* and $\mathfrak{D}_j^{\mathfrak{F}}$ commute elementwise. Set $\mathfrak{F}_3 = [\mathfrak{F}_2, \mathfrak{B}^*]\mathfrak{C}_i \triangleleft \mathfrak{F}$. Since B centralizes $\overline{\mathfrak{F}}_2/D(\overline{\mathfrak{F}}_2)$, (8.8) implies that Ω normalizes \mathfrak{F}_3 . Thus, $\mathfrak{F}_3 \triangleleft \mathfrak{C}$. Since $\mathfrak{D}_j^{\mathfrak{F}}$ and \mathfrak{B}^* commute elementwise, $[\mathfrak{F}_2, \mathfrak{D}_j]$ centralizes \mathfrak{F}_3 . Hence, $[\mathfrak{F}_2, \mathfrak{D}_j]^q$ centralizes $\mathfrak{F}_3^q = \mathfrak{F}_3$, Q being an element of $\Omega - \Omega'$. Since $\mathfrak{F}_2 = [\mathfrak{F}_2, \mathfrak{D}_j][\mathfrak{F}_2, \mathfrak{D}_j]^q$, it follows that \mathfrak{F}_2 centralizes \mathfrak{F}_3 .

Let $\tilde{\mathfrak{F}}_3 = \mathfrak{F}_3 \cap C(\Omega)$. Thus, $\mathfrak{F}_3 = \tilde{\mathfrak{F}}_3 \times \mathfrak{C}_i$. Clearly, $N_{\mathfrak{C}}(\Omega)$ normalizes $\tilde{\mathfrak{F}}_3$; so does \mathfrak{F}_2 since \mathfrak{F}_2 centralizes \mathfrak{F}_3 . Since $\mathfrak{C} = \mathfrak{F}_2 N_{\mathfrak{C}}(\Omega)$, we have $\tilde{\mathfrak{F}}_3 \triangleleft \mathfrak{C}$. Since Ω contains an involution, no subgroup of $\tilde{\mathfrak{F}}_3$ is in $\mathcal{Z}^*(\mathfrak{F})$. Hence, $\tilde{\mathfrak{F}}_3$ is cyclic. Since \mathfrak{F}_3 is isomorphic to a subgroup of $\overline{\mathfrak{F}}_2$, it follows that $\tilde{\mathfrak{F}}_3$ is of order 1 or 3.

Suppose $[\mathfrak{F}_2, B] \subseteq \mathfrak{C}_i$. Then (8.3) forces $[\mathfrak{F}_2, B] = 1$. This violates (8.6), (8.10), (8.13). Thus, $\tilde{\mathfrak{F}}_3$ is of order 3 and $\mathfrak{F}_3 = [\mathfrak{F}_2, B]\mathfrak{C}_i$, and $[\mathfrak{F}_2, B]$ is of order 3. Now (8.10) and (8.13) yield that $d = 1$.

Suppose $D(\overline{\mathfrak{F}}_2) = 1$. Then by (8.11), B centralizes $\overline{\mathfrak{F}}_2$. This conflicts with (8.10) and (8.13). Hence, $D(\overline{\mathfrak{F}}_2) \neq 1$, so that

$$(8.14) \quad |\mathfrak{F}_2| = 3^5.$$

Since $\mathfrak{B}_i \tilde{\mathfrak{F}}_3$ is a normal subgroup of \mathfrak{C} centralized by Ω , we get $\mathfrak{B}_i = \tilde{\mathfrak{F}}_3$, as Ω centralizes no element of $\mathcal{Z}^*(3)$. Hence,

$$(8.15) \quad Z(\mathfrak{F}_2) = \mathfrak{D}_i.$$

Since \mathfrak{F}_2 is the normal closure of $[\mathfrak{F}_2, \mathfrak{D}_j]$ in \mathfrak{C} , and since $\mathfrak{D}_j^{\mathfrak{U}}$ is of exponent 3, it follows that \mathfrak{F}_2 is generated by elements of order 3. Since \mathfrak{F}_2 is of class 2, it follows that

$$(8.16) \quad \mathfrak{F}_2 \text{ is of exponent } 3.$$

Since $\mathfrak{B}_i \subset \mathfrak{F}_2$, the group $\mathfrak{F}_2/\mathfrak{B}_i$ is of order 3^4 and is inverted by the involution of Ω . Hence, $\mathfrak{F}_2 \subseteq \mathfrak{B}_i$. Since \mathfrak{F}_2 is non abelian it follows that

$$(8.17) \quad \mathfrak{F}_2 = \mathfrak{B}_i.$$

We next show that $B \in \mathfrak{F}_2$. Namely, $\mathfrak{B} = CU$, so that $[C, U] = [C, CU] = [C, B]$. As we have already seen, C centralizes $\overline{\mathfrak{F}}_2$, that is, $[C, U] \in \mathfrak{C}_i$. Since $[C, U] = [C, B]$, (8.3) implies that $[C, B] = 1$. Since

we have also shown that $[\mathfrak{F}_2, B]$ has order $3^d = 3$, it follows that U is not in $Z(\mathfrak{F}_2)$. Since $\mathfrak{F}_2/\mathfrak{F}'_2$ is an irreducible Ω -module, it follows that C centralizes \mathfrak{F}_2 , as C centralizes an element of \mathfrak{F}_2 (namely, U) which does not map into \mathfrak{F}'_2 . Since C and U commute, and since B and U have order 1 or 3, it follows that C has order 1 or 3. If $C \notin \mathfrak{F}_2$, then $\Omega_1(C_{\mathfrak{F}_0}(\mathfrak{F}_2)) \cap C(\Omega)$ is noncyclic, so that Ω centralizes an element of $\mathcal{U}^*(3)$. Since this is not the case, we conclude that

$$(8.18) \quad B \in \mathfrak{F}_2 .$$

We will next show that $\mathfrak{F}_0 = \mathfrak{F}_2$.

Since $B \in \mathfrak{F}_2$, $[\mathfrak{F}_0, B]$ is a subgroup of \mathfrak{F}_2 centralized by \mathfrak{D}_j . Suppose $[\mathfrak{F}_0, B] \not\subseteq \mathfrak{F}'_2 (= \mathfrak{F}_i)$. Let B_i be a generator for \mathfrak{F}_i , A a generator for \mathfrak{X} . Choose P in \mathfrak{F}_0 so that $[P, B] = B_i^a A^b$ with $b \neq 0$. Clearly, $a \neq 0$, since $A_{\mathfrak{G}}(\langle B, A \rangle)$ is a 3'-group. Since $[\mathfrak{F}_2, B] = \mathfrak{F}'_2 = \mathfrak{F}_i$, we may choose P_2 in \mathfrak{F}_2 so that $[P_2, B] = B_i^{-a}$. Then $[PP_2, B] = A^b$, which is impossible. Hence, $[\mathfrak{F}_0, B] = \mathfrak{F}'_2$. Hence, $[\mathfrak{F}_1, \mathfrak{F}_2] = 1$, by the three subgroups lemma. Here $\mathfrak{F}_1 = \mathfrak{F}_0 \cap C(\Omega)$. Hence, $\mathfrak{F}_1 \triangleleft \mathfrak{E}$, so \mathfrak{F}_1 is cyclic, as Ω centralizes no element of $\mathcal{U}^*(3)$. If $|\mathfrak{F}_1| > 3$, it is easy to verify that $\mathcal{O}^1(\mathfrak{F}_1) \triangleleft \langle \mathfrak{N}_i, \mathfrak{N}_j \rangle$ against the nonsolvability of $\langle \mathfrak{N}_i, \mathfrak{N}_j \rangle$. Hence, \mathfrak{F}_1 is of order 3, so that $\mathfrak{F}_1 = \mathfrak{F}_i$. Hence,

$$(8.19) \quad \mathfrak{F}_0 = \mathfrak{F}_2 \text{ is of order } 3^5 ,$$

$$(8.20) \quad \mathfrak{F} \text{ is of order } 3^6 .$$

With the preceding information at our disposal, we turn our attention to $\mathfrak{M} = \mathfrak{N}_i$ once again. Let $\tilde{\mathfrak{E}}$ be a $S_{(2,3)'}-$ subgroup of \mathfrak{M} permutable with \mathfrak{F} . Then $\tilde{\mathfrak{E}}$ centralizes \mathfrak{D}_i , for otherwise $2^3 \cdot 3 \cdot 13$ divides $A_{\mathfrak{M}}(\mathfrak{D}_i)$, forcing nonsolvability of $A_{\mathfrak{M}}(\mathfrak{D}_i)$. Since $|O_2(\mathfrak{M}) : \mathfrak{D}_i| \leq 9$, $\tilde{\mathfrak{E}}$ also centralizes $O_3(\mathfrak{M})/\mathfrak{D}_i$. Hence, $\tilde{\mathfrak{E}}$ centralizes $O_3(\mathfrak{M})$, or equivalently,

$$(8.21) \quad \mathfrak{M} \text{ is a } 2, 3\text{-group} .$$

Let \mathfrak{T} be a S_2 -subgroup of \mathfrak{N}_i containing Ω . Since $\mathfrak{E}_i \in \mathcal{U}^*(\mathfrak{F})$, no element of $\mathfrak{T}^\#$ centralizes \mathfrak{E}_i . Thus,

$$(8.22) \quad \mathfrak{F}_0 = C(\mathfrak{E}_i) .$$

Since $\mathfrak{F}_i \subseteq \mathfrak{F}$, it follows that $C(\mathfrak{F}) \subseteq \mathfrak{N}_i = \mathfrak{M}$. By Lemma 7.4, $|C(\mathfrak{F})|$ is odd. From (8.21) we conclude that

$$(8.23) \quad \mathfrak{F} = C(\mathfrak{F}) .$$

By construction, $\mathfrak{D}_i \subseteq \mathfrak{F}_0$. Suppose j_0 is an index such that $\mathfrak{D}_{j_0} \not\subseteq \mathfrak{F}_0$. Then $[\mathfrak{D}_i, \mathfrak{D}_{j_0}] \neq 1$. In this case, two applications of Lemma 8.5 imply that $|\mathfrak{D}_i| = |\mathfrak{D}_{j_0}| = 27$ and that $[\mathfrak{F}, \mathfrak{D}_{j_0}]$ is of order 3. Hence, $[\mathfrak{F}, \mathfrak{D}_{j_0}] =$

$[\mathfrak{D}_i, \mathfrak{D}_{j_0}]$ so that \mathfrak{D}_{j_0} centralizes $\mathfrak{F}_0/\mathfrak{D}_i$. This contradicts (8.6) and (8.19). Hence, no such j_0 exists, that is,

$$(8.24) \quad \mathfrak{D} \subseteq \mathfrak{F}_0 .$$

Our previous information shows that $Z(\mathfrak{F}) = \mathfrak{B}$. Hence, $N(\mathfrak{F})$ normalizes \mathfrak{B} , so permutes the groups $\mathfrak{B}^{N(\mathfrak{B}_k)}$ among themselves, $1 \leq k \leq 4$. By definition of \mathfrak{D} , we get

$$(8.25) \quad N(\mathfrak{F}) \subseteq N(\mathfrak{D}) .$$

Suppose $\mathfrak{D} \triangleleft \mathfrak{F}$. Since $\mathfrak{D} \triangleleft \mathfrak{F}$, it follows that $\mathfrak{D} \triangleleft \langle \mathfrak{F}, \mathfrak{F} \rangle$. We can choose N in $N(\mathfrak{D})$ so that $\mathfrak{F}^N = \mathfrak{F}$. Hence, $\mathfrak{D}^N = \mathfrak{D}$ and $\mathfrak{F}^{GN} = \mathfrak{F}$. Let $H = GN$. Then $\mathfrak{D} = \mathfrak{D}^H$ and $H \in N(\mathfrak{F})$. By (8.25), we get $\mathfrak{D} = \mathfrak{D}^H = \mathfrak{D}$. This conflicts with (8.24), since by construction $\mathfrak{D} \not\subseteq \mathfrak{F}_0$. Thus,

$$(8.26) \quad \mathfrak{D} \not\triangleleft \mathfrak{F} .$$

Suppose \mathfrak{F}^* is a S_3 -subgroup of \mathfrak{N}_i and that $\mathfrak{F}^* \subseteq \mathfrak{N}_i$. Thus, $Z(\mathfrak{F}^*) \subseteq \mathfrak{D}_i \cap \mathfrak{D}_j = \mathfrak{A}$. This is impossible since $|\mathfrak{A}| = 3$, $|Z(\mathfrak{F}^*)| = 9$. We conclude that

$$(8.27) \quad \mathfrak{N}_i \cap \mathfrak{N}_j \text{ contains no } S_3\text{-subgroup of } \mathfrak{G} .$$

Since $\mathfrak{D} \not\triangleleft \mathfrak{F}$, (8.20) implies that $|\mathfrak{D}| \leq 3^4$. Suppose $|\mathfrak{D}| \leq 3^3$. Then $\mathfrak{D} \supseteq \mathfrak{D}_i$ implies $|\mathfrak{D}| = 3^3$ and $\mathfrak{D} = \mathfrak{D}_i$. Thus, for each k , $1 \leq k \leq 4$, we have $\mathfrak{B} \subseteq \mathfrak{D}_k \subseteq \mathfrak{D}_i$. If $\mathfrak{B} = \mathfrak{D}_k$, then \mathfrak{N}_k normalizes $C(\mathfrak{B})$. By (8.23), we have $\mathfrak{F} \triangleleft \mathfrak{N}_k$, so by (8.25), we have $\mathfrak{N}_k \subseteq N(\mathfrak{D})$. Thus, if $|\mathfrak{D}| = 3^3$, then $\langle \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{N}_4 \rangle \subseteq N(\mathfrak{D})$. Since $\mathfrak{A} \subseteq Z(\mathfrak{F})$, it follows that $\mathfrak{A} = \mathfrak{B}_k$ for some k . Hence, $N(\mathfrak{A}) \subseteq \mathfrak{N}_i$. But \mathfrak{A} is a subgroup of \mathfrak{D}_j , so there is a S_3 -subgroup \mathfrak{F}^* of \mathfrak{N}_j which contains \mathfrak{A} in its center. This violates (8.27). Hence, $|\mathfrak{D}| = 3^4$. Since $\mathfrak{F}_0 \supset \mathfrak{D} \supset \mathfrak{D}_i$, we conclude that

$$(8.28) \quad \mathfrak{D} \text{ is elementary of order } 3^4 .$$

Let \mathfrak{C} be any subgroup of \mathfrak{D}_j which is of order 3 and is not contained in \mathfrak{F}_0 . Thus, $\mathfrak{D}_j = \mathfrak{C} \times \mathfrak{B}_j \times \mathfrak{A}$. Since $\mathfrak{D} \subseteq \mathfrak{F}$, it follows that $\mathfrak{D} \subseteq C_{\mathfrak{F}}(\mathfrak{D}_j) = \mathfrak{C} \cdot C_{\mathfrak{F}_0}(\mathfrak{D}_j)$, and as we have already shown, $C_{\mathfrak{F}_0}(\mathfrak{B}_j) = \mathfrak{D}_i \mathfrak{B}_j$ (that is, $\mathfrak{B}_j \not\subseteq Z(\mathfrak{F}_0)$). Since \mathfrak{D}_j does not centralize \mathfrak{C} , it follows that $C_{\mathfrak{F}_0}(\mathfrak{D}_j) = \mathfrak{B}_j \times \mathfrak{B}_i \times \mathfrak{A}$. Since $\mathfrak{D} \not\triangleleft \mathfrak{F}$, it follows that $N_{\mathfrak{F}}(\mathfrak{D}) = \mathfrak{D} \cdot \mathfrak{C}$. Choose P in $\mathfrak{F}_0 - N_{\mathfrak{F}}(\mathfrak{D})$. Since

$$[P, \mathfrak{B}_j, \mathfrak{B}_i \mathfrak{A}] \subseteq \mathfrak{B}_i \subseteq \mathfrak{D} ,$$

it follows that $[P, C] \notin \mathfrak{D}$, where C is a generator for \mathfrak{C} . Hence, $[P, C] = DE_i$ with E_i in $\mathfrak{C} - \mathfrak{A}$ and D in $\mathfrak{D} \cap \mathfrak{F}_0$. Hence, $[P, C, C] = [DE_i, C] = [E_i, C]$ is a generator for \mathfrak{A} . This is a subtle and important

bit of information, since it shows that the \mathbb{C} module $\mathfrak{P}_0/\mathfrak{P}'_0$ has an indecomposable constituent of dimension 3. Thus,

$$(8.29) \quad \begin{array}{l} \text{the indecomposable direct factors of } \mathfrak{P}_0/\mathfrak{P}'_0 \\ \text{as } \mathbb{C}\text{-modules are of dimensions 1 and 3.} \end{array}$$

We note

$$(8.30) \quad \mathfrak{P} = V(\text{ecl}_{\mathbb{C}}(\mathfrak{D}); \mathfrak{P}) .$$

Namely, \mathfrak{Q} does not normalize \mathfrak{D} , $\mathfrak{D}^{\mathfrak{Q}} = \mathfrak{P}_0$. Since $\mathfrak{P} = \langle \mathfrak{P}_0, \mathfrak{D} \rangle$, (8.30) holds. This fact has an important consequence. Namely, if $\mathfrak{P} \subseteq \mathfrak{M}^* \in \mathcal{MS}(\mathbb{G})$ and $\mathfrak{P} \not\triangleleft \mathfrak{M}^*$, then \mathfrak{M}^* satisfies Hypothesis 8.1. If this were not so, then $\mathfrak{P} \subseteq C_{\mathfrak{M}^*}(\Omega_1(\mathbf{Z}(\mathfrak{P})))^{\mathfrak{M}^*}$. Now (8.23) implies that $\mathfrak{P} \triangleleft \mathfrak{M}^*$. Thus,

$$(8.31) \quad \begin{array}{l} \text{if } \mathfrak{P} \subseteq \mathfrak{M}^* \in \mathcal{MS}(\mathbb{G}), \text{ then either } \mathfrak{P} \triangleleft \mathfrak{M}^* \\ \text{or } \mathfrak{M}^* \text{ satisfies Hypothesis 8.1.} \end{array}$$

Let $\tilde{\mathfrak{M}}$ be an element of $\mathcal{MS}(\mathbb{G})$ which contains $N(\mathfrak{A} \cdot)$ and let $\mathfrak{P}_0 = \mathcal{O}_3(\tilde{\mathfrak{M}})$. We argue that

$$(8.32) \quad \mathfrak{P} \not\triangleleft N(\mathfrak{A} \cdot) .$$

Namely, $\mathfrak{P} \subseteq N(\mathfrak{A} \cdot)$. Also, $N(\mathfrak{A} \cdot)$ contains a S_3 -subgroup of \mathfrak{N}_i . By (8.27), this implies that $N(\mathfrak{A} \cdot)$ has more than one S_3 -subgroup so (8.32) holds. By (8.31), it follows that $|\mathfrak{P}_0| = 3^5$ and that $\tilde{\mathfrak{M}} = N(\mathfrak{x})$, where \mathfrak{x} is some subgroup of $\mathbf{Z}(\mathfrak{P})$ of order 3. Clearly, $\mathfrak{x} \neq \mathfrak{B}_i$, since $N(\mathfrak{A} \cdot) \not\subseteq \tilde{\mathfrak{M}}$. On the other hand, if I is the involution of \mathfrak{D} , then $I \in \tilde{\mathfrak{M}}$. Since $\mathfrak{A} \cdot$ and \mathfrak{B}_i are the only subgroups of $\mathbf{Z}(\mathfrak{P})$ of order 3 which are normalized by I , it follows that $\mathfrak{x} = \mathfrak{A} \cdot$.

Since $\mathfrak{M} \neq \tilde{\mathfrak{M}}$, so also $\mathfrak{P}_0 \neq \tilde{\mathfrak{P}}_0$. Hence, (8.20) implies that $\mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0$ is of order 3^4 . Now (8.24) implies that $\mathfrak{D} \subseteq \mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0$, so by (8.28), we have

$$(8.33) \quad \mathfrak{D} = \mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0 .$$

Since $\tilde{\mathfrak{M}} = N(\mathfrak{A} \cdot)$, we have $\mathfrak{D} \cdot \subseteq \mathcal{O}_3(\tilde{\mathfrak{M}})$. Hence,

$$(8.34) \quad \tilde{\mathfrak{P}}_0 = \langle \mathfrak{D}, \mathfrak{D} \cdot \rangle .$$

Since I inverts $\mathfrak{A} \cdot$, we have $I \in \tilde{\mathfrak{M}}$. Let $\tilde{\mathfrak{Q}}$ be a S_2 -subgroup of $\mathcal{O}^{3'}(\tilde{\mathfrak{M}})$ which is normalized by I . Thus, $\tilde{\mathfrak{Q}}$ is a quaternion group, and by (8.22) (with $\tilde{\mathfrak{M}}$ in the role of \mathfrak{M}), we get that

$$(8.35) \quad \tilde{\mathfrak{Q}} \langle I \rangle \text{ is a } S_2\text{-subgroup of } \tilde{\mathfrak{M}} .$$

Let J be the involution of $\tilde{\mathfrak{Q}}$ and set

$$(8.36) \quad \mathfrak{G} = \langle I, J \rangle .$$

Thus, \mathfrak{G} is a four-group and $\mathfrak{G} \subseteq N(\mathfrak{P})$. Since $\mathfrak{B}_i = \mathbf{Z}(\mathfrak{P}) \cap C(I)$, it follows that $\mathfrak{G} \subseteq \mathfrak{M}$. Hence,

$$(8.37) \quad \mathfrak{P} \triangleleft \mathfrak{P}\mathfrak{G} = \mathfrak{M} \cap \tilde{\mathfrak{M}} .$$

Notice that by (8.22), we have $\mathfrak{M} = \mathfrak{E}\langle J \rangle$. Consider $C_{\mathfrak{M}}(I) = C_{\mathfrak{E}}(I)\langle J \rangle$. By (8.5), it follows that $\mathfrak{D} \triangleleft C_{\mathfrak{E}}(I)$. Thus, J normalizes \mathfrak{D} so that

$$(8.38) \quad \mathfrak{D}\langle J \rangle \text{ is a } S_2\text{-subgroup of } \mathfrak{M} .$$

Let Q be an element of \mathfrak{D} of order 4 which normalizes \mathfrak{G} and let \tilde{Q} be an element of $\tilde{\mathfrak{D}}$ of order 4 which normalizes \mathfrak{G} . By (8.22), it follows that $N_{\mathfrak{M}}(\mathfrak{D})/N_{\mathfrak{P}_0}(\mathfrak{D}) \cong GL(2, 3)$. Similarly for $\tilde{\mathfrak{M}}$. Hence, neither Q nor \tilde{Q} centralizes \mathfrak{G} , that is,

$$(8.39) \quad \langle Q, \mathfrak{G} \rangle \text{ and } \langle \tilde{Q}, \mathfrak{G} \rangle \text{ are dihedral groups of order } 8 .$$

We set

$$(8.40) \quad I_1 = JQ , \quad I_2 = IQ .$$

Thus, I_1 and I_2 are involutions and

$$(8.41) \quad \begin{aligned} I_1 J I_1 &= J I , & I_1 I I_1 &= I , \\ I_2 J I_2 &= J , & I_2 I I_2 &= I J . \end{aligned}$$

Finally, we get

$$(8.42) \quad \mathfrak{B}_0 = \langle I_1, I_2 \rangle \subseteq N(\mathfrak{G}) .$$

We next show that \mathfrak{P}_0 is complemented in \mathfrak{M} . It is clear from the structure of $\mathfrak{M} = \mathfrak{N}_i$ that $C_{\mathfrak{M}}(I)$ covers $\mathfrak{N}_i/\mathfrak{C}_i = \mathfrak{N}_i/\mathfrak{P}_0$ and that $C_{\mathfrak{M}}(I) \cap \mathfrak{P}_0 = \mathfrak{B}_0$. Hence \mathfrak{M} will split over \mathfrak{P}_0 if $C_{\mathfrak{M}}(I)$ splits over \mathfrak{B}_0 . Since \mathfrak{B}_i is an abelian 3-group, this occurs if and only if a S_3 -subgroup of $C_{\mathfrak{M}}(I)$ splits over \mathfrak{B}_i , hence, if and only if $C_{\mathfrak{P}}(I)$ is elementary of order 3^2 . Regarding I as an element of $\tilde{\mathfrak{M}}$, we know from the structure of this group that $C_{\mathfrak{P}}(I) = C_{\tilde{\mathfrak{P}}_0}(I)$. But $\tilde{\mathfrak{P}}_0$ has exponent 3, by (8.16) and (8.19). Since the structure of \mathfrak{M} implies that $|C_{\mathfrak{P}}(I)| = 3^2$, we have proved that

$$(8.43) \quad \mathfrak{M} \text{ splits over } \mathfrak{P}_0; \tilde{\mathfrak{M}} \text{ splits over } \tilde{\mathfrak{P}}_0 .$$

We define

$$(8.44) \quad \mathfrak{x}_6 = \mathfrak{U}^{\cdot}, \mathfrak{x}_5 = \mathfrak{B}_i, \mathfrak{x}_4 = \mathfrak{U}^{\cdot q}, \mathfrak{x}_3 = \tilde{\mathfrak{x}}_3^{\cdot} .$$

Since $\langle \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6 \rangle = \mathfrak{B}^{\mathfrak{M}}$ and $\langle \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6 \rangle = \mathfrak{B}^{\tilde{\mathfrak{M}}}$, (8.28) implies that

$$(8.45) \quad \mathfrak{D} = \langle \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6 \rangle .$$

We set

$$(8.46) \quad \mathfrak{x}_1 = \mathfrak{x}_3^q, \mathfrak{x}_2 = \mathfrak{x}_4^{\tilde{q}}.$$

It then follows that

$$(8.47) \quad \begin{aligned} &\mathfrak{x}_{i+1} \cdots \mathfrak{x}_6 \text{ is a subgroup of } \mathfrak{P} \text{ of} \\ &\text{order } 3^{6-i}, i = 0, \dots, 5. \end{aligned}$$

Further, by construction,

$$(8.48) \quad \mathfrak{S} \text{ normalizes } \mathfrak{x}_i, 1 \leq i \leq 6.$$

We now set up a 6 by 2 array whose (i, j) entry is $\mathfrak{x}_i^{I_j}$, in case $\mathfrak{x}_i^{I_j} \subseteq \mathfrak{P}$, and is - otherwise.

$$(8.49) \quad \begin{array}{c|cc} & I_1 & I_2 \\ \hline \mathfrak{x}_1 & \mathfrak{x}_3 & - \\ \mathfrak{x}_2 & - & \mathfrak{x}_4 \\ \mathfrak{x}_3 & \mathfrak{x}_1 & \mathfrak{x}_5 \\ \mathfrak{x}_4 & \mathfrak{x}_6 & \mathfrak{x}_2 \\ \mathfrak{x}_5 & \mathfrak{x}_5 & \mathfrak{x}_3 \\ \mathfrak{x}_6 & \mathfrak{x}_4 & \mathfrak{x}_6 \end{array} .$$

We will eventually determine \mathfrak{M} and $\tilde{\mathfrak{M}}$ in terms of generators and relations. To do this, a number of choices must be made, and some care is required to guarantee that these choices are possible. We have already chosen the groups $\mathfrak{x}_i, 1 \leq i \leq 6$, each of order 3 and each normalized by \mathfrak{S} . Since $\mathfrak{x}_1 \not\subseteq \tilde{\mathfrak{P}}_0$, and since \mathfrak{x}_1 centralizes J , we have $\mathfrak{x}_1 \subseteq C_{\mathfrak{M}}(J) = \tilde{\mathfrak{Q}} \langle I \rangle \mathfrak{x}_1 \mathfrak{x}_6$. Hence, \mathfrak{x}_1 normalizes $\tilde{\mathfrak{Q}}$. We therefore may choose a generator X_1 of \mathfrak{x}_1 such that $X_1 \tilde{Q}$ has order 3. Namely, let X_1 be any generator for \mathfrak{x}_1 . Then $(X_1 \tilde{Q})^3 \in \langle J \rangle$, so either $(X_1 \tilde{Q})^3 = 1$ or $(X_1 \tilde{Q})^3 = J$. Since $\tilde{Q}J = \tilde{Q}^{-1}$, in the second case we get $(X_1 \tilde{Q}^{-1})^3 = 1$, or equivalently, $(\tilde{Q} X_1^{-1})^3 = 1$, or equivalently, $(X_1^{-1} \tilde{Q})^3 = 1$. Thus, we may assume that

$$(8.50) \quad (X_1 \tilde{Q})^3 = 1.$$

For the same reason, we may choose a generator X_2 for \mathfrak{x}_2 such that

$$(8.51) \quad (X_2 Q)^3 = 1.$$

We set $X_3 = X_1^q, X_4 = X_2^{\tilde{q}}, X_5 = X_3^{\tilde{q}}, X_6 = X_4^q$. Notice that

$$(8.52) \quad \langle X_i \rangle = \mathfrak{x}_i, 1 \leq i \leq 6.$$

It is now convenient to draw up a table listing the action of \mathfrak{S} on each \mathfrak{x}_i . This information is available since we know the action

$ab \neq 0$. It follows from (8.29) that $[X_1, X_2, X_2]$ does not lie in \mathfrak{X}_5 , so $d \neq 0$. By symmetry, $c \neq 0$. To determine the values a through f explicitly, we make use of the following identities:

$$\begin{aligned} [AB, C] &= [A, C][A, C, B][B, C] \\ [A, BC] &= [A, C][A, B][A, B, C] \\ [A^{-1}, C] &= [A, C, A^{-1}]^{-1}[A, C]^{-1} \\ [A, B^{-1}] &= [A, B, B^{-1}]^{-1}[A, B]^{-1} \\ [A^{-1}, B^{-1}] &= [A, B^{-1}, A^{-1}]^{-1}[A, B^{-1}]^{-1}. \end{aligned}$$

Since X_2Q has order 3, we have

$$Q^{-1}X_2Q = IQX_2Q = IX_2^{-1}Q^{-1}X_2^{-1} = X_2^{-1}QX_2^{-1}.$$

Using this relation, conjugate (8.59) by Q , to obtain

$$[X_3, X_2^{-1}QX_2^{-1}] = X_1^{-c}X_6^dX_5^eX_4^{-f}.$$

Since X_2 and X_3 commute, we have

$$[X_3, X_2^{-1}QX_2^{-1}] = [X_3, QX_2^{-1}].$$

By the preceding identities, $[X_3, QX_2^{-1}] = [X_3, Q][X_3, Q, X_2^{-1}]$. Now

$$[X_3, Q] = X_3^{-1}Q^{-1}X_3Q = X_3^{-1}X_1^{-1},$$

so that

$$[X_3, QX_2^{-1}] = X_3^{-1}X_1^{-1}[X_3^{-1}X_1^{-1}, X_2^{-1}].$$

Since \mathfrak{X}_2 and \mathfrak{X}_3 commute, we have $[X_3^{-1}X_1^{-1}, X_2^{-1}] = [X_1^{-1}, X_2^{-1}]$. Now by the preceding identities, we have

$$\begin{aligned} [X_1^{-1}, X_2^{-1}] &= [X_1, X_2^{-1}, X_1^{-1}]^{-1}[X_1, X_2^{-1}]^{-1} \\ &= [[X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1}, X_1^{-1}]^{-1} \\ &\quad \times ([X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1})^{-1}. \end{aligned}$$

Since $[X_1, X_2, X_2^{-1}] \in \mathfrak{C}_i$, it follows that

$$[[X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1}, X_1^{-1}]^{-1} = [[X_1, X_2]^{-1}, X_1^{-1}]^{-1}.$$

We get that $[X_1^{-1}, X_2^{-1}] = X_5^{ac}X_3^cX_4^dX_5^eX_6^{f+bd}$. Since $X_3^{-1}X_1^{-1} = X_1^{-1}X_3^{-1}X_5^{-a}$, we see that $[X_3, X_2^{-1}QX_2^{-1}] = X_1^{-1}X_3^{-1}X_5^{-a+ac}X_3^cX_4^dX_5^eX_6^{f+bd}$. This gives us the following equations: $c = 1, d = -f, f + bd = d$. Conjugating (8.59) successively by I, J, IJ and using the fact that $d \neq 0$ yield the values $b = -1, a = e, d = -f$. No more information is forthcoming from \mathfrak{M} , so we conjugate (8.59) by \tilde{Q} and work in $\tilde{\mathfrak{M}}$. We state the result of these calculations:

$$(8.60) \quad a = -1, b = -1, c = 1, d = -1, e = -1, f = 1 .$$

Let $\mathfrak{R}^* = \langle \mathfrak{X}_2, \mathfrak{X}_5 \rangle$ and note that $\mathfrak{R}^* = C_{\mathfrak{F}}(I)$. By construction, $\mathfrak{X}_2 \subseteq O_3(\tilde{\mathfrak{M}})$ and $\mathfrak{X}_3 \subseteq Z(O_3(\tilde{\mathfrak{M}}))$. Hence,

$$C_{\mathfrak{F}}(\mathfrak{X}_2) = C_{\mathfrak{F}}(\mathfrak{R}^*) = \langle X_2, X_3, X_5, X_6 \rangle ,$$

so that $|\mathfrak{F}:C_{\mathfrak{F}}(\mathfrak{R}^*)| = 9$. With \mathfrak{R}^* in the role of \mathfrak{A} in Lemma 7.6 (c), it follows that \mathfrak{R}^* centralizes every abelian subgroup of $\mathfrak{N}(\mathfrak{R}^*; 2)$.

Since $O^3(\mathfrak{M}) \cap C(I) = \mathfrak{R}^*\mathfrak{Q}$, it follows that \mathfrak{Q} is normalized by \mathfrak{R}^* but is not centralized by \mathfrak{R}^* . Let \mathfrak{X}^* be a $S_{2,3}$ -subgroup of $C(I)$ which contains $\mathfrak{R}^*\mathfrak{Q}$. Then \mathfrak{X}^* contains an element $\mathfrak{Z}(2)$. Let \mathfrak{X}_2 be a S_2 -subgroup of \mathfrak{X}^* which contains \mathfrak{Q} . By Lemma 7.5, there is an element \mathfrak{M}_1 of $\mathcal{MS}(\mathfrak{G})$ such that $(\mathfrak{R}^*, \mathfrak{X}^*, \mathfrak{X}_2, \mathfrak{R} = O_2(\mathfrak{M}_1), \mathfrak{M}_1)$ satisfies all parts of Lemma 7.5 with \mathfrak{R}^* in the role of \mathfrak{B} , \mathfrak{R} in the role of \mathfrak{S} , \mathfrak{M}_1 in the role of \mathfrak{M} . Since by (e) of Lemma 7.5, $\mathfrak{Q} \subseteq \mathfrak{R}$, it follows that $\mathfrak{M}_1 = C(I)$. Hence, $J \in \mathfrak{M}_1$.

The next task is shown that

$$(8.61) \quad N(\mathfrak{D}) \subseteq N(\mathfrak{F}) .$$

By our preceding results, $\mathfrak{D} \triangleleft \mathfrak{F}$. It is straightforward to verify that $N_{\mathfrak{M}}(\mathfrak{D}) \subseteq N_{\mathfrak{M}}(\mathfrak{F})$. Let $\mathfrak{M}^* \in \mathcal{MS}(\mathfrak{G})$ with $N(\mathfrak{D}) \subseteq \mathfrak{M}^*$. If $\mathfrak{F} \triangleleft \mathfrak{M}^*$, we have our desired containment. Otherwise, \mathfrak{M}^* satisfies Hypothesis 8.1. Hence, $N(\mathfrak{D}) = N_{\mathfrak{M}^*}(\mathfrak{D}) \subseteq N_{\mathfrak{M}^*}(\mathfrak{F}) \subseteq N(\mathfrak{F})$, as desired.

We next show that

$$(8.62) \quad \text{if } X \in \mathfrak{X}_2^\#, Y \in \mathfrak{X}_5^\#, \text{ then } |C(XY)|_3 = 3^4 .$$

Let $Z = XY$. Let $X^* = X^{\tilde{q}}$, $Y^* = Y^{\tilde{q}}$, $Z^* = X^*Y^*$. Then $X^* \in \mathfrak{X}_4$, $Y^* \in \mathfrak{X}_3$, so it suffices to show that \mathfrak{D} is a S_3 -subgroup of $C(Z^*)$. Suppose false. Let $\tilde{\mathfrak{D}}$ be a S_3 -subgroup of $C(\mathfrak{X}^*)$ which contains \mathfrak{D} , and let $\mathfrak{D} \subseteq \mathfrak{D}^* \subseteq \tilde{\mathfrak{D}}$, with $|\mathfrak{D}^*:\mathfrak{D}| = 3$. Then $\mathfrak{D}^* \subseteq N(\mathfrak{D}) \subseteq N(\mathfrak{F})$, so $\mathfrak{D}^* \subseteq \mathfrak{F}$. However, $\mathfrak{D} = C_{\mathfrak{F}}(Z^*)$. Notice that we have shown that $\mathfrak{D} \triangleleft C(Z^*)$. Namely, \mathfrak{D} is a S_3 -subgroup of $C(Z^*)$, and since $\mathfrak{D} \in \mathcal{L}_{ev_3}(P)$, we have $O_3(C(Z^*)) = 1$ so that

$$(8.63) \quad \mathfrak{D} \text{ is a normal } S_3\text{-subgroup of } C(Z^*) .$$

Retaining the preceding notation we will show that $\langle I \rangle = C_{\mathfrak{R}}(Z)$. Suppose false. Since $\langle I \rangle = C_{\mathfrak{R}}(\mathfrak{R}^*)$, it follows that $1 \neq [C_{\mathfrak{R}}(Z), \mathfrak{R}^*]$. This violates the fact that $C(Z)$ is 3-closed by (8.63).

We next observe that $\mathfrak{X}_2 \sim \mathfrak{X}_6$ so that $|C(\mathfrak{X}_2)|_2 = |C(\mathfrak{X}_6)|_2 = 8$. These equalities together with the preceding paragraph show that \mathfrak{R} is extra special of width 2 and that

$$(8.64) \quad \mathfrak{R} \text{ is the central product of quaternion groups } \mathfrak{Q}, \mathfrak{Q}_1, \text{ where } \mathfrak{Q} = C_{\mathfrak{R}}(\mathfrak{X}_5), \mathfrak{Q}_1 = C_{\mathfrak{R}}(\mathfrak{X}_2) .$$

This choice of notation conforms with our previous definition of \mathfrak{Q} .

Since \mathfrak{R}^* maps onto a S_3 -subgroup of $\mathfrak{A}_{\mathfrak{G}}(\mathfrak{R})$, it follows that $\mathfrak{R}^*\mathfrak{R} \triangleleft \mathfrak{M}_1$. Let

$$(8.65) \quad \mathfrak{Q} = N_{\mathfrak{M}_1}(\mathfrak{R}^*) ,$$

so that $\mathfrak{Q} \cap \mathfrak{R} = \langle I \rangle$, and $\mathfrak{M}_1 = \mathfrak{R}\mathfrak{Q}$. \mathfrak{Q} acts as a permutation group on the subgroups of \mathfrak{R}^* of order 3. By the previous arguments, \mathfrak{x}_2 and \mathfrak{x}_3 are permuted among themselves. Let

$$(8.66) \quad \mathfrak{Q}^* = N_{\mathfrak{Q}}(\mathfrak{x}_2) = N_{\mathfrak{Q}}(\mathfrak{x}_3)$$

so that $|\mathfrak{Q}:\mathfrak{Q}^*| \leq 2$. Also, if $L \in \mathfrak{Q}^*$ and L centralizes \mathfrak{x}_3 , then $L \in \mathfrak{R}^*\langle I \rangle$. Hence, $\mathfrak{Q}^* = \mathfrak{R}^*\mathfrak{Q}$, and $|\mathfrak{M}_1:\mathfrak{R}\mathfrak{Q}^*| \leq 2$. Let \mathfrak{Q}_2 be a S_2 -subgroup of \mathfrak{Q} which contains \mathfrak{Q} . Thus, $|\mathfrak{Q}_2| = 4$ or 8 .

We must now show that

$$(8.67) \quad \mathfrak{Q} = \mathfrak{Q}^* .$$

Suppose false. Since $\mathfrak{R}^{*\tilde{Q}} = \langle \mathfrak{x}_3, \mathfrak{x}_4 \rangle$, it follows that $\mathfrak{Q}^{\tilde{Q}}$ normalizes $\langle \mathfrak{x}_3, \mathfrak{x}_4 \rangle$. From (8.63), we conclude that $\mathfrak{D} \text{ char } C(\mathfrak{x}_3\mathfrak{x}_4)$. Hence, $N(\mathfrak{x}_3\mathfrak{x}_4) \subseteq N(\mathfrak{D})$. Now by (8.61), we have $N(\mathfrak{D}) \subseteq N(\mathfrak{P})$. Thus $\mathfrak{Q}^{\tilde{Q}}$ normalizes \mathfrak{P} .

It is a straightforward consequence of (8.55) through (8.60) that $\mathfrak{P}_0 \cup \tilde{\mathfrak{P}}_0$ is the set of elements of \mathfrak{P} of order at most 3. Hence, \mathfrak{P} contains exactly $3^6 - 2 \cdot 3^5 + 3^4 = 4 \cdot 3^4$ elements of order 9. Thus, some involution I_0 of $\mathfrak{Q}_2^{\tilde{Q}}$ centralizes an element P of \mathfrak{P} of order 9. It is clear from (8.53) that $I_0 \in \mathfrak{Q}$.

If $X \in \mathfrak{x}_3^*\mathfrak{x}_6^*$, we will show that $C(X) \subseteq N(\mathfrak{P})$. Suppose false. Let $\mathfrak{M}^* \in \mathcal{MS}(\mathfrak{G})$ with $C(X) \subseteq \mathfrak{M}^*$. We may apply all the preceding results to \mathfrak{M}^* in place of \mathfrak{M} and conclude that $O_3(\mathfrak{M}^*)$ is of exponent 3 and order 3^5 . However, \mathfrak{P}_0 and $\tilde{\mathfrak{P}}_0$ are the only subgroups of \mathfrak{P} meeting these conditions, so $C(X) \subseteq \mathfrak{M}$ or $C(X) \subseteq \tilde{\mathfrak{M}}$, from which the desired containment is obvious. In particular,

$$(8.68) \quad C(P^3) \subseteq N(\mathfrak{P}) .$$

Let \mathfrak{N}_2 be a S_2 -subgroup of $N(\mathfrak{P})$ which contains $\mathfrak{Q}_2^{\tilde{Q}}$. By (8.23) \mathfrak{N}_2 is faithfully represented on $\mathfrak{B} = Z(\mathfrak{P}) = \langle \mathfrak{x}_5, \mathfrak{x}_6 \rangle$. It is clear that $\text{Aut}(\mathfrak{P})$ is a 2, 3-group, so we conclude that

$$(8.69) \quad N(\mathfrak{P}) = \mathfrak{P}\mathfrak{Q}^{\tilde{Q}} .$$

It now follows from (8.68), (8.69), and (8.53) that

$$(8.70) \quad C(P^3) = \mathfrak{P}\langle I_0 \rangle .$$

By hypothesis, $C(I_0)$ is solvable. Let $\mathfrak{F} = O_2(C(I_0))$. Suppose $\langle P \rangle$ acts faithfully on \mathfrak{F} . Then $m(\mathfrak{F}) \geq 6$, since P has order 9. But \mathfrak{M}_1

contains a S_2 -subgroup of \mathfrak{G} , and since S_2 -subgroups of \mathfrak{M}_1 are extensions of \mathfrak{R} by a 4 group, it follows that every 2-subgroup of \mathfrak{G} is generated by 4-elements (naturally, this uses the action of the 4-group on \mathfrak{R}). Hence, P^3 centralizes \mathfrak{F} . By (8.70), we get $\mathfrak{F} = \langle I_0 \rangle$.

By Lemma 5.38 (a)(ii), $C(I_0)$ contains an element \mathfrak{U} of $\mathcal{Z}(2)$. Since $O_2(C(I_0)) = \langle I_0 \rangle$, we get that $O_{2,2'}(C(I_0)) = \langle I_0 \rangle \times O_{2'}(C(I_0))$, so that by Lemma 7.1, \mathfrak{U} centralizes $O_{2,2'}(C(I_0))$. But $C(I_0)$ is solvable, so that $O_{2,2'}(C(I_0))$ contains its centralizer. Thus, $\mathfrak{U} \subseteq O_{2,2'}(C(I_0))$, an absurdity. This contradiction establishes (8.67). Notice that (8.67) is equivalent to

$$(8.71) \quad \mathfrak{M}_1 = \mathfrak{R}\mathfrak{R}^*\langle J \rangle .$$

Since J inverts \mathfrak{R}^* , it follows that $\mathfrak{Q}\langle J \rangle$ and $\mathfrak{Q}_1\langle J \rangle$ are both isomorphic to S_2 -subgroups of $GL(2, 3)$. This implies that

$$(8.72) \quad C_{\mathfrak{M}_1}(J) \text{ is elementary of order } 8 .$$

The hard work is now completed. We may now determine the Weyl group. Recall that $I_1 = JQ, I_2 = I\tilde{Q}$, so that I_1 and I_2 are involutions. Let $W = I_1I_2$. Thus W^3 centralizes \mathfrak{G} . Since W centralizes no element of \mathfrak{G}^* , W^3 is not in \mathfrak{G}^* . Since $W^3 \in O(\mathfrak{G}) \subseteq \mathfrak{M}_1$, and since the structure of $C_{\mathfrak{M}_1}(J)$ is given in (8.72), it follows that $W^6 = 1$, so that W is of order 3 or 6.

From (8.49), we get that $\mathfrak{X}_1^{W^3} = \mathfrak{X}_1^{I^2} \neq \mathfrak{X}_1$, and conclude that W is of order 6. Thus,

$$(8.73) \quad W_0 = W^3 \text{ is an involution in the center of } \langle I_1, I_2 \rangle = \mathfrak{B}_0 .$$

We argue that

$$(8.74) \quad \mathfrak{P} \cap \mathfrak{P}^{W_0} = 1 .$$

Since $\mathfrak{P} \cap \mathfrak{P}^{W_0}$ is normalized by \mathfrak{G} and by W_0 , (8.53) implies that if $\mathfrak{P}^* = \mathfrak{P} \cap \mathfrak{P}^{W_0}$, then

$$\mathfrak{P}^* = (\mathfrak{P}^* \cap \langle \mathfrak{X}_1, \mathfrak{X}_6 \rangle)(\mathfrak{P}^* \cap \langle \mathfrak{X}_2, \mathfrak{X}_5 \rangle)(\mathfrak{P}^* \cap \langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle) .$$

If $X \in \mathfrak{X}_3^*\mathfrak{X}_4^*$, we know that $C(X) \subseteq N(\mathfrak{D})$. This fact, coupled with (8.49) implies that $\mathfrak{P}^* = 1$, so that (8.74) holds.

Let $\mathfrak{B} = \mathfrak{P}\mathfrak{G}$. (No confusion with previous notation is to be feared.) We then get that $\mathfrak{M} = \mathfrak{B} \cup \mathfrak{B}I_1\mathfrak{B}, \tilde{\mathfrak{M}} = \mathfrak{B} \cup \mathfrak{B}I_2\mathfrak{B}$. Hence, (8.49) implies that conditions (i') and (iv) of Théorème 1 of [40] are satisfied. Hence, $\mathfrak{B}\mathfrak{B}_0\mathfrak{B} = \mathfrak{G}_0$ is a group and if we let \mathfrak{B}_X be the largest subset of \mathfrak{B} such that $\mathfrak{B}_X^X \subseteq \mathfrak{P}^{W_0}$, it follows easily from (8.74) that each element of \mathfrak{G}_0 has a unique representation of the shape $BXB_X, B \in \mathfrak{B}, X \in \mathfrak{B}_0, B_X \in \mathfrak{B}_X$. Thus, $|\mathfrak{G}_0| = |E_2(3)|$, by an easy calculation. Hence, (8.41), (8.50), (8.51), (8.53), (8.54), (8.57), (8.58), (8.59), (8.60), (8.73) determine the multiplication table of \mathfrak{G}_0 . Thus, if \mathfrak{G}^* is any group which satisfies

the hypothesis of Theorem 8.1 and also satisfies Hypothesis 8.1, it follows that \mathbb{G}^* contains a subgroup isomorphic to \mathbb{G}_0 . Since we may take $\mathbb{G}^* = E_2(3)$, it follows that $\mathbb{G}_0 \cong E_2(3)$, and so $i(\mathbb{G}_0) = 1$. Clearly, \mathbb{G}_0 contains \mathfrak{M}_1 , so that \mathbb{G}_0 contains the centralizer of each of its involutions. Hence, $i(\mathbb{G}) = 1$, by Lemma 5.35.

Since $E_2(3)$ does not satisfy $E_{7,13}$ (by Sylow's theorem), it follows from Lemma 5.35 that $\mathbb{G}_0 = \mathbb{G} \cong E_2(3)$.

The remaining lemmas are proved under the following hypothesis:

HYPOTHESIS 8.2. Whenever $\mathfrak{F} \subseteq \mathfrak{M} \in \mathcal{MS}(\mathbb{G})$ and $\mathfrak{B} = \Omega_1(\mathbf{Z}(\mathfrak{F}))^{\mathfrak{M}}$, then $V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{F}) \subseteq C_{\mathfrak{M}}(\mathfrak{B})$.

We must derive a contradiction from this hypothesis. When this is done, the proof of Theorem 8.1 will be complete.

LEMMA 8.7. *If \mathfrak{X} is a 2, 3-subgroup of \mathbb{G} and \mathfrak{X}_3 is a S_3 -subgroup of \mathfrak{X} , then $V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{X}_3) \triangleleft \mathfrak{X}$.*

Proof. We assume without loss of generality that $\mathfrak{X}_3 \subseteq \mathfrak{F}$. First, suppose $\mathfrak{X}_3 = \mathfrak{F}$. Let $\mathfrak{X} \subseteq \mathfrak{M} \in \mathcal{MS}(\mathbb{G})$, and let \mathfrak{X}^* be a $S_{2,3}$ -subgroup of \mathfrak{M} containing \mathfrak{X} . Let $\mathfrak{B} = \Omega_1(\mathbf{Z}(\mathfrak{F}))^{\mathfrak{M}}$, $\mathbb{C} = C_{\mathfrak{M}}(\mathfrak{B})$. As $\mathbb{C} \triangleleft \mathfrak{M}$, $\mathbb{C} \cap \mathfrak{X}^*$ is a $S_{2,3}$ -subgroup of \mathbb{C} . By Hypothesis 8.2, $\mathfrak{B}^* \subseteq \mathbb{C} \cap \mathfrak{X}^*$, where $\mathfrak{B}^* = \mathfrak{B}(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{F})$. Since $\mathfrak{B} \subseteq \mathfrak{B}$, Lemmas 7.4 and 5.38 imply that $|\mathbb{C} \cap \mathfrak{X}^*|$ is odd. Hence, $\mathbb{C} \cap \mathfrak{X}^* \triangleleft \mathfrak{X}^*$ implies $\mathfrak{B}^* = V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathbb{C} \cap \mathfrak{X}^*) \triangleleft \mathfrak{X}^*$.

We may now assume that $\mathfrak{X}_3 \subset \mathfrak{F}$. We proceed by induction on $|\mathfrak{F}|/|\mathfrak{X}_3|$. Let $\mathfrak{B}^* = V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{X}_3)$. As \mathfrak{B}^* is generated by conjugates of \mathfrak{B} , it follows that \mathfrak{B}^* centralizes $O_2(\mathfrak{X})$. Hence, if $\mathfrak{B}^* \neq 1$, then $O_2(\mathfrak{X}) = 1$, so that $O_{2,3}(\mathfrak{X}) = O_3(\mathfrak{X})$. If $\mathfrak{B}^* = 1$, the lemma is trivial, so suppose $\mathfrak{B}^* \neq 1$. In particular, $O_3(\mathfrak{X}) \neq 1$. If \mathfrak{X}_3 is not a S_3 -subgroup of $N(O_3(\mathfrak{X}))$, let \mathfrak{X}^* be a $S_{2,3}$ -subgroup of $N(O_3(\mathfrak{X}))$ containing \mathfrak{X} , and let \mathfrak{X}_3^* be a S_3 -subgroup of \mathfrak{X}^* which contains \mathfrak{X}_3 . Then $V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{X}_3^*) \triangleleft \mathfrak{X}^*$. In particular, $[\mathfrak{B}^*, \mathfrak{X}]$ is a 3-group, so $\mathfrak{B}^* \triangleleft \mathfrak{X}$. Hence, we may assume that \mathfrak{X}_3 is a S_3 -subgroup of $N(O_3(\mathfrak{X}))$.

Let $\mathfrak{B}_0 = \Omega_1(\mathbf{Z}(O_3(\mathfrak{X})))$, so that $\mathfrak{B} \subseteq \mathfrak{B}_0$. Since $|C_{\mathfrak{F}}(\mathfrak{B}_0)|$ is odd, it follows that $O_3(\mathfrak{X}) = C_{\mathfrak{X}}(\mathfrak{B}_0)$. Suppose $\mathfrak{B}^* \not\subseteq O_2(\mathfrak{X})$. Choose G in \mathbb{G} so that $\mathfrak{D}^G \subseteq \mathfrak{B}^*$ but $\mathfrak{D}^G \not\subseteq O_3(\mathfrak{X})$, and for any subset \mathfrak{S} of \mathbb{G} , let $\mathfrak{S}^\bullet = \mathfrak{S}^G$.

It is a straightforward consequence of Hypothesis 8.2 that $\mathfrak{D}^\bullet = 1$.

As \mathfrak{D}^\bullet acts nontrivially on $\mathbf{Q}_3(\mathfrak{X})$, we let \mathfrak{D} be a \mathfrak{D}^\bullet -invariant subgroup of $\mathbf{Q}_3(\mathfrak{X})$ minimal subject to $[\mathfrak{D}^\bullet, \mathfrak{D}] \neq 1$. Let $\mathfrak{D}_i = C_{\mathfrak{D}}(\mathfrak{D}_i)$, so that $|\mathfrak{D}^\bullet : \mathfrak{D}_i| = 3$. Thus, $\mathfrak{B}_0 = C_{\mathfrak{B}_0}(\mathfrak{D}_i)$ is invariant under \mathfrak{D}^\bullet and \mathfrak{D} .

Let $N = \{i \mid 1 \leq i \leq 4, \mathfrak{B}_i \subseteq \mathfrak{D}_i, \mathfrak{D}_i \not\subseteq \mathfrak{D}_i\}$. If $\mathfrak{B} \subseteq \mathfrak{D}_i$, then it is obvious that $N \neq \emptyset$. If $\mathfrak{B} \subseteq \mathfrak{D}_i$, then no \mathfrak{D}_i is contained in $\mathfrak{D}_i, 1 \leq i \leq 4$. Since $\mathfrak{B} \cap \mathfrak{D}_i$ is of order 3 in this case, we again conclude that $N \neq \emptyset$. Choose $i \in N$. Thus, $\mathfrak{D}^\bullet = \langle \mathfrak{D}_i, \mathfrak{D}_i \rangle$ and $|\mathfrak{D}_i : \mathfrak{D}_i \cap \mathfrak{D}_i| =$

3. Since Ω is faithfully represented on $\tilde{\mathfrak{W}}_0$, it follows that $[\mathfrak{D}; \tilde{\mathfrak{W}}_0] \neq 1$. By Lemma 8.5, $[\Omega, \tilde{\mathfrak{W}}_0]$ is of order 9 and is not centralized by \mathfrak{D}^\cdot . Since $\mathfrak{B} \subseteq \mathfrak{W}_0$, so also $\mathfrak{B} \subseteq \tilde{\mathfrak{W}}_0$. Hence, Ω centralizes some B of $\mathfrak{B}^\#$, so if $\Omega = \Omega^*/C_{\mathfrak{Z}}(\mathfrak{W}_0)$, then $\langle \mathfrak{D}^\cdot, \Omega^* \rangle \subseteq C(B)$. By the preceding argument, $[\Omega^*, \mathfrak{D}^\cdot]$ is a 3-group, violating the nontrivial action of \mathfrak{D}^\cdot on Ω . Thus, $\mathfrak{B}^* \subseteq O_3(\mathfrak{X})$, and so $\mathfrak{B}^* \triangleleft \mathfrak{X}$, completing the proof of this lemma.

For the remainder of this section, we let

$$\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{B}), \mathfrak{N} = N(\mathfrak{B}) .$$

LEMMA 8.8. (i) \mathfrak{N} contains no element of $\mathcal{S}(2)$. (See Definition 2.9.)

(ii) If \mathfrak{X}_0 is any 2-subgroup of \mathfrak{N} , then $A_{\mathfrak{N}}(\mathfrak{X}_0)$ does not contain a subgroup of type (3, 3).

(iii) If \mathfrak{C} is any subgroup of $Z(\mathfrak{B})$ of type (3, 3), then $(\mathfrak{N}, \mathfrak{C}) \in \mathcal{N}$ for all $\mathfrak{N} \in \mathcal{Z}(2)$. (See Definition 7.2).

(iv) If \mathfrak{X}_1 is an abelian 2-subgroup of \mathfrak{N} , the $A_{\mathfrak{N}}(\mathfrak{X}_1)$ is a 3'-group.

Proof. We first prove (iii). We invoke Lemma 7.4, so that (iii) will hold if we can show that \mathfrak{C} centralizes every element of $\mathfrak{N}(\mathfrak{C}; 2)$. Suppose $\Omega \in \mathfrak{N}(\mathfrak{C}; 2)$ is minimal subject to $[\Omega, \mathfrak{C}] \neq 1$. Let $\mathfrak{C}_0 = C_{\mathfrak{C}}(\Omega) \neq 1$. Let \mathfrak{X} be a $S_{2,3}$ -subgroup of $C(\mathfrak{C}_0)$ containing $\mathfrak{C}\Omega$. Since $C(\mathfrak{C}_0) \cong \mathfrak{B}$, it follows that if \mathfrak{X}_0 is a $S_{2,3}$ -subgroup of $C(\mathfrak{C}_0)$ containing \mathfrak{B} , then $\mathfrak{B} \subseteq O_3(\mathfrak{X}_0)$. By Lemma 0.7.5, we have $\mathfrak{C} \subseteq O_3(\mathfrak{X})$, so that $[\Omega, \mathfrak{C}] \subseteq \Omega \cap O_3(\mathfrak{X}) = 1$. (iii) is proved.

Let $\mathfrak{X} \in \mathcal{S}(2)$, $\mathfrak{X} \subseteq \mathfrak{N}$. We may assume that \mathfrak{X} is a noncyclic abelian group of order 8. Since $\mathfrak{B} \subseteq Z(\mathfrak{B})$, $Z(\mathfrak{B})$ is noncyclic. Hence, \mathfrak{X} contains an involution I such that $C(I) \cap Z(\mathfrak{B})$ is noncyclic. Thus, $C(I)$ contains an element of $\mathcal{Z}(2)$ and also a subgroup \mathfrak{C} of $Z(\mathfrak{B})$ of type (3, 3). By hypothesis, $C(I)$ is solvable, in violation of (iii). (i) is proved.

(ii) is a straightforward consequence of (i).

To prove (iv), let \mathfrak{X}_1 be an abelian 2-subgroup of \mathfrak{N} minimal subject to $3 \mid |A_{\mathfrak{N}}(\mathfrak{X}_1)|$. Thus, \mathfrak{X}_1 is a four-group, and the involutions of \mathfrak{X}_1 are all \mathfrak{N} -conjugate. Thus, (iii) implies that $C(I) \cap Z(\mathfrak{B})$ is cyclic for all $I \in \mathfrak{X}_1^\#$. This implies that $|\Omega_1(Z(\mathfrak{B}))| \leq 3^3$. Since the reverse inequality holds by (B), we find that $Z(\mathfrak{B}) \cap \mathfrak{X}_1$ is cyclic. This is not the case, since $\mathfrak{B} \subseteq Z(\mathfrak{B}) \cap \mathfrak{X}_1$. (iv) is proved.

LEMMA 8.9. If \mathfrak{X} is a 2, 3-subgroup of \mathfrak{G} and \mathfrak{X} contains a conjugate of \mathfrak{B} , then \mathfrak{X} is contained in conjugate of \mathfrak{N} .

Proof. We assume without loss of generality that \mathfrak{X} is a maximal

2, 3-subgroup of \mathfrak{G} , and that $\mathfrak{B} \subseteq \mathfrak{X}$. Since \mathfrak{B} centralizes $O_2(\mathfrak{X})$, it follows that $O_2(\mathfrak{X}) = 1$, and so $O_3(\mathfrak{X}) \neq 1$. Let \mathfrak{X}_3 be a S_3 -subgroup of \mathfrak{X} . By maximality of \mathfrak{X} , \mathfrak{X}_3 is a S_3 -subgroup of $N(O_3(\mathfrak{X}))$. We assume without loss of generality that $\mathfrak{X}_3 \subseteq \mathfrak{P}$. This implies that $\mathfrak{B} \subseteq Z(O_3(\mathfrak{X}))$.

If \mathfrak{X} contains a conjugate of \mathfrak{D} , we are done by Lemma 8.7. We therefore suppose that for each G in \mathfrak{G} , $\mathfrak{D}^G \not\subseteq \mathfrak{X}$.

Suppose $1 \leq i \leq 4$, and $\mathfrak{D}_i \cap O_3(\mathfrak{X}) = \mathfrak{D}_i \cap \mathfrak{X}_3$. We conclude that $\mathfrak{D}_i \subseteq O_3(\mathfrak{X})$. Since $\mathfrak{D} \not\subseteq \mathfrak{X}_3$, we may choose i with $1 \leq i \leq 4$ such that $\mathfrak{D}_i \cap O_3(\mathfrak{X}) \subset \mathfrak{D}_i \cap \mathfrak{X}_3$. Set $\mathfrak{F} = \mathfrak{D}_i \cap O_3(\mathfrak{X})$, $\mathfrak{F}^* = \mathfrak{D}_i \cap \mathfrak{X}_3$. The index i is fixed in the following discussion. We note that \mathfrak{F} and \mathfrak{F}^* are normal elementary subgroups of \mathfrak{X}_3 .

Let \mathfrak{Q} be a \mathfrak{F}^* -invariant subgroup of $Q_3(\mathfrak{X})$ minimal subject to $[\mathfrak{F}^*, \mathfrak{Q}] \neq 1$. Thus, \mathfrak{F}^* acts irreducibly on the Frattini quotient group of \mathfrak{Q} . We remark that \mathfrak{Q} is available, since $O_2(\mathfrak{X}) = 1$.

Let $\mathfrak{B}_0 = \Omega_1(Z(O_3(\mathfrak{X})))$, so that $\mathfrak{B} \subseteq \mathfrak{B}_0$.

Choose $Q \in \mathfrak{Q} - \mathfrak{Q}'$. We will show that $\mathfrak{B}^Q \cap C(\mathfrak{D}_i) = 1$. Suppose false, and that B in \mathfrak{B}^* satisfies $B^Q \in C(\mathfrak{D}_i)$. Hence, for D in \mathfrak{F}^* ($\subseteq \mathfrak{D}_i$), we have $B^{QD} = B^Q$, or $B^{QDQ^{-1}} = B$. Hence, $QDQ^{-1}D^{-1}$ centralizes B for each D in \mathfrak{F}^* . This implies that \mathfrak{Q} centralizes B . Apply Lemma 8.7 to $C(B)$ and conclude that if $\mathfrak{Q} = \mathfrak{Q}^*/O_3(\mathfrak{X})$, then $[\mathfrak{Q}^*, \mathfrak{F}^*]$ is a 3-group. As this violates the nontrivial action of \mathfrak{F}^* on \mathfrak{Q} , the assertion follows.

Since $\mathfrak{B}^Q \subseteq O_3(\mathfrak{X}) \subseteq \mathfrak{X}_3 \subseteq \mathfrak{P}$, we have $\mathfrak{B}^Q \subseteq N(\mathfrak{D}_i)$. Since \mathfrak{D}_i is 3-reducible in $N(\mathfrak{D}_i)$, it follows that \mathfrak{B}^Q is faithfully represented on $\mathfrak{R} = O_3(N(\mathfrak{D}_i)/C(\mathfrak{D}_i))$. On the other hand, if $B \in \mathfrak{B}^*$, then $[C(B)^Q, \mathfrak{B}^Q, \mathfrak{B}^Q] = 1$. This implies that \mathfrak{B}^Q centralizes every 2'-subgroup of \mathfrak{R} which \mathfrak{B}^Q normalizes. Thus, there is a 2-subgroup \mathfrak{X}_0 of $N(\mathfrak{D}_i)$ such that $A_{N(\mathfrak{D}_i)}(\mathfrak{X}_0)$ contains a subgroup of type (3, 3). This violates Lemma 8.8 by $D_{2,3}$ in $N(\mathfrak{D}_i)$. The proof is complete.

LEMMA 8.10. *If \mathfrak{C} is any subgroup of \mathfrak{G} of type (3, 3), then \mathfrak{C} centralizes every abelian subgroup in $\mathfrak{N}(\mathfrak{C}; 2)$.*

Proof. Suppose \mathfrak{Q} is a four-group in $\mathfrak{N}(\mathfrak{C}; 2)$ with $[\mathfrak{Q}, \mathfrak{C}] \neq 1$. Let $\mathfrak{C}_0 = C_{\mathfrak{G}}(\mathfrak{Q})$. Let \mathfrak{X} be a $S_{2,3}$ -subgroup of $C(\mathfrak{C}_0)$ which contains $\mathfrak{C}\mathfrak{Q}$. By Lemma 8.9, $\mathfrak{X}^G \subseteq \mathfrak{N}$ for some G in \mathfrak{G} . Lemma 8.8 (iv) is violated.

LEMMA 8.11. *Hypothesis 7.2 is satisfied with $p = 2$. Furthermore, \mathfrak{M} has the following properties:*

- (i) S_3 -subgroups of \mathfrak{M} are noncyclic.
- (ii) \mathfrak{M} is a 2, 3-group.
- (iii) \mathfrak{M} contains no elementary subgroup of order 27.
- (iv) $m(\mathfrak{M}_0) \leq 2$ for every 3-subgroup \mathfrak{M}_0 of \mathfrak{M} .

Proof. Let \mathfrak{X} be a 2, 3-subgroup of \mathfrak{G} which contains elements of $\mathcal{S}(2)$ and $\mathcal{S}(3)$; \mathfrak{X} is available by hypothesis (vii) of Theorem 8.1. We assume without loss of generality that \mathfrak{X} is a maximal 2, 3-subgroup of \mathfrak{G} . By Lemma 8.8 (i), \mathfrak{X} is contained in no conjugate of \mathfrak{N} . By Lemma 8.9, \mathfrak{X} contains no conjugate of \mathfrak{B} . This fact together with maximality of \mathfrak{X} implies that $O_3(\mathfrak{X}) = 1$.

Let \mathfrak{C} be a subgroup of \mathfrak{X} of type (3, 3) and let \mathfrak{X}_3 be a S_3 -subgroup of \mathfrak{X} containing \mathfrak{C} . By Lemma 8.10, \mathfrak{C} centralizes $Z(O_2(\mathfrak{X}))$. Hence, $\Omega_1(\mathfrak{X}_3)$ centralizes $Z(O_2(\mathfrak{X}))$. By Lemma 8.9, each $S_{2,3}$ -subgroup of $N(\Omega_1(\mathfrak{X}_3))$ is contained in a conjugate of \mathfrak{N} . Hence, \mathfrak{X}_2 centralizes $Z(O_2(\mathfrak{X}))$ by Lemma 8.8 (iv). By hypothesis (iv) of Theorem 8.1, $\mathfrak{X} \cdot C(Z(O_2(\mathfrak{X})))$ is solvable, so by maximality of \mathfrak{X} , we conclude that \mathfrak{X} is a $S_{2,3}$ -subgroup of $\mathfrak{X}C(Z(O_2(\mathfrak{X})))$. Hence, we can choose a S_2 -subgroup \mathfrak{P}_2 of \mathfrak{G} such that $\mathfrak{P}_2 \cap \mathfrak{X} = \mathfrak{X}_2$ is a S_2 -subgroup of \mathfrak{X} , and be guaranteed that $Z(\mathfrak{P}_2) \subseteq Z(O_2(\mathfrak{X}))$. Hence, $\mathfrak{X} \subseteq C(Z(\mathfrak{P}_2))$, so by maximality of \mathfrak{X} , we have $\mathfrak{X}_2 = \mathfrak{P}_2$.

By Lemma 7.4, $\Omega_1(Z(O_2(\mathfrak{X}))) = \Omega_1(Z(\mathfrak{X}_2))$ is of order 2 and

$$N(\Omega_1(Z(\mathfrak{X}_2))) = \mathfrak{M} \in \mathcal{MS}(G).$$

By construction, $\mathfrak{X}_3 \subseteq \mathfrak{M}$, so (i) is satisfied. By Lemma 7.5, $O_2(\mathfrak{M}) = \mathfrak{H}$ is of symplectic type with $w \leq 4$. (ii) is an easy consequence of this fact together with (i).

Suppose \mathfrak{C} is an elementary subgroup of \mathfrak{M} of order 27. Clearly, the width of \mathfrak{H} is at least 3. By Lemma 7.5, no element of $\mathfrak{C}^\#$ centralizes any four-subgroup of \mathfrak{H} . This is obviously impossible.

It remains to prove (iv). By Lemma 7.5 (c), \mathfrak{M}_0 is isomorphic to a subgroup \mathfrak{M}_1 of $(Z_3 \wr Z_3) \times Z_3$. By Lemma 8.11 (iii), the intersection of \mathfrak{M}_1 with the normal abelian subgroup \mathfrak{A} such that $m(\mathfrak{A}) = 4$ in $(Z_3 \sim Z_3) \times Z_3$, is of order at most 3^2 . It follows that \mathfrak{M}_0 is either trivial, abelian of type (3), (3, 3) or $(3^2, 3)$, or non abelian of order 3^3 . In all cases, $m(\mathfrak{M}_0) \leq 2$. The proof is complete.

Let \mathfrak{C} be a subgroup of \mathfrak{M} of type (3, 3), let \mathfrak{X}_3 be a S_3 -subgroup of \mathfrak{M} containing \mathfrak{C} , and let $\mathfrak{H}_1 = [\mathfrak{H}, \mathfrak{C}]$, where $\mathfrak{H} = O_2(\mathfrak{M})$. Let I be the involution of \mathfrak{H}' . Choose C in $\mathfrak{C}^\#$ so that $C_{\mathfrak{H}_1}(C) = \mathfrak{D}$ is not centralized by \mathfrak{C} . We may assume that $C_{\mathfrak{M}}(C) \subseteq \mathfrak{N}$, since replacing \mathfrak{M} by a suitable conjugate guarantees this. Let \mathfrak{X} be a $S_{2,3}$ -subgroup of \mathfrak{N} containing $\mathfrak{C}\mathfrak{D}$. This notation is fixed throughout the concluding argument.

LEMMA 8.12. (i) \mathfrak{D} is a quaternion group.

(ii) \mathfrak{N} is a 2, 3-group.

(iii) $\mathfrak{N} \in \mathcal{MS}(\mathfrak{G})$ and \mathfrak{N} is the only element of $\mathcal{MS}(\mathfrak{G})$ which contains \mathfrak{B} .

(iv) \mathfrak{N} is the only element of $\mathcal{MS}(\mathfrak{G})$ which contains \mathfrak{C} .

Proof. Since \mathfrak{H}_1 is extra special, so is Ω . Since $\Omega \subseteq O^3(\mathfrak{M})$, Lemma 8.8 and Lemma 5.27 imply (i)

By (i) and Lemma 8.8, Ω is a S_2 -subgroup of $O^3(\mathfrak{M})$. Clearly, since \mathfrak{M} is a 2, 3-group, $N_{\mathfrak{M}}(\Omega)$ is a 2, 3-group. Since $\mathbb{C}\Omega \subseteq O^3(\mathfrak{M})$, it follows that $\Omega \subseteq O^3(\mathfrak{M})'$. Hence, Ω has a normal complement \mathfrak{R} in $O^3(\mathfrak{M})'$. To prove (ii), it suffices to show that \mathfrak{R} is a 3-group. Let \mathfrak{R}_0 be a S_3 -subgroup of \mathfrak{R} normalized by Ω . Then I inverts \mathfrak{R}_0 since \mathfrak{M} is a 2, 3-group. Choose \mathfrak{F} char $O_3(\mathfrak{R})$ with $\ker(\mathfrak{R} \rightarrow \text{Aut}(\mathfrak{F}))$ a 3-group, and with \mathfrak{F} of exponent 3. Such an \mathfrak{F} is available by Lemma 5.18 and 0.3.6. As Ω is nonabelian, \mathfrak{R}_0 is noncyclic. It follows readily that I centralizes a subgroup of $\mathfrak{F}/D(\mathfrak{F})$ of order 27. This implies that $C_{\mathfrak{F}}(I)$ contains an elementary subgroup of order 27, in violation of Lemma 8.11. (ii) is proved.

Let $\mathfrak{P} \subseteq \mathfrak{N}_1 \in \mathcal{MS}(\mathfrak{G})$. By Hypothesis 8.2, it suffices to show that $C_{\mathfrak{N}_1}(\mathfrak{B}) \subseteq \mathfrak{N}$, where $\mathfrak{B} = \Omega_1(\mathbf{Z}(\mathfrak{P}))^{\mathfrak{N}_1}$. Since $C_{\mathfrak{N}_1}(\mathfrak{B}) \subseteq N(\Omega_1(\mathbf{Z}(\mathfrak{P})))$, and since $I \in N(\mathfrak{P}) \subseteq N(\Omega_1(\mathbf{Z}(\mathfrak{P})))$, we may replace \mathfrak{N}_1 by an element of $\mathcal{MS}(\mathfrak{G})$ which contains $N(\Omega_1(\mathbf{Z}(\mathfrak{P})))$ and so assume that $I \in \mathfrak{N}_1$.

Let \mathfrak{B} be a S_3 -subgroup of \mathfrak{N}_1 which contains I . By Lemma 8.8 (i) and Lemma 8.9, it follows that \mathfrak{B} has a normal 2-complement \mathfrak{R} . Since \mathfrak{M} is a 2, 3-group, I inverts \mathfrak{R} . Suppose by way of contradiction that $\mathfrak{R} \neq 1$. Since $O_3(\mathfrak{N}_1) = 1$, $\mathfrak{R}\langle I \rangle$ is faithfully represented as automorphisms of $O_3(\mathfrak{N}_1)$. By Lemma 8.11 (iv), the only possibility is that $|\mathfrak{R}| = 5$, that $C(\mathfrak{R}) \cap O_3(\mathfrak{N}_1) \cong D(O_2(\mathfrak{N}_1))$ and that $O_3(\mathfrak{N}_1)/C(\mathfrak{R}) \cap O_3(\mathfrak{N}_1)$ is elementary of order 3^4 . Since \mathfrak{R} is an S -subgroup of \mathfrak{N}_1 , it follows that $\mathfrak{R}O_5(\mathfrak{N}_1)/O_5(\mathfrak{N}_1)$ is a chief factor of \mathfrak{N}_1 . Hence, $I \notin \mathfrak{N}_1'$. This implies that $O_3(\mathfrak{N}_1) = \mathfrak{B}$, so that $\mathfrak{N}_1 \subseteq N(\mathfrak{B}) = \mathfrak{N}$. Hence, $\mathfrak{N}_1 = \mathfrak{N}$. This is absurd since $I \in \mathfrak{N}'$. This contradiction forces $\mathfrak{R} = 1$, that is, \mathfrak{N}_1 is a 2, 3-group.

Since $|C(\Omega_1(\mathbf{Z}(\mathfrak{P})))|$ is odd, it follows that $C_{\mathfrak{N}_1}(\mathfrak{B}) = C_{\mathfrak{P}}(\mathfrak{B}) \subseteq \mathfrak{P} \subseteq \mathfrak{N}$. Thus, (iii) holds.

We turn to (iv). Let $\mathcal{S} = \{\mathfrak{P}_0 \mid \text{(i) } \mathfrak{P}_0 \text{ is a 3-subgroup of } N, \text{(ii) } \mathfrak{P}_0 \cong \mathfrak{B}^N \text{ for some } N \text{ in } \mathfrak{N}, \text{(iii) } \mathfrak{P}_0 \text{ is contained in a solvable subgroup of } \mathfrak{G} \text{ which is not contained in } \mathfrak{N}\}$. Suppose by way of contradiction that $\mathcal{S} \neq \emptyset$. Choose \mathfrak{P}_0 in \mathcal{S} with $|\mathfrak{P}_0|$ maximal. We assume without loss of generality that $\mathfrak{P}_0 \subseteq \mathfrak{P}$. Let \mathfrak{R} be a solvable subgroup of \mathfrak{G} which contains \mathfrak{P}_0 and is minimal subject to $\mathfrak{R} \not\subseteq \mathfrak{N}$. Since $\mathfrak{P} \in \mathcal{S}$, it follows that $\mathfrak{P}_0 \subset \mathfrak{P}$, so maximality of $|\mathfrak{P}_0|$ forces $N_{\mathfrak{P}}(\mathfrak{P}_0) \in \mathcal{S}$. In particular, $N(\mathfrak{P}_0) \subseteq \mathfrak{N}$. This implies that \mathfrak{P}_0 is a S_p -subgroup of \mathfrak{R} . Minimality of \mathfrak{R} yields that $\mathfrak{R} = \mathfrak{P}_0\mathfrak{R}_1$, where \mathfrak{R}_1 is a q -group for some prime $q \neq 3$.

Since $\mathfrak{B}^N \subseteq \mathfrak{P}_0$ for some N in \mathfrak{N} , it follows that $O_3(\mathfrak{R}) \subseteq \mathfrak{N}$, as $O_3(\mathfrak{R})$ is generated by its subgroups $O_3(\mathfrak{R}) \cap C(B)$, $B \in (\mathfrak{B}^N)^{\#}$.

Suppose $q = 2$. Then by Lemma 8.9, $\mathfrak{R} \subseteq \mathfrak{N}^G$ for some G in \mathfrak{G} . Hence $\mathfrak{P}_0 \subseteq \mathfrak{N}^G$. Let \mathfrak{P}^* be a S_3 -subgroup of $\mathfrak{N} \cap \mathfrak{N}^G$ which contains \mathfrak{P}_0 .

Maximality of $|\mathfrak{P}_0|$ forces $\mathfrak{P}_0 = \mathfrak{P}^*$. But then since $N(\mathfrak{P}_0) \subseteq \mathfrak{N}$, we get that \mathfrak{P}_0 is a S_3 -subgroup of \mathfrak{N}^G . This is absurd. Hence, $q \neq 2$.

It is a consequence of [43] that $\mathfrak{R} = O_3(\mathfrak{R})\mathfrak{A}_1\mathfrak{A}_2$, where

$$\mathfrak{A}_1 = C_{\mathfrak{R}}(\mathbf{Z}(\mathfrak{P}_0)), \mathfrak{A}_2 = N_{\mathfrak{R}}(\mathbf{J}(\mathfrak{P}_0)).$$

Maximality of $|\mathfrak{P}_0|$ forces $N(\mathbf{Z}(\mathfrak{P}_0)) \subseteq \mathfrak{N}$, $N(\mathbf{J}(\mathfrak{P}_0)) \subseteq \mathfrak{N}$, so $\mathfrak{R} \subseteq \mathfrak{N}$. This establishes (iv).

We may now complete the proof of Theorem 8.1. Choose C_1 in $\mathbb{C}^\#$. Then $C(C_1) \cong \mathfrak{B}$, so that $C(C_1) \subseteq \mathfrak{N}$. Hence, $\mathfrak{G} \subseteq \mathfrak{N}$, in violation of Lemma 8.8 (ii).

9. A characterization of $S_4(3)$.

THEOREM 9.1. *$S_4(3)$ is the only simple group \mathfrak{G} with the following properties:*

- (i) \mathfrak{G} contains an elementary subgroup of order 27.
- (ii) If \mathfrak{P} is a S_3 -subgroup of \mathfrak{G} and $\mathfrak{A} \in \mathcal{S}_{m_3}(\mathfrak{P})$, then $\mathfrak{N}(\mathfrak{A})$ is trivial.
- (iii) The center of a S_3 -subgroup of \mathfrak{G} is cyclic.
- (iv) The normalizer of every nonidentity 3-subgroup of \mathfrak{G} is solvable.
- (v) S_2 -subgroups of \mathfrak{G} contain normal elementary subgroups of order 8.
- (vi) If \mathfrak{X} is a S_2 -subgroup of \mathfrak{G} and $\mathfrak{B} \in \mathcal{S}_{m_2}(\mathfrak{X})$, then $\mathfrak{N}(\mathfrak{B})$ is trivial.
- (vii) The centralizer of every involution of \mathfrak{G} is solvable.
- (viii) $2 \sim 3$. (See Definition 2.9.).

After careful translation, it can be shown that Dickson [12] lists several properties of $S_4(3)$. Namely, $A(4, 3)$ is Dickson's notation for $S_4(3)$ (pp. 89-100). Now in § 194 (pp. 109-191), Dickson sets $FO(m, p^n) = O'(m, p^n)$ (for m odd), so by § 189 (pp. 179-183), $A(4, 3) \cong FO(5, 3) \cong S_4(3)$. Thus, by § 270 (pp. 292-293), $S_4(3)$ has a subgroup of index 27 which is a split extension of an elementary group of order 16 by A_5 . So $S_4(3)$ is not an N -group. That $S_4(3)$ satisfies the hypothesis of Theorem 9.1 is left as an exercise. We remark that (viii) holds for $S_4(3)$, the centralizers of suitable involutions exhibiting $2 \sim 3$.

Throughout most of this section the following notation is used: \mathfrak{P} is a S_3 -subgroup of \mathfrak{G} ,

$$\mathfrak{Z} = \Omega_1(\mathbf{Z}(\mathfrak{P})), \mathfrak{N} = N(\mathfrak{Z}), \mathfrak{G} = O_3(\mathfrak{N}).$$

By hypothesis (iii), $|\mathfrak{Z}| = 3$, and by hypothesis (ii), $O_3(\mathfrak{N}) = 1$. By hypothesis (iv), \mathfrak{N} is solvable, so by Lemma 1.2.3 of [26], $C_{\mathfrak{N}}(\mathfrak{G}) = \mathbf{Z}(\mathfrak{G})$.

Clearly, $C_{\mathfrak{N}}(\mathfrak{G}) = C(\mathfrak{G})$.

We remark that \mathfrak{G} satisfies Hypothesis 7.4 and also satisfies Hypothesis 7.1 for $p = 2$ and for $p = 3$.

HYPOTHESIS 9.1. \mathfrak{G} is the central product of a cyclic group and a nonabelian group of order 27 and exponent 3.

LEMMA 9.1. *Assume that Hypothesis 9.1 is satisfied. Then*

- (i) $|\mathfrak{P}:\mathfrak{G}| = 3$.
- (ii) $|\mathfrak{G}| = 27$.
- (iii) $O^3(\mathfrak{N})/\mathfrak{G} \cong SL(2, 3)$.

Proof. We remark that $GL(2, 3)$ contains no noncyclic abelian subgroup of order 8.

As $\mathfrak{N}/\mathfrak{G}$ is faithfully represented on $\Omega_1(\mathfrak{G})/D(\Omega_1(\mathfrak{G}))$, it follows that \mathfrak{N} is a 2, 3-group, and $\mathfrak{N}/\mathfrak{G}$ is isomorphic to a subgroup of $GL(2, 3)$. As \mathfrak{G} contains no elementary subgroup of order 27 and \mathfrak{P} does, (i) holds.

Let \mathfrak{A} be a normal elementary subgroup of \mathfrak{P} of order 27. Since $|\mathfrak{P}:\mathfrak{G}| = 3$, it follows that $O^3(\mathfrak{N})/\mathfrak{G} \cong SL(2, 3)$, yielding (iii). Let \mathfrak{Q} be a S_2 -subgroup of $O^3(\mathfrak{N})$ so that \mathfrak{Q} is a quaternion group. Let $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{G}$. Let I be the involution of \mathfrak{Q} . As I inverts every element of $\Omega_1(\mathfrak{G})/\mathfrak{G}$, it follows that I normalizes \mathfrak{A}_0 . Since I also normalizes \mathfrak{P} , it follows that I normalizes $C_{\mathfrak{P}}(\mathfrak{A}_0) = \langle \mathfrak{A}, Z(\mathfrak{G}) \rangle$. Hence, I normalizes $\Omega_1(C_{\mathfrak{P}}(\mathfrak{A}_0)) = \mathfrak{A}$. Since I centralizes the factor $O^3(\mathfrak{N})/O^3(\mathfrak{N})'\mathfrak{G} \cong \mathfrak{P}/\mathfrak{G}$, it follows that $\mathfrak{A} - \mathfrak{A}_0$ contains an element A_1 such that $A_1^I = A_1$. Since I also centralizes \mathfrak{G} , it follows that $C_{\mathfrak{N}}(I) = \langle A_1 \rangle \times \mathfrak{G} = \mathfrak{A}_1$. Also, $C_{\mathfrak{P}}(I) = \mathfrak{A}_1\mathfrak{G}_1$, where $\mathfrak{G}_1 = Z(\mathfrak{G})$, and it is clear that $C_{\mathfrak{P}}(I)$ is a S_3 -subgroup of $C_{\mathfrak{N}}(I)$.

Suppose $|\mathfrak{G}_1| > 3$. Thus, $|\mathfrak{P}| > 3^4$, so Lemma 7.6 is at our disposal. If $G \in \mathfrak{G}$ and $\mathfrak{Z}_1^G \subseteq \mathfrak{P}$, then \mathfrak{Z}_1^G centralizes \mathfrak{G} , and so $\Omega_1(\mathfrak{Z}_1^G) = \Omega_1(\mathfrak{Z}_1) = \mathfrak{G}$, so that $G \in \mathfrak{N}$, $\mathfrak{Z}_1^G = \mathfrak{Z}_1$. We may therefore apply Theorem 14.4.2 of [21] and conclude that $\mathfrak{P} \subseteq \mathfrak{N}'$. Since $\text{Aut}(\mathfrak{Z}_1)$ is abelian, this implies that $\mathfrak{Z}_1 = Z(\mathfrak{P})$. We may therefore appeal to Lemma 7.6 (d) and conclude that if \mathfrak{A}^* is any subgroup of \mathfrak{A} of type (3, 3), then \mathfrak{A}^* centralizes every element of $\mathfrak{N}(\mathfrak{A}^*; 2)$. Taking $\mathfrak{A}^* = \mathfrak{A}_1$, Lemma 7.4 is violated. This completes the proof of (ii).

LEMMA 9.2. *Assume that Hypothesis 9.1 is satisfied. Let $\mathfrak{A} \in \mathcal{L}_{27}(\mathfrak{P})$ and let I be an involution of \mathfrak{N} . Then*

- (i) S_2 -subgroups of \mathfrak{N} are quaternion.
- (ii) If $\mathfrak{A}_0 = C_{\mathfrak{N}}(I)$, then
 - (a) $|\mathfrak{A}_0| = 9$.
 - (b) \mathfrak{A}_0 contains a subgroup \mathfrak{A}_1 of order 3 such that $C(\mathfrak{A}_1) \not\subseteq \mathfrak{N}$.

- (iii) If $\mathfrak{M} = C(I)$, then $O_2(\mathfrak{M})$ is extra special of width 2, $O_2(\mathfrak{M}) = 1$, and $|\mathfrak{M}:O_2(\mathfrak{M})|_2 = 2$.
- (iv) $A_{\mathfrak{G}}(\mathfrak{A}) \cong \Sigma_4$.

Proof. Let \mathfrak{Q} be a S_2 -subgroup of $O^{3'}(\mathfrak{N})$. By Lemma 9.1 (i), \mathfrak{Q} is a quaternion group. It clearly suffices to prove the lemma on the assumption that I is the involution of \mathfrak{Q} .

By Lemma 9.1 and hypothesis (i) of Theorem 9.1, the group \mathfrak{F} is $Z_3 \wr Z_3$. Hence, \mathfrak{A} char \mathfrak{F} . Since I normalizes \mathfrak{F} , it therefore normalizes \mathfrak{A} . This implies (ii)(a), since I centralizes $Z(\mathfrak{G})$ and $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$.

Clearly, \mathfrak{M} contains an element of $\mathcal{U}(2)$. It is equally clear from (B) and Lemma 9.1 that

$$(*) \quad \begin{array}{l} \text{if } \mathfrak{X} \text{ is any noncyclic subgroup of } \mathfrak{A}, \text{ then } \mathfrak{X} \\ \text{centralizes every abelian subgroup of } \mathfrak{N}(\mathfrak{X}; 2). \end{array}$$

Let \mathfrak{X} be a $S_{2,3}$ -subgroup of \mathfrak{M} which contains $\langle \mathfrak{A}_0, \mathfrak{Q} \rangle$. Let \mathfrak{X}_2 be a S_2 -subgroup of \mathfrak{X} which contains \mathfrak{Q} . We may apply Lemma 7.5 with \mathfrak{A}_0 in the role of \mathfrak{B} . Thus, there is an element $\tilde{\mathfrak{M}}$ of $\mathcal{MS}(\mathfrak{G})$ satisfying the conclusions of Lemma 7.5. By Lemma 7.5 (e), we get $\mathfrak{Q} \subseteq O_2(\tilde{\mathfrak{M}})$. Hence, $\langle I \rangle \triangleleft \tilde{\mathfrak{M}}$, so $\tilde{\mathfrak{M}} = C(I) = \mathfrak{M}$. Since $\langle I \rangle$ is a S_2 -subgroup of $C(\mathfrak{A}_0)$, it follows that

$$(9.1) \quad O_2(\mathfrak{M}) \text{ is extra special of width } w = 2, 3, \text{ or } 4.$$

Thus, (ii)(b) holds.

Again, let \mathfrak{X} be a noncyclic subgroup of \mathfrak{A} . Suppose that $|C(\mathfrak{X}) \cap N(\mathfrak{A})|$ is even. Then of course $|\mathfrak{X}| = 9$, as \mathfrak{A} is a self-centralizing subgroup of \mathfrak{G} . Let J be an involution of $C(\mathfrak{X}) \cap N(\mathfrak{A})$. Then (*) and Lemma 7.5 yield that J and I are conjugate in \mathfrak{G} . Since \mathfrak{X} is faithfully represented on $O_2(C(J))$, we can choose a subgroup \mathfrak{Y} of \mathfrak{X} of order 3 such that

$$(9.2) \quad \mathfrak{X} \text{ does not centralize } C(\mathfrak{Y}) \cap O_2(C(J)).$$

Thus, $C(\mathfrak{Y})$ is not 3-closed. Thus, \mathfrak{A} is not a S_3 -subgroup of $C(\mathfrak{Y})$. This implies that

$$(9.3) \quad C(\mathfrak{Y}) \text{ contains a } S_3\text{-subgroup of } \mathfrak{G}.$$

Let $\tilde{\mathfrak{F}}$ be a S_3 -subgroup of $C(\mathfrak{Y})$ which contains \mathfrak{A} . Thus $\langle \tilde{\mathfrak{F}}, J \rangle \subseteq C(\mathfrak{Y})$. Thus, J normalizes both \mathfrak{A} and $O_3(C(\mathfrak{Y}))$, so J normalizes $\langle \mathfrak{A}, O_3(C(\mathfrak{Y})) \rangle$. Thus, Lemma 9.1 yields that

$$(9.4) \quad \begin{array}{l} \text{if } \mathfrak{X} \text{ is any noncyclic subgroup of } \mathfrak{A}, \text{ then each involution} \\ \text{of } C(\mathfrak{X}) \cap N(\mathfrak{A}) \text{ normalizes some } S_3\text{-subgroup of } N(\mathfrak{A}). \end{array}$$

By (9.3) with the pair $(\mathfrak{A}_1, \mathfrak{A}_0)$ in the role of $(\mathfrak{Y}, \mathfrak{X})$, we conclude that $C(\mathfrak{A}_1)$ contains a S_3 -subgroup \mathfrak{P}^* of \mathfrak{G} with $\mathfrak{A} \subset \mathfrak{P}^*$. Hence, $N(\mathfrak{A})$ is not 3-closed, since \mathfrak{P} and \mathfrak{P}^* are distinct S_3 -subgroups of $N(\mathfrak{A})$.

Set $\tilde{\mathfrak{N}} = N(\mathfrak{A})$. Clearly,

$$\mathfrak{A} = O_3(\tilde{\mathfrak{N}}) = C(\mathfrak{A}), 1 = O_3(\tilde{\mathfrak{N}}), I \in \tilde{\mathfrak{N}}.$$

Suppose $13 \mid |\tilde{\mathfrak{N}}|$. Since $I \in \tilde{\mathfrak{N}}$, it follows that I centralizes a S_{13} -subgroup of $\tilde{\mathfrak{N}}$, since the nonidentity 13-elements of $GL(3, 3)$ are nonreal. However, $13 \nmid |\mathfrak{M}|$, since $4 \geq w$. Hence, $\tilde{\mathfrak{N}}$ is a 2, 3-group.

Let \mathfrak{X}_0 be a S_2 -subgroup of $O_{3,2}(\tilde{\mathfrak{N}})$, and let $\mathfrak{R} = N_{\tilde{\mathfrak{N}}}(\mathfrak{X}_0)$. Thus, $\tilde{\mathfrak{N}} = \mathfrak{A}\mathfrak{R}$, $\mathfrak{A} \cap \mathfrak{R} = 1$, so that $\mathfrak{R} \cong A_{\mathfrak{G}}(\mathfrak{A})$. Suppose J is an involution of \mathfrak{X}_0 and $\mathfrak{A} \cap C(J) = \mathfrak{X}$ is noncyclic. Thus, $|\mathfrak{X}| = 9$. By (9.4), J normalizes some S_3 -subgroup \mathfrak{P} of $\tilde{\mathfrak{N}}$. Since $\mathfrak{P} \supset \mathfrak{A}$, $\mathfrak{P} \cap \mathfrak{R}$ is of order 3. Hence, $[\mathfrak{P} \cap \mathfrak{R}, J] \subseteq \mathfrak{X}_0 \cap \mathfrak{P} = 1$, so that $\mathfrak{P} \cap \mathfrak{R}$ centralizes J . Hence, $\mathfrak{P} \cap \mathfrak{R}$ normalizes \mathfrak{X} , that is, $\mathfrak{X} \triangleleft \mathfrak{P}$. Hence, $\mathfrak{X} \in \mathcal{Z}(3)$. Thus, J centralizes elements of $\mathcal{Z}(3)$ and $\mathcal{Z}(2)$. This violates the solvability of $C(J)$. Hence,

(9.5) no element of \mathfrak{X}_0^* centralizes a noncyclic subgroup of \mathfrak{A} .

Since $|Z(\mathfrak{P})| = 3$, $\mathfrak{P} \cap \mathfrak{R}$ is indecomposable on \mathfrak{A} . Hence, \mathfrak{X}_0 acts on \mathfrak{A} as a multiple of the sum of the \mathfrak{R} -conjugates of a fixed F_3 -irreducible representation ρ . If \mathfrak{X}_0 is nonabelian, then 2 divides $\deg \rho$. Hence, 2 divides $3 = m(\mathfrak{A})$, a contradiction. Hence, \mathfrak{X}_0 is abelian. If \mathfrak{X}_0 is not elementary, then $\deg \rho \neq 1$ or 3. So $\deg \rho = 2$, which again gives a contradiction. Hence, \mathfrak{X}_0 is elementary. Now (9.5) implies that $|\mathfrak{X}_0| \leq 4$, so we must have equality, since $\mathfrak{X}_0 = O_2(\mathfrak{R})$ and $|\mathfrak{P} \cap \mathfrak{R}| = 3$. Since $I \in \mathfrak{N} \cap N(\mathfrak{P})$, it follows that $\mathfrak{R} \cong \Sigma_4$, which establishes (iv) and also (i).

It remains to show that $w = 2$ and that $|\mathfrak{M}: O_2(\mathfrak{M})|_2 = 2$, since by (9.1) we know that as $O_2(\mathfrak{M})$ is extra special.

Suppose \mathfrak{A} is any subgroup of \mathfrak{A} of order 3 which is conjugate to \mathfrak{Z} , $\mathfrak{A}^*_{\mathfrak{G}} \mathfrak{Z}$. We contend that $\mathfrak{A}^*_{N(\mathfrak{A})} \mathfrak{Z}$. Namely, let \mathfrak{P}^* be a S_3 -subgroup of $C(\mathfrak{A}^*)$ which contains \mathfrak{A} . Then \mathfrak{P} and \mathfrak{P}^* both normalize \mathfrak{A} , since $|\mathfrak{P}^* : \mathfrak{A}| = 3$. We may thus choose N in $N(\mathfrak{A})$ such that $\mathfrak{P}^{*N} = \mathfrak{P}$; since $\mathfrak{Z} = Z(\mathfrak{P})$, we necessarily have $\mathfrak{A}^{*N} = \mathfrak{Z}$, as desired.

Since $\mathfrak{P} \langle I \rangle$ normalizes \mathfrak{Z} , we obtain all $N(\mathfrak{A})$ -conjugates of \mathfrak{Z} by transformation with elements of \mathfrak{X}_0 . We will show that \mathfrak{Z} and \mathfrak{A}_1 are the only $N(\mathfrak{A})$ -conjugates of \mathfrak{Z} which are in \mathfrak{A}_0 . If $K \in \mathfrak{X}_0^*$ and $\mathfrak{Z}^K \subseteq \mathfrak{A}_0$, then since no element of \mathfrak{X}_0^* normalizes \mathfrak{Z} , we conclude that K normalizes \mathfrak{A}_0 . It is clear that \mathfrak{X}_0 does not normalize \mathfrak{A}_0 , so our assertion follows.

It is an immediate consequence of the preceding paragraph that $w = 2$. That is, only \mathfrak{Z} and \mathfrak{A}_1 centralize elements of $O_2(\mathfrak{M}) - \langle I \rangle$. Since $N(\mathfrak{A}_0) \subseteq N(\mathfrak{A})$, we have $|\mathfrak{M}: O_2(\mathfrak{M})|_2 = 2$, and the proof is complete.

We now change notation somewhat in order to conform with more standard notation. Let $\mathfrak{B}_1 = N(\mathfrak{B})$, $\mathfrak{B}_2 = N(\mathfrak{A})$, and let $\mathfrak{B} = \mathfrak{P}\langle I \rangle$, $\mathfrak{G} = \langle I \rangle$. Let \mathfrak{Q}_1 be a S_2 -subgroup of \mathfrak{B}_1 which contains I , and let \mathfrak{Q}_2 be a S_2 -subgroup of \mathfrak{B}_2 which contains I . Thus, \mathfrak{Q}_1 is a quaternion group and \mathfrak{Q}_2 is a dihedral group of order 8. Let $\mathfrak{X}_2 = \mathfrak{Q}_2 \cap \mathcal{O}^{3'}(\mathfrak{B}_2)$, so that \mathfrak{X}_2 is a four-group.

Let $\mathfrak{C}_2 = N_{\mathfrak{B}_2}(\mathfrak{X}_2)$. Thus, \mathfrak{C}_2 is a complement to \mathfrak{A} in \mathfrak{B}_2 and $\mathfrak{C}_2 \cong \Sigma_4$. Let $\mathfrak{X}_1 = \mathfrak{P} \cap \mathfrak{C}_2$, so that \mathfrak{X}_1 is of order 3 and is inverted by I . Since $\mathcal{O}_3(\mathfrak{B}_1)$ contains all elements of \mathfrak{P} which are inverted by I , we have $\mathfrak{X}_1 \subseteq \mathcal{O}_3(\mathfrak{B}_1)$. Since \mathfrak{Q}_1 permutes transitively the subgroups of $\mathcal{O}_3(\mathfrak{B}_1)/\mathfrak{B}$ of order 3, we may choose Q in \mathfrak{Q}_1 so that $\mathfrak{X}_2 = \mathfrak{X}_1^Q$ lies in \mathfrak{A} . Thus, I inverts \mathfrak{X}_2 , since Q centralizes I . Let $\langle J \rangle = \mathfrak{X}_2 \cap C(I)$, that is, let J be a generator for $Z(\mathfrak{Q}_2)$. Since $\mathfrak{A} \cap C(I)$ is of order 9, \mathfrak{X}_2 is the only subgroup of \mathfrak{A} of order 3 which is inverted by I , so that $\mathfrak{X}_2^J = \mathfrak{X}_2$. Let $\mathfrak{X}_4 = \mathfrak{B}$ and set $\mathfrak{X}_3 = \mathfrak{X}_1^J$. We may now draw up the following table:

	J	Q
\mathfrak{X}_1	—	\mathfrak{X}_2
\mathfrak{X}_2	\mathfrak{X}_2	\mathfrak{X}_1
\mathfrak{X}_3	\mathfrak{X}_4	—
\mathfrak{X}_4	\mathfrak{X}_3	\mathfrak{X}_4 .

Let X_i be a generator for \mathfrak{X}_i , so that we have the following table:

	I
X_1	X_1^{-1}
X_2	X_2^{-1}
X_3	X_3
X_4	X_4 .

Let $\mathfrak{N} = \langle J, Q \rangle$. Since $\mathfrak{N} \subseteq \mathfrak{M}$, the structure of \mathfrak{N} may be easily determined. Let $\mathfrak{Q}_1, \mathfrak{Q}_1^*$ be the quaternion subgroups of $\mathcal{O}_3(\mathfrak{M})$, \mathfrak{Q}_1 being as above. As J normalizes $\langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle$ and $\langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle$ is a S_3 -subgroup of \mathfrak{M} , it follows that $\mathfrak{Q}_1^J = \mathfrak{Q}_1^*$. Hence, $(JQ)^2 = JQJQ$ is an involution distinct from I . This means that $\mathfrak{N}/\langle I \rangle$ is a dihedral group of order 8 with involutory generators $J\langle I \rangle, Q\langle I \rangle$.

Finally, notice that $\mathfrak{B} = \mathfrak{P}\langle I \rangle = N(\mathfrak{P})$.

Since $(JX_1)^3 = 1$ and since $(\mathfrak{Q}X_3)^3 \in \langle I \rangle$, it is straightforward to deduce from the first table that $\mathfrak{B}\mathfrak{N}\mathfrak{B} = \mathfrak{G}_1$ is a group. We will determine the multiplication table of \mathfrak{G}_1 . First, we assume without

loss of generality that $(QX_3)^3 = 1$, since replacement of Q by $Q^{-1} = QI$ will achieve this if $(QX_3)^3 = I$. Since I inverts X_1 and centralizes X_3 , it follows easily that I neither inverts nor centralizes $[X_1, X_3]$. Thus, we may choose X_2, X_4 as generators for $\mathfrak{x}_2, \mathfrak{x}_4$ respectively such that

$$(9.6) \quad [X_1, X_3] = X_2X_4.$$

By construction, $\mathfrak{x}_4 = \mathfrak{z} = \mathbf{Z}(\mathfrak{B})$, so to complete the determination of \mathfrak{B} , we must compute $[X_1, X_2]$. Conjugation of (9.6) by I yields $[X_1^{-1}, X_3] = X_2^{-1}X_4$, from which we find easily that $[X_1, X_2] = X_4$.

Let $X_1^Q = X_2^a$. Since $(QX_3)^3 = 1$, an easy calculation (conjugation of (9.6) by Q) shows that $a = +1$. Since $J \in \mathfrak{X}_2 \triangleleft \mathfrak{C}_2$, it follows that $C_{\mathfrak{B}}(J)$ has order 3. Since J normalizes but does not invert $\langle X_3, X_4 \rangle$, it follows that $C_{\mathfrak{B}}(J) \subseteq \langle X_3, X_4 \rangle$. Hence, $X_2^J = X_2^{-1}$. Let $X_3^J = X_4^b$. Since $(X_1J)^3 = 1$, an easy calculation (conjugation of (9.6) by J) shows that $b = -1$.

Set $W_0 = (JQ^2)$. We argue that

$$(9.7) \quad \mathfrak{B} \cap \mathfrak{B}^{W_0} = 1.$$

Suppose by way of contradiction that 9.7 is false. Since W_0 is an involution, $\mathfrak{D} = \mathfrak{B} \cap \mathfrak{B}^{W_0}$ is normalized by W_0 . Since $W_0 \in \mathbf{Z}(\mathfrak{N})$, it follows that $\mathfrak{G} = \langle I \rangle$ also normalizes \mathfrak{D} . Since $C_{\mathfrak{B}}(I) = \langle X_3, X_4 \rangle$ and since $W_0 \in \mathfrak{M}$, it follows from the construction of \mathfrak{N} that I inverts \mathfrak{D} . Thus, $\mathfrak{D} \subset \mathcal{O}_3(\mathfrak{B}_1)$, since $\mathcal{O}_3(\mathfrak{B}_1)$ contains all the elements of \mathfrak{B} which are inverted by I . As \mathfrak{D} is abelian, and as I centralizes $\mathfrak{x}_4 = \mathbf{Z}(\mathcal{O}_3(\mathfrak{B}_1))$, it follows that $|\mathfrak{D}| = 3$. There are exactly 4 subgroups of $\mathcal{O}_3(\mathfrak{B}_1)$ of order 3 which are inverted by I ; they are all of the shape $\mathfrak{x}_2^{Q^*}$ for some Q^* in \mathfrak{D}_1 . Since I normalizes \mathfrak{x}_2 and since $\mathfrak{D}_1 = \langle Q \rangle \cup \langle Q^{X_3} \rangle \cup \langle Q^{X_3^{-1}} \rangle$, we may assume that $\mathfrak{D} = \mathfrak{x}_2^{Q^*}$, where Q^* is one of $1, Q, X_3^{-1}QX_3, X_3QX_3^{-1}$. Since W_0 normalizes \mathfrak{D} , we get that $Q^*W_0Q^{*-1} \in N(\mathfrak{x}_2)$. Since $Q^* \in \mathfrak{M}$ and $W_0 \in \mathcal{O}_2(\mathfrak{M})$, we get that $Q^*W_0Q^{*-1} \in N(\mathfrak{x}_2) \cap \mathcal{O}_2(\mathfrak{M}) = \mathfrak{Z}$, say. Since I inverts X_2 , it follows that $I \notin D(\mathfrak{Z})$. Thus, \mathfrak{Z} is elementary. But 1 and $\langle I \rangle$ are the only elementary subgroups of $\mathcal{O}_2(\mathfrak{M})$ which admit $\langle X_3, X_4 \rangle$, so $Q^*W_0Q^{*-1} = \langle I \rangle$. This is not the case, since $Q^* \in \mathfrak{M}$, $I \in \mathbf{Z}(\mathfrak{M})$, and $I \neq W_0$. This proves (9.7).

Set $\mathfrak{B} = \{1, Q, J, QJ, JQ, QJQ, JQJ, W_0\}$, a set of representatives for the cosets of \mathfrak{G} in \mathfrak{N} . For each W in \mathfrak{B} , let $\mathfrak{B}_W = \langle \mathfrak{x}_i \mid 1 \leq i \leq 4, \mathfrak{x}_i^{W^{-1}} \subseteq \mathfrak{B}^{W_0} \rangle$. It follows that condition (iii) of Théorème I of [36] is satisfied, so by that theorem, so is condition (ii), that is, if $W_1, W_2 \in \mathfrak{B}$ and $BW_1B = BW_2B$, then $W_1 = W_2$. In view of our preceding information, we conclude that each element of \mathfrak{G}_1 has a normal form of the shape $PHWP'$, where $P \in \mathfrak{B}, H \in \mathfrak{G}, W \in \mathfrak{B}, P' \in \mathfrak{B}_W$. Furthermore, it is clear that the normal forms for $PHWP'J$ and $PHWP'Q$ are determined by our information. This implies immediately that if \mathfrak{G}^* is any group which satisfies the hypothesis of Theorem 9.1 and Hy-

pothesis 9.1, then \mathfrak{G}^* contains a subgroup \mathfrak{G}_1^* isomorphic to \mathfrak{G}_1 . Taking $\mathfrak{G}^* = S_4(3)$, a comparison of orders yields $\mathfrak{G}_1 \cong S_4(3)$. In particular, $i(\mathfrak{G}_1) = 2$ and I, W_0 are representatives for the two classes of involutions of \mathfrak{G}_1 . Since $\mathfrak{M} = \langle Q_1, J, X_3, X_4 \rangle$, we have $\mathfrak{M} \subseteq \mathfrak{G}_1$. We will show that $C(W_0) \subseteq \mathfrak{G}_1$. Let $\mathfrak{R} = O_2(C_{\mathfrak{G}_1}(W_0))$. Then \mathfrak{R} is elementary of order 2^4 and \mathfrak{R} is characteristic in a S_2 -subgroup of $C_{\mathfrak{G}_1}(W_0)$. Thus, it suffices to show that $N(\mathfrak{R}) = N_{\mathfrak{G}_1}(\mathfrak{R})$. Since $N_{\mathfrak{G}_1}(\mathfrak{R})$ is an extension of \mathfrak{R} by A_5 and since $\mathfrak{R} = C(\mathfrak{R})$, it follows that $A_{\mathfrak{G}_1}(\mathfrak{R})$ is a subgroup of $\text{Aut}(\mathfrak{R}) = L_4(2)$ which contains a subgroup isomorphic to A_5 and has S_2 -subgroups of order 4. Hence, $A_{\mathfrak{G}_1}(\mathfrak{R}) = A_{\mathfrak{G}_1}(\mathfrak{R}) \cong A_5$. Hence, \mathfrak{G}_1 contains the centralizer of each of its involutions. By Lemma 5.35, $\mathfrak{G} = \mathfrak{G}_1$. Thus, Theorem 9.1 is proved in case Hypothesis 9.1 is satisfied.

We now revert to our previous notation.

HYPOTHESIS 9.2. \mathfrak{G} is of symplectic type and width $w \geq 2$.

HYPOTHESIS 9.3. (i) \mathfrak{G} is extra special of order 3^5 .

(ii) $|\mathfrak{P}| = 3^6$.

(iii) \mathfrak{Z} is not weakly closed in \mathfrak{G} .

Lemmas 9.3 through 9.10 are all proved under Hypothesis 9.2. [Notice that Hypothesis 9.3 trivially implies Hypothesis 9.2.

LEMMA 9.3. (i) $C(\mathfrak{Z})$ does not contain a four-group.

(ii) If \mathfrak{Q} is any abelian 2-subgroup of \mathfrak{R} , then $A_{\mathfrak{R}}(\mathfrak{Q})$ is a 2-group.

(iii) If \mathfrak{A} is any subgroup of \mathfrak{G} of type $(3, 3)$ which contains \mathfrak{Z} , then $|C(\mathfrak{A})|$ is odd.

(iv) If \mathfrak{A} is any subgroup of \mathfrak{G} of type $(3, 3)$ which contains \mathfrak{Z} , then $\mathfrak{A} \in \mathcal{E}(3)$.

Proof. Clearly, (i) implies (ii), and (iii) implies (i). Suppose (iv) holds, but I is an involution in $C(\mathfrak{A})$. By Lemma 5.37, $C(I)$ contains an element of $\mathcal{Z}(2)$. By Lemma 7.4, $C(I)$ is nonsolvable. Hence, (iv) implies (iii). To complete the proof of the lemma, it suffices to prove (iv). However, (iv) is a consequence of Lemma 7.2.

LEMMA 9.4. Suppose $B \in \Omega_1(\mathfrak{G}) - \mathfrak{Z}$ and $\mathfrak{G}_0 = C_{\mathfrak{G}}(B)$. Then

$$C(\mathfrak{G}_0) = Z(\mathfrak{G}_0) = \langle B \rangle \times Z(\mathfrak{G}) .$$

Proof. Since $\mathfrak{Z} \subset \mathfrak{G}_0$, it follows that $C(\mathfrak{G}_0) = C_{\mathfrak{R}}(\mathfrak{G}_0)$. Since a S_3 -subgroup of \mathfrak{R} is faithfully represented on \mathfrak{G} , it follows that $C(\mathfrak{G}_0)$ is a 3-group. It suffices to show that $C(\mathfrak{G}_0) \subseteq \mathfrak{G}$. Suppose false and

$C \in C(\mathfrak{G}_0)$, $C \notin \mathfrak{G}$. We may assume that $C^3 \in \mathfrak{G}$. In this case, $\langle C \rangle / \langle C^3 \rangle$ is faithfully represented on $Q_3(\mathfrak{N})$ and by Lemma 5.30, it follows that $[Q_3(\mathfrak{N}), \langle C \rangle] = \tilde{\mathfrak{Q}}$ is a quaternion group. Let \mathfrak{Q} be a subgroup of \mathfrak{N} incident with $\tilde{\mathfrak{Q}}$. Clearly, $\mathfrak{G} = C_{\mathfrak{G}}(\mathfrak{Q})[\mathfrak{Q}, \mathfrak{G}]$ and $C_{\mathfrak{G}}(\mathfrak{Q})$ commutes elementwise with $[\mathfrak{Q}, \mathfrak{G}]$. By Lemma 9.3 (iii), \mathfrak{Q}' centralizes no noncyclic subgroup of \mathfrak{G} . It follows that $C_{\mathfrak{G}}(\mathfrak{Q}) = Z(\mathfrak{G})$ is cyclic. However, $w \geq 2$ and C centralizes \mathfrak{G}_0 .

LEMMA 9.5. *Hypothesis 9.3 is not satisfied.*

Proof. Suppose false.

Let $\mathfrak{X} = \{\mathfrak{Z}_1 \mid \mathfrak{Z}_1 \subseteq \mathfrak{G}, \mathfrak{Z}_1 \sim \mathfrak{Z}, \mathfrak{Z}_1 \neq \mathfrak{Z}\}$. By Hypothesis 9.3 (iii), $\mathfrak{X} \neq \emptyset$. Since $\mathfrak{G} \triangleleft \mathfrak{N}$, \mathfrak{X} is invariant in \mathfrak{N} . Choose $\mathfrak{Z}_1 \in \mathfrak{X}$ such that $C_{\mathfrak{F}}(\mathfrak{Z}_1)$ is a S_3 -subgroup of $C_{\mathfrak{N}}(\mathfrak{Z}_1)$. Let $\mathfrak{F}_0 = C_{\mathfrak{F}}(\mathfrak{Z}_1)$.

If $\mathfrak{F}_0 = C_{\mathfrak{G}}(\mathfrak{Z}_1)$, then \mathfrak{Z} char \mathfrak{F}_0 . This is impossible since \mathfrak{F}_0 is not a S_3 -subgroup of $C(\mathfrak{Z}_1)$. Hence, $|\mathfrak{F}_0| = 3^5$.

Let $\mathfrak{D} = \langle \mathfrak{Z}, \mathfrak{Z}_1 \rangle$, so that $\mathfrak{D} \subseteq Z(\mathfrak{F}_0)$. If $\mathfrak{D} \subset Z(\mathfrak{F}_0)$, then choose $Z \in Z(\mathfrak{F}_0) - \mathfrak{G}$, so that Z centralizes a 3-dimensional subspace of $\mathfrak{G}/\mathfrak{G}'$. This implies that some involution of $O'(\mathfrak{N})$ has a noncyclic fixed point set on \mathfrak{G} , in violation of Lemma 9.3 (iii). Hence, $\mathfrak{D} = Z(\mathfrak{F}_0)$.

Let \mathfrak{F}^* be a S_3 -subgroup of $C(\mathfrak{Z}_1)$ which contains \mathfrak{F}_0 . Thus, $\langle \mathfrak{F}, \mathfrak{F}^* \rangle \subseteq N(\mathfrak{F}_0) \subseteq N(\mathfrak{D})$, so $O'(A_{\mathfrak{G}}(\mathfrak{D})) \cong SL(2, 3)$.

By Lemma 9.3, $|C(\mathfrak{D})|$ is odd. Since \mathfrak{F}_0 is a S_3 -subgroup of $C(\mathfrak{D})$ and since $O_3(C(\mathfrak{D})) = 1$, it follows that $\mathfrak{F}_0 = C(\mathfrak{D})$. Hence, $N(\mathfrak{F}_0) = N(\mathfrak{D})$. Let $\mathfrak{M} = N(\mathfrak{D})$.

Let \mathfrak{Q} be a S_2 -subgroup of $O'(\mathfrak{M})$. Thus, \mathfrak{Q} is a quaternion group. Let J be the involution of \mathfrak{Q} . Let \mathfrak{Q}^* be a S_2 -subgroup of $O'(\mathfrak{M})$. Thus \mathfrak{Q}^* is a quaternion group. Let I be the involution of \mathfrak{Q}^* . Since J inverts $\mathfrak{G}/\mathfrak{G}'$, $J \in \mathfrak{M}$. Since I inverts \mathfrak{D} , $I \in \mathfrak{N}$. We assume without loss of generality that I normalizes \mathfrak{Q} and J normalizes \mathfrak{Q}^* .

Since J neither inverts nor centralizes \mathfrak{D} , it is clear that $A_{\mathfrak{G}}(\mathfrak{D}) \cong GL(2, 3)$ and so $\langle J, \mathfrak{Q}^* \rangle$ is isomorphic to a S_2 -subgroup of $GL(2, 3)$. Let Q^* be an element of \mathfrak{Q}^* of order 4 which is inverted by J .

We will show that $\langle I, \mathfrak{Q} \rangle$ is isomorphic to a S_2 -subgroup of $GL(2, 3)$. Since $I \in \mathfrak{N}$, we need only prove that I inverts $\mathfrak{F}_0/\mathfrak{F}_0 \cap \mathfrak{G}$. Suppose false. Then I centralizes $\mathfrak{F}_0/\mathfrak{F}_0 \cap \mathfrak{G}$. We know that $\mathfrak{D} \subseteq \mathfrak{F}_0$, because \mathfrak{M} operates irreducibly on \mathfrak{D} and \mathfrak{D} contains $\mathfrak{Z} = (\mathfrak{F}_0 \cap \mathfrak{G})'$. Since \mathfrak{Q}^* is faithfully represented on \mathfrak{F}_0 , there must be a 2-dimensional subspace of $\mathfrak{F}_0/D(\mathfrak{F}_0)$ which I inverts and \mathfrak{Q}^* leaves invariant. Since I centralizes $\mathfrak{F}_0/\mathfrak{F}_0 \cap \mathfrak{G}$ and $|\mathfrak{F}_0| = 3^5$, we conclude that $\mathfrak{D} = \mathfrak{F}_0' = D(\mathfrak{F}_0)$, and that $\mathfrak{F}_0 \cap \mathfrak{G}/\mathfrak{D}$ is the subspace of $\mathfrak{F}_0/\mathfrak{F}_0'$ inverted by I . This forces I to invert both $\mathfrak{F}_0 \cap \mathfrak{G}/\mathfrak{D}$ and \mathfrak{D} , that is, to invert $\mathfrak{F}_0 \cap \mathfrak{G}$. So $\mathfrak{F}_0 \cap \mathfrak{G}$ is abelian, which is false.

Let Q be an element of \mathfrak{Q} of order 4 which is inverted by I .

Set $\mathfrak{x}_6 = \mathfrak{B}$. Since J normalizes \mathfrak{D} and centralizes \mathfrak{x}_6 , we can choose an element X_5 of \mathfrak{D} of order 3 such that $X_5^J = X_5^{-1}$. Let $\mathfrak{x}_5 = \langle X_5 \rangle$, $\mathfrak{x}_4 = \mathfrak{x}_5^Q$, $X_4 = X_5^Q$. Then we have relations $X_5^I = X_5^{-1}$, $X_5^J = X_5^{-1}$, $X_4^I = X_4$, $X_4^J = \mathfrak{x}_4^{-1}$. Suppose $[X_4, X_5] \neq 1$. The following argument is designed to exclude this possibility.

Let $[X_4, X_5] = X_6$ so that X_6 is a generator for \mathfrak{x}_6 . Since $X_4 \notin \mathfrak{P}_0$, it follows that $\mathfrak{P}_0 \cap C(I)$ is of order 3 with generator X_3 , say. Thus, $\langle [X_3, X_4] \rangle = C_{\mathfrak{P}_0}(I)$ is of order 9, so that $[X_3, X_4] = 1$. As \mathfrak{S} contains all the elements of \mathfrak{P} which are centralized by I , we have $X_3 \in H$. Since $\langle X_3 \rangle = C_{\mathfrak{P}_0}(I)$, it follows that J normalizes $\langle X_3 \rangle$, so that $X_3^J = X_3^{-1}$, as J inverts $\mathfrak{S}/\mathfrak{B}$. Let $X_2 = X_3^Q$. Since $[X_3, X_4] = 1$, so also $[X_2, X_4^Q] = 1$. But $X_4^Q = X_5^{Q^2} = X_5^{-1}$, so X_2 centralizes X_5 , that is, $X_2 \in \mathfrak{P}_0$. Let $X_1 = X_2^{Q^*}$, and let $\mathfrak{x}_i = \langle \mathfrak{x}_i \rangle$, $1 \leq i \leq 6$. We obtain the following data:

Table 1			Table 2		
	J	I		Q	Q^*
X_1	X_1	X_1^{-1}	X_1	—	X_2^{-1}
X_2	X_2^{-1}	X_2^{-1}	X_2	X_5^{-1}	X_1
X_3	X_3^{-1}	X_3	X_3	X_2	X_3
X_4	X_4^{-1}	X_4	X_4	X_5^{-1}	—
X_5	X_5^{-1}	X_5^{-1}	X_5	X_4	X_6^{-a}
X_6	X_6	X_6^{-1}	X_6	X_6	X_5^a .

Here $a^2 = 1$, and the last two entries in Table 2 are at our disposal since Q^* normalizes $\langle I, J \rangle$ and since $\mathfrak{x}_5, \mathfrak{x}_6$ are the only subgroups of \mathfrak{D} of order 3 which admit $\langle I, J \rangle$. In addition we have the following commutation relations:

$$[X_i, X_6] = 1, 1 \leq i \leq 6, [X_4, X_5] = X_6,$$

$$[X_i, X_5] = 1, 1 \leq i \leq 3, [X_3, X_4] = [X_2, X_4] = 1.$$

Furthermore, $[X_2, X_3] = X_6^b$, so by Table 2, we get $[X_1, X_3] = X_5^{ab}$. Here $b^2 = 1$, for if $b = 0$, we get $X_3 \in Z(\mathfrak{S})$, which is not the case. The as yet undetermined commutation relations are:

$$[X_1, X_4] = X_2^x X_3^y X_5^z X_6^t, \quad [X_1, X_2] = X_3^c X_5^d X_6^e.$$

Use Table 1 and conjugate the second relation by J , obtaining $e = bc$. Then conjugation by I yields $d = abc$. Conjugation of the first relation by J yields $t = xyb + z$. Conjugation of the first relation by I yields $y = cx$.

Assume $c \neq 0$. Then

$$\mathfrak{P}'_0 = \langle X_3, X_5, X_6 \rangle, [\mathfrak{P}'_0, \mathfrak{P}_0] = \langle X_5, X_6 \rangle = \mathfrak{D} = Z(\mathfrak{P}_0).$$

We see that \mathfrak{P}'_0 is elementary abelian. If $A \in \mathfrak{P}_0, B \in \mathfrak{P}'_0$, then $(AB)^3 = A^3 B^{A^2+A+1}$. But $cl(\mathfrak{P}_0) = 3$ and so $B^{A^2+A+1} = B^3[B, A]^3 = 1$. Hence,

there is a map φ of $\mathfrak{P}_0/\mathfrak{P}'_0$ given by $\varphi(A\mathfrak{P}'_0) = A^3$.

Clearly, $\varphi(X_1\mathfrak{P}'_0) = 1$. But \mathfrak{M} operates as $GL(2, 3)$ on \mathfrak{D} . Since $|\mathfrak{P}_0/\mathfrak{P}'_0| = 3^2$, this forces \mathfrak{M} to operate as $GL(2, 3)$ on $\mathfrak{P}_0/\mathfrak{P}'_0$. In particular, the four subgroups of $\mathfrak{P}_0/\mathfrak{P}'_0$ of order 3 are all conjugate under \mathfrak{M} . Hence, $\varphi(A\mathfrak{P}'_0) = 1$ for all A in \mathfrak{P}_0 and \mathfrak{P}_0 is of exponent 3.

By [21, p. 324], the order of the Burnside group of exponent 3 on 2 generators is 27. Since \mathfrak{P}_0 must be a homomorphic image of this group, we get a contradiction, as $|\mathfrak{P}_0| = 3^5 > 27$. So $c = 0$.

Since $c = 0$, so also $d = e = y = 0$. Since $y = 0$, we also have $t = z$. Conjugation of the first relation by Q yields $[Q^{-1}X_1Q, X_5^{-1}] = X_3^{-z}X_4^zX_6^z$. Now $C(J) \cap O^{3'}(\mathfrak{N}) = \langle X_1, X_6, \Omega \rangle$, so $C(J) \cap O^{3'}(\mathfrak{N})$ is 2-closed, that is, X_1 normalizes Ω . Hence, $(QX_1)^3 = J^u, u = 0$ or 1. Hence, $Q^{-1}X_1Q = JX_1^{-1}Q^{-1}X_1^{-1}J^u$. The previous commutation relation now yields $x = 0$.

Since $x = 0$, it follows that X_4 centralizes $\mathfrak{P}_0/\mathfrak{D}$. Hence, Ω^* is forced to centralize $\mathfrak{P}_0/\mathfrak{D}$. This is not the case, since $\mathfrak{D} \subseteq \mathfrak{P}'_0$. We conclude that $[X_4, X_5] = 1$.

Since I centralizes X_4 , it follows that $\langle X_4, X_5, X_6 \rangle = \mathfrak{C} \triangleleft \mathfrak{M}$. Namely, $\mathfrak{D} \triangleleft \mathfrak{M}$, so we need to show that $\mathfrak{C}/\mathfrak{D} \triangleleft \mathfrak{M}/\mathfrak{D}$. Since X_4 centralizes \mathfrak{D} , we have $X_4 \in \mathfrak{P}_0$. Since $\mathfrak{P}_0/\mathfrak{D}$ is of order 27 and admits Ω^* as a group of automorphisms, it follows that $\mathfrak{C}/\mathfrak{D} = C_{\mathfrak{P}_0/\mathfrak{D}}(I) \triangleleft \mathfrak{M}/\mathfrak{D}$. Thus, $\langle \mathfrak{M}, Q \rangle \subseteq N(\mathfrak{C})$. Since $\langle \mathfrak{P}, Q \rangle = O^{3'}(\mathfrak{N})$, it follows that both \mathfrak{M} and $O^{3'}(\mathfrak{N})$ are subgroups of $N(\mathfrak{C})$.

Let $\mathfrak{C}^* = O_3(N(\mathfrak{C}))$. Thus,

$$\mathfrak{C} \subseteq \mathfrak{C}^* \subseteq O_3(\mathfrak{M}) \cap O_3(\mathfrak{N}) = \mathfrak{P}_0 \cap \mathfrak{S}.$$

Suppose $\mathfrak{C}^* = \mathfrak{P}_0 \cap \mathfrak{S}$. Then $\mathfrak{B} = \mathfrak{C}^{*'} \triangleleft N(\mathfrak{C})$, against $O^{3'}(\mathfrak{N}) \subset N(\mathfrak{C})$. Hence, $\mathfrak{C}^* \subset \mathfrak{P}_0 \cap \mathfrak{S}$. Since $|\mathfrak{C}^*| = 3^3$ and $|\mathfrak{P}_0 \cap \mathfrak{S}| = 3^4$, it follows that $\mathfrak{C}^* = \mathfrak{C}$. Thus, $N(\mathfrak{C})/\mathfrak{C}$ is isomorphic to a subgroup of $\text{Aut}(\mathfrak{C})$ which (a) is solvable, (b) contains a S_3 -subgroup of $\text{Aut}(\mathfrak{C})$, (c) is 3-reduced. There are no such groups. The proof of the lemma is complete.

LEMMA 9.6. *Let \mathfrak{B} be a subgroup of \mathfrak{S} of type (3, 3). Then $\mathfrak{B} \in \mathcal{D}$. (See Definition 7.3.)*

Proof. We first show that if $B \in \mathfrak{B}$, then

$$(9.8) \quad \text{for some } N \text{ in } \mathfrak{N}, B \text{ centralizes an element of } \mathcal{U}(\mathfrak{P}^N).$$

Let

$$\mathfrak{S}_1 = \Omega_1(\mathfrak{S}), \mathfrak{B} = \mathfrak{S}_1/D(\mathfrak{S}_1), \mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{B}).$$

Suppose $|\mathfrak{B}_0| > 3$. Then $\mathfrak{B}_0 = \mathfrak{B}/D(\mathfrak{S}_1)$ and every subgroup of \mathfrak{B} which contains $D(\mathfrak{S}_1)$ is normal in \mathfrak{B} . Let $\mathfrak{B}_1 = \mathfrak{B} \cap C(B)$ so that $|\mathfrak{B}:\mathfrak{B}_1| \leq 3$.

Since $|\mathfrak{B}_0| > 3$, so also $|\mathfrak{B}_1| \geq 9$. Thus, B centralizes an element of $\mathcal{U}(\mathfrak{B})$ in this case. We may assume that $|\mathfrak{B}_0| = 3$.

Suppose $\mathfrak{N}/\mathfrak{G}$ has a normal subgroup $\mathfrak{N}/\mathfrak{G} = \mathfrak{X}$ of odd order $\neq 1$. Let k be a field of characteristic 3 which contains all $|\mathfrak{X}|^{\text{th}}$ roots of 1. Let $\tilde{\mathfrak{B}} = k \otimes_{F_3} \mathfrak{B}$. Thus, $\tilde{\mathfrak{B}}$ admits $\mathfrak{N}/\mathfrak{G}$ and $k \otimes \mathfrak{B}_0$ is the set of all fixed points of $\mathfrak{B}/\mathfrak{G}$ on $\tilde{\mathfrak{B}}$. Let $\tilde{\mathfrak{B}} = \bigoplus_{\rho} \tilde{\mathfrak{B}}(\rho)$, where $\tilde{\mathfrak{B}}(\rho)$ is the largest \mathfrak{X} -submodule of $\tilde{\mathfrak{B}}$ on which \mathfrak{X} acts as a multiple of the irreducible representation ρ . Since $\tilde{\mathfrak{B}}$ inherits the non singular symplectic structure of \mathfrak{B} , it follows that ρ and ρ^* appear with the same multiplicity in $\tilde{\mathfrak{B}}$, ρ^* denoting the contragredient representation of ρ . Since $|\mathfrak{B}|$ is odd, $\tilde{\mathfrak{B}}(\rho)$ and $\tilde{\mathfrak{B}}(\rho^*)$ are not conjugate under \mathfrak{B} . Hence, \mathfrak{B}_0 is not 1-dimensional in this case.

We may now assume that

$$(9.9) \quad F(\mathfrak{N}/\mathfrak{G}) \text{ is a 2-group .}$$

If $\mathfrak{G} = \mathfrak{B}$, then (9.8) is obvious, so suppose $\mathfrak{G} \subset \mathfrak{B}$. Set $\mathfrak{N}^* = C(\mathfrak{B})$, so that $|\mathfrak{N}:\mathfrak{N}^*| \leq 2$. By Lemma 9.3 (i), together with (9.9), we conclude that \mathfrak{N} is a 2, 3-group, and that a S_2 subgroup of \mathfrak{N}^* is quaternion. Hence, $|\mathfrak{B}:\mathfrak{G}| = 3$. Since $|\mathfrak{B}_0| = 3$, we get that the width of \mathfrak{G} is 1, against Hypothesis 9.2. Thus, (9.8) holds.

Suppose $\mathfrak{B} \notin \mathcal{S}$. Then $\mathfrak{N}(\mathfrak{B}; 2)$ contains a four-group Ω which is not centralized by \mathfrak{B} . Hence, $[\mathfrak{B}, \Omega] = \Omega$, and $\mathfrak{B}_0 = C_{\mathfrak{B}}(\Omega)$ is of order 3.

Let $\mathfrak{C} = C(\mathfrak{B}_0)$, $\mathfrak{G}_0 = C_{\mathfrak{B}}(\mathfrak{B}_0)$. By Lemma 7.2 applied to $\langle \mathfrak{B}_0, \mathfrak{B} \rangle$, it follows that \mathfrak{B} centralizes $O_3(\mathfrak{C})$. Hence, $[O_3(\mathfrak{C}), \mathfrak{G}_0] \subseteq \mathfrak{G} \cap O_3(\mathfrak{C}) = 1$. This implies that $O_3(\mathfrak{C}) = 1$ by Lemma 9.4.

Let \mathfrak{B}_0 be a S_3 -subgroup of \mathfrak{C} containing \mathfrak{G}_0 and let \mathfrak{B}^* be a S_3 -subgroup of \mathfrak{G} containing \mathfrak{B}_0 . Then $\mathfrak{B}^* = \mathfrak{B}^{\mathfrak{G}}$, so that with $\mathfrak{B}^* = \mathfrak{B}^{\mathfrak{G}}$, it follows that $\mathfrak{B}^* \subseteq Z(O_3(\mathfrak{C}))$. Let $\mathfrak{B} = \mathfrak{B}^{*\mathfrak{C}}$, so that \mathfrak{B} is 3-reducible in \mathfrak{C} . Set $\mathfrak{C}_1 = C_{\mathfrak{C}}(\mathfrak{B})$. We argue that $\mathfrak{C}_1 \cap \Omega = 1$. If not, then $\Omega \subseteq C(\mathfrak{B})$, as Ω is an irreducible \mathfrak{B} -module. Hence, $\Omega \subseteq C(\mathfrak{B}^*)$, against Lemma 9.3 (i). Hence,

$$(9.10) \quad \mathfrak{C}_1 \cap \Omega = 1 .$$

By (B), elements of $\mathfrak{B} - \mathfrak{B}_0$ have minimal polynomial $(x - 1)^3$ on \mathfrak{B} .

We next argue that $\mathfrak{B} \subseteq \mathfrak{C}_1$. If not, then \mathfrak{G}_0 contains an extra special subgroup of width $w - 1$ disjoint from \mathfrak{C}_1 . We get that $m(\mathfrak{B}) \geq 2 \cdot 3^{w-1}$. Since $m(\mathfrak{B} \cap \mathfrak{G}^{\mathfrak{C}}) \leq w + 1$, we have $m(\mathfrak{B}/\mathfrak{B} \cap \mathfrak{G}^{\mathfrak{C}}) \geq 2 \cdot 3^{w-1} - w - 1$. By [32], it follows that $w^2 \geq 2 \cdot 3^{w-1} - w - 1$. This is false for $w \geq 3$, so $w = 2$. Thus, $C(\mathfrak{B})/\mathfrak{G}$ is isomorphic to a subgroup of $GL(4, 3)$ which (a) is solvable, (b) is 3-reduced, (c) has an elementary subgroup of order 27. There are no such groups. We conclude that $\mathfrak{B} \subseteq \mathfrak{C}_1$.

Since $\mathfrak{B} \subseteq \mathfrak{C}_1$, we have $\mathfrak{B} \subseteq \mathfrak{N}$, so that $[\mathfrak{B}, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{B}$. Hence by (B),

$$(9.11) \quad [\mathfrak{B}, \mathfrak{B}, \mathfrak{B}] = \mathfrak{Z} .$$

Since $\mathfrak{B} \subseteq \mathbf{Z}(\mathbb{C}_1)$, we get

$$(9.12) \quad \mathfrak{Z} \subseteq \mathbf{Z}(\mathbb{C}_1) ,$$

which implies that

$$(9.13) \quad \mathbb{C}_1 \subseteq \mathfrak{N} .$$

By (9.13), we get $[\mathbb{C}_1, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{Z}$, and in particular,

$$(9.14) \quad [\mathbf{O}_3(\mathbb{C}), \mathfrak{B}, \mathfrak{B}] = \mathfrak{Z} ,$$

equality holding by (9.11) and the obvious containment $\mathfrak{B} \subseteq \mathbf{O}_3(\mathbb{C})$.

Now (9.14) and (9.11) yield

$$(9.15) \quad \mathbf{O}_3(\mathbb{C}) = \mathfrak{B}_1 \times \mathfrak{B}_2 ,$$

where

$$(9.16) \quad \mathfrak{Z} \subset \mathfrak{B}_1 = [\mathfrak{Q}, \mathbf{O}_3(\mathbb{C})], \text{ and } \mathfrak{B}_1 \text{ is elementary of order } 27 ,$$

$$(9.17) \quad \mathfrak{B}_2 = \mathbf{C}(\mathfrak{Q}) \cap \mathbf{O}_3(\mathbb{C}) .$$

Let $\mathbb{C}_2 = \mathbf{O}_{3'}(\mathbb{C} \bmod \mathbb{C}_1)$, $\mathbb{C}_3 = \mathbb{C}_2 \mathbf{Z}(\mathfrak{H})$. Thus, $\mathbb{C}_1 \mathbf{Z}(\mathfrak{H})$ contains a S_3 -subgroup of \mathbb{C}_3 . Thus, $\mathbf{Z}(\mathfrak{H})$ is normal in a S_3 -subgroup of \mathbb{C}_3 . By Lemma 5.22, we get $\mathbf{Z}(\mathfrak{H}) \subseteq \mathbf{O}_3(\mathbb{C}_3)$. Hence, $\mathbf{Z}(\mathfrak{H}) \subseteq \mathbf{O}_3(\mathbb{C})$. From (9.16), we conclude that $\mathbf{Z}(\mathfrak{H}) = \mathfrak{Z}$, that is,

$$(9.18) \quad \mathfrak{H} \text{ is extra special .}$$

We argue that $\mathbf{O}_{3,3'}(\mathbb{C})$ does not centralize $\mathbf{Z}(\mathbf{O}_3(\mathbb{C}))$. If it does, then since $\mathfrak{Z} \subseteq \mathbf{Z}(\mathbf{O}_3(\mathbb{C}))$, it follows that $\mathbf{O}_{3,3'}(\mathbb{C}) \subseteq \mathfrak{N}$, so $[\mathbf{O}_{3,3'}(\mathbb{C}), \mathfrak{H}_0] \subseteq \mathfrak{H}$, which implies that $\mathfrak{H}_0 \subseteq \mathbf{O}_3(\mathbb{C})$, which in turn gives $\mathfrak{Q} = [\mathfrak{B}, \mathfrak{Q}] \subseteq \mathbf{O}_3(\mathbb{C})$. Since $\mathfrak{B}_0 \subseteq \mathbf{Z}(\mathbb{C})$, it follows that

$$[\mathbf{Z}(\mathbf{O}_3(\mathbb{C})), \mathbf{O}_{3,3'}(\mathbb{C})] \text{ and } \mathbf{C}(\mathbf{O}_{3,3'}(\mathbb{C})) \cap \mathbf{Z}(\mathbf{O}_3(\mathbb{C}))$$

are disjoint nontrivial normal abelian subgroups of \mathbb{C} . In particular, if \mathfrak{P}_0 is a S_3 -subgroup of \mathbb{C} containing \mathfrak{H}_0 , then $\mathbf{Z}(\mathfrak{P}_0)$ is not cyclic. By Lemma 9.4, we get that $\Omega_1(\mathbf{Z}(\mathfrak{P}_0)) = \mathfrak{B}_0 \times \mathfrak{Z}$, and in particular, $\mathfrak{P}_0 \subseteq \mathfrak{N}$.

Since $\mathfrak{P}_0 \subseteq \mathfrak{N}$, we get that $[\mathfrak{P}_0, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{Z}$. Thus, if $B \in \mathfrak{B}$, the minimal polynomial of B on the Frattini quotient group of $\mathbf{O}_{3,3',3}(\mathbb{C})/\mathbf{O}_{3,3'}(\mathbb{C})$ divides $(x - 1)^2$. By (B), it follows that \mathfrak{Q} centralizes $\mathbf{O}_{3,3',3}(\mathbb{C})/\mathbf{O}_{3,3'}(\mathbb{C})$, and so $\mathfrak{Q} \subseteq \mathbf{O}_{3,3'}(\mathbb{C})$.

Let $\mathfrak{R} = \langle \mathfrak{Q}, \mathfrak{H}_0 \rangle \subseteq \mathbb{C}$, let $\mathfrak{R}_0 = \mathbf{C}_{\mathfrak{R}}(\mathbf{O}_3(\mathbb{C}))$ and for any subset \mathfrak{S} of \mathfrak{R} , let $\bar{\mathfrak{S}} = \mathfrak{S}\mathfrak{R}_0/\mathfrak{R}_0$.

We argue that $\bar{\mathfrak{Q}} \subseteq \mathbf{O}_3(\bar{\mathfrak{R}})$. Namely, $\mathfrak{Q} \subseteq \mathbf{O}_{3,3'}(\mathbb{C})$, and so $\mathfrak{Q} \subseteq \mathbf{O}_{3,3'}(\mathfrak{R})$,

Thus, it suffices to show that $[O_3(\mathfrak{K}), \mathfrak{Q}] \subseteq \mathfrak{R}_0$. But

$$[O_3(\mathfrak{K}), \mathfrak{Q}] \subseteq O_3(\mathfrak{K}) \cap O_{3,3'}(\mathbb{C}) \subseteq O_3(\mathfrak{K}) \cap O_3(\mathbb{C}),$$

and so

$$\begin{aligned} [O_3(\mathfrak{K}), \mathfrak{Q}] &= [O_3(\mathfrak{K}), \mathfrak{Q}, \mathfrak{Q}] \subseteq [O_3(\mathfrak{K}) \cap O_3(\mathbb{C}), \mathfrak{Q}] \subseteq [O_3(\mathbb{C}), \mathfrak{Q}] \\ &= \mathfrak{W}_1 \subseteq Z(O_3(\mathbb{C})), \end{aligned}$$

whence $[O_3(\mathfrak{K}), \mathfrak{Q}] \subseteq \mathfrak{K} \cap Z(O_3(\mathbb{C})) \subseteq \mathfrak{R}_0$.

Case 1. $\langle \overline{\mathfrak{Q}}, \overline{\mathfrak{Q}^H} \rangle$ is abelian for all $H \in \mathfrak{G}_0$.

Since $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}}, \mathfrak{B}]$ admits the abelian group \mathfrak{H}_0 , and since $\mathfrak{Q} \subseteq [\mathfrak{Q}^{\mathfrak{H}_0}, \mathfrak{B}]$, it follows that $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}}, \mathfrak{B}] = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$. Since $\mathfrak{W}_2 = C(\mathfrak{Q}) \cap O_3(\mathbb{C})$ admits the abelian group $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$, (B) implies that $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ centralizes \mathfrak{W}_2 . Hence, $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ is isomorphic to an elementary 2-subgroup of $\text{Aut}(\mathfrak{W}_1)$. Since $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}}, \mathfrak{B}] = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$, we get that $\overline{\mathfrak{Q}} = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$, so that $\overline{\mathfrak{Q}}$ is a S_2 -subgroup of \mathfrak{K} .

Let $\mathfrak{K}_1 = O_3(\mathfrak{K} \text{ mod } \mathfrak{R}_0)$. Thus, $\mathfrak{K}_1 \cap \mathfrak{G}_0$ is of index 3 in \mathfrak{G}_0 and $\langle \mathfrak{K}_1 \cap \mathfrak{G}_0, \mathfrak{B} \rangle = \mathfrak{G}_0$. Since $|\mathfrak{K}_0|$ is odd, it follows that \mathfrak{Q} is a S_2 -subgroup of \mathfrak{K} . Let $\mathfrak{L} = \mathfrak{K}_1 O_3(\mathbb{C})$ and let \mathfrak{L}_3 be a S_3 -subgroup of \mathfrak{L} which contains $\mathfrak{K}_1 \cap \mathfrak{G}_0$ and is normalized by \mathfrak{G}_0 . Since $|\mathfrak{L}|$ is odd, it follows that S_2 -subgroups of $N(\mathfrak{L}_3) \cap \mathfrak{L}\mathfrak{K}$ are four-groups. If $\mathfrak{Z} \subseteq D(\mathfrak{L}_3)$, then by (B), S_2 -subgroups of $N(\mathfrak{L}_3) \cap \mathfrak{L}\mathfrak{K}$ centralize \mathfrak{L}_3 . This is not the case, as \mathfrak{Q} does not centralize \mathfrak{W}_1 . Hence, $\mathfrak{Z} \not\subseteq D(\mathfrak{L}_3)$. In particular, $\mathfrak{Z} \not\subseteq D(\mathfrak{K}_1 \cap \mathfrak{G}_0)$. But $\mathfrak{K}_1 \cap \mathfrak{G}_0$ is of index 3 in \mathfrak{G}_0 . Since \mathfrak{G} is extra special, it follows that $w = 2$. Clearly, $\mathfrak{G} \subset \mathfrak{F}$, since $O_3(\mathbb{C})$ contains an elementary subgroup of order 3^4 . On the other hand, Lemma 9.3 implies that $|\mathfrak{F} : \mathfrak{G}| \leq 3$. Hence, $|\mathfrak{F} : \mathfrak{G}| = 3$, and $|\mathfrak{F}| = 3^6$. Since \mathfrak{B}_0 is obviously not conjugate to \mathfrak{Z} , it follows that $O_3(\mathbb{C})$ is elementary of order 3^4 and $\mathfrak{F}_0 = O_3(\mathbb{C})\mathfrak{G}_0$, $|\mathfrak{G}_0 \cap O_3(\mathbb{C})| = 27$. Clearly, $O_3(\mathbb{C}) \text{ char } \mathfrak{F}_0$, since $O_3(\mathbb{C})$ is the only elementary subgroup of its order in \mathfrak{F}_0 .

Let $\mathfrak{M} = N(O_3(\mathbb{C}))$ so that \mathfrak{M} contains a S_3 -subgroup \mathfrak{F} of \mathfrak{G} with $\mathfrak{F} \supset \mathfrak{F}_0$. Since $\mathfrak{Z} = Z(\mathfrak{F}_0) \cap \mathfrak{F}_0 \text{ char } \mathfrak{F}_0$, we have $\mathfrak{Z} \triangleleft \mathfrak{F}$. In particular, $\mathfrak{G} \subset \mathfrak{M}$. We therefore assume without loss of generality that $\mathfrak{F} = \mathfrak{F}$.

It is clear that $O_3(\mathbb{C}) = O_3(\mathfrak{M})$ and that \mathfrak{M} is a 2, 3-group. It is equally clear that $l_3(\mathfrak{M}) = 2$, so that $\mathfrak{Q} \subseteq O_{3,2}(\mathfrak{M})$. Hence, \mathfrak{B} is a S_3 -subgroup of $N(\mathfrak{Q}) \cap \mathfrak{M}$, so we can choose a subgroup \mathfrak{B}_1 of \mathfrak{B} of order 3 such that $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$ and such that \mathfrak{B}_1 normalizes some S_2 -subgroup \mathfrak{X}_0 of $O_{3,2}(\mathfrak{M})$. Let $\mathfrak{A} = N(\mathfrak{X}_0) \cap \mathfrak{F}$. Thus, \mathfrak{A} is elementary of order 9, $\mathfrak{B}_1 \subset \mathfrak{A}$ and $\mathfrak{F} = \mathfrak{A}O_3(\mathfrak{M})$, $\mathfrak{A} \cap O_3(\mathfrak{M}) = 1$. Since $O_3(\mathfrak{M}) \cap C(\mathfrak{B}_1)$ is of order 9, it follows that $C(\mathfrak{B}_1) \cap \mathfrak{F}$ is of order 3^4 . Hence, $C(\mathfrak{B}_1) \cap \mathfrak{F} = C(\mathfrak{B}_1) \cap \mathfrak{G}$; since \mathfrak{A} is elementary we get $\mathfrak{A} \subset \mathfrak{G}$.

We now choose \mathfrak{A}_1 of order 3 in \mathfrak{A} so that \mathfrak{A} does not centralize $C_{\mathfrak{X}_0}(\mathfrak{A}_1)$. Let $\mathfrak{X}_1 = [C_{\mathfrak{X}_0}(\mathfrak{A}_1), \mathfrak{A}]$. Thus, \mathfrak{X}_1 is faithfully represented on $C(\mathfrak{A}_1) \cap O_3(\mathfrak{M}) = \mathfrak{R}$. It is straightforward to verify that $|\mathfrak{R}| = 9$ and

that \mathfrak{X}_1 is a quaternion group. Hence, $\mathfrak{R} = [O_3(\mathfrak{M}), \mathfrak{X}_1]$, so $\mathfrak{R} \subseteq \mathfrak{G}$. Since $\mathfrak{Z} \subset \mathfrak{R}$, it follows that \mathfrak{Z} is not weakly closed in \mathfrak{G} . As this violates Lemma 9.5, we conclude that Case 1 does not hold.

Case 2. There is an element H of \mathfrak{G}_0 such that $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle$ is nonabelian.

Set $\mathfrak{W} = \langle \mathfrak{W}_1, \mathfrak{W}_1^H \rangle$, so that $\langle \mathfrak{D}, \mathfrak{D}^H \rangle$ normalizes $\widetilde{\mathfrak{W}}$ and centralizes $O_3(\mathbb{C})/\widetilde{\mathfrak{W}}$. Since $\mathfrak{W}_1 \cap \mathfrak{G} \supset \mathfrak{Z}$, it follows that $|\mathfrak{W}_1 \cap \mathfrak{W}_1^H| \geq 9$. Clearly, $\mathfrak{W}_1 \neq \mathfrak{W}_1^H$, since $\langle \mathfrak{D}, \mathfrak{D}^H \rangle$ is nonabelian. Hence, $\widetilde{\mathfrak{W}}$ is elementary of order 3^4 . Since $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle$ is injected into $\text{Aut}(\widetilde{\mathfrak{W}})$ under the restriction map, it follows readily that $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle$ is the central product of two quaternion groups, each of which necessarily admits \mathfrak{B} . In particular, $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle'$ is of order 2 and inverts $\overline{\mathfrak{W}_1} \cap \mathfrak{G}$. Since no involution of \mathfrak{G} centralizes $\overline{\mathfrak{W}_1} \cap \mathfrak{G}$, it follows that $\overline{\mathfrak{D}}^{\mathfrak{G}_0}$ is extra special of order 32. Hence, $[\overline{\mathfrak{D}}^{\mathfrak{G}_0}, O_3(\mathbb{C})]$ is elementary of order 3^4 . This implies that $O_3(\mathbb{C})$ contains $[\overline{\mathfrak{D}}^{\mathfrak{G}_0}, O_3(\mathbb{C})] \times \mathfrak{B}_0$, an elementary subgroup of order 3^5 . Hence, $w \geq 3$.

Write $O_3(\mathbb{C}) = \mathfrak{x} \times \mathfrak{y}$, where

$$\mathfrak{x} = [O_3(\mathbb{C}), \overline{\mathfrak{D}}^{\mathfrak{G}_0}] \quad \text{and} \quad \mathfrak{y} = O_3(\mathbb{C}) \cap C(\overline{\mathfrak{D}}^{\mathfrak{G}_0}).$$

Thus, \mathfrak{G}_0 normalizes both \mathfrak{x} and \mathfrak{y} . Suppose $Y \in \mathfrak{y} \cap \mathfrak{G}$. Then

$$[Y, \mathfrak{G}_0] \subseteq \mathfrak{Z} \cap \mathfrak{y} = 1, \quad \text{so} \quad Y \in Z(\mathfrak{G}_0) = \mathfrak{Z} \times \mathfrak{B}_0.$$

Hence, $\mathfrak{y} \cap \mathfrak{G} = \mathfrak{B}_0$. Since $[\mathfrak{y}, \mathfrak{G}_0] \subseteq \mathfrak{G}$, it follows that $[\mathfrak{y}, \mathfrak{G}_0] \subseteq \mathfrak{B}_0$. Since $\overline{\mathfrak{D}}^{\mathfrak{G}_0}$ is absolutely irreducible on \mathfrak{x} , it follows that

$$[C_{\mathfrak{G}_0}(\overline{\mathfrak{D}}^{\mathfrak{G}_0}), O_3(\mathbb{C})] \subseteq \mathfrak{B}_0, \quad \text{so} \quad C_{\mathfrak{G}_0}(\overline{\mathfrak{D}}^{\mathfrak{G}_0}) \subseteq O_3(\mathbb{C}),$$

since $O_3(\mathbb{C}) = O_3(\mathbb{C} \bmod \mathfrak{B}_0)$.

Clearly, $|\mathfrak{G}_0 : C_{\mathfrak{G}_0}(\overline{\mathfrak{D}}^{\mathfrak{G}_0})| = 3^a$, $a = 1$ or 2 , since $\overline{\mathfrak{D}}^{\mathfrak{G}_0}$ is extra special of order 32. If $a = 1$, then $\mathfrak{G}_0 \cap O_3(\mathbb{C})$ is of index 9 in \mathfrak{G} , so is nonabelian since $w \geq 3$. This is impossible, since $\mathfrak{Z} \not\subseteq D(O_3(\mathbb{C}))$.

Suppose $a = 2$. Set $\mathfrak{A} = \mathfrak{G}_0 \cap O_3(\mathbb{C})$. Since \mathfrak{A} is abelian, $w = 3$. Thus, $\mathfrak{A} \in \text{Sen}(\mathfrak{G})$. Let $\mathfrak{A}_1 = \mathfrak{x} \cap \mathfrak{A}$, so that $27 \geq |\mathfrak{A}_1| \geq 9$. Suppose $|\mathfrak{A}_1| = 9$. Let \mathfrak{A}_2 be a complement to \mathfrak{A}_1 in \mathfrak{x} , so that $|\mathfrak{A}_2| = 9$, and $\mathfrak{A}_2 \cap \mathfrak{G} = 1$. Since \mathfrak{A}_2 centralizes \mathfrak{A} , we get $[\mathfrak{G}, \mathfrak{A}_2] \subseteq \mathfrak{G} \cap C(\mathfrak{A}) = \mathfrak{A}$, so that $[\mathfrak{G}, \mathfrak{A}_2, \mathfrak{A}_2] = 1$. Thus, $[Q_3^*(\mathfrak{N}), \mathfrak{A}_2]$ is a 2-group on which \mathfrak{A}_2 is faithfully represented. This violates Lemma 9.3. Hence, \mathfrak{A}_1 is of order 3^3 , so that $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{B}_0$. Suppose $\mathfrak{B}_0 \subset \mathfrak{y}$. Let \mathfrak{y}_1 be a subgroup of \mathfrak{y} of order 9 which contains \mathfrak{B}_0 . Then $\mathfrak{x}\mathfrak{y}_1$ is abelian of order 3^6 , and $[\mathfrak{G}, \mathfrak{x}\mathfrak{y}_1] \subseteq \mathfrak{G} \cap C(\mathfrak{A}) = \mathfrak{A}$, so that $[\mathfrak{G}, \mathfrak{x}\mathfrak{y}_1, \mathfrak{x}\mathfrak{y}_1] = 1$. It follows that $[Q_3^*(\mathfrak{N}), \mathfrak{x}\mathfrak{y}_1]$ is a 2-group on which $\mathfrak{x}\mathfrak{y}_1/\mathfrak{A}$ is faithfully represented. This again violates Lemma 9.3, so $\mathfrak{y} = \mathfrak{B}_0$.

Since $\mathfrak{U} = \mathfrak{P}_0$ and $a = 2$, $O_3(\mathbb{C})$ is elementary of order 3^5 and $|\mathfrak{P}_0| = 3^7$. Lemma 5.2 implies that if U is any element of $\mathbb{C}/O_3(\mathbb{C})$ of order 3, then $C(U) \cap O_3(\mathbb{C})$ is of order at most 3^3 .

Suppose by way of contradiction that \mathfrak{U} is an elementary subgroup of \mathfrak{P}_0 of order 3^5 which is distinct from $O_3(\mathbb{C})$. By the previous paragraph, we conclude that $\mathfrak{U} \cap O_3(\mathbb{C})$ is of order 3^3 , and that if $U \in \mathfrak{U} - O_3(\mathbb{C})$, then $O_3(\mathbb{C}) \cap C(U) = O_3(\mathbb{C}) \cap \mathfrak{U}$. Let \mathfrak{U}_0 be a complement to $\mathfrak{U} \cap O_3(\mathbb{C})$ in \mathfrak{U} . Thus, \mathfrak{U}_0 is faithfully represented on $Q_3^1(\mathbb{R})$, the central product of two quaternion groups. Let \mathfrak{R} be a quaternion subgroup of $Q_3^1(\mathbb{R})$, and let $\mathfrak{U}_1 = C(\mathfrak{R}) \cap \mathfrak{U}_0$. Thus, \mathfrak{U}_1 is of order 3. By Lemma 3.7 of [20], \mathfrak{R} is faithfully represented on $O_3(\mathbb{C}) \cap C(\mathfrak{U}_1)$. This is absurd, since \mathfrak{U}_0 centralizes $O_3(\mathbb{C}) \cap C(\mathfrak{U}_1)$. We conclude that $O_3(\mathbb{C})$ is the only elementary subgroup of its order in \mathfrak{P}_0 .

Since $|\mathfrak{P}_0| = |\mathfrak{G}| = 3^7$ and since \mathfrak{P}_0 is obviously not extra special it follows that \mathfrak{P}_0 is not a S_3 -subgroup of \mathfrak{G} . Hence, \mathfrak{P}_0 is not a S_3 -subgroup of $N(O_3(\mathbb{C}))$. Hence, $A_{\mathfrak{G}}(O_3(\mathbb{C}))$ is a solvable subgroup of $GL(5, 3)$ with S_3 -subgroups of order at least 27 and with no nonidentity normal 3-subgroups. There are no such groups. The proof of Lemma 9.6 is complete.

LEMMA 9.7. *Every involution in \mathfrak{R} centralizes \mathfrak{Z} .*

Proof. Suppose false. Let $\mathfrak{G}_1 = \Omega_1(\mathfrak{G})$, so that \mathfrak{G}_1 is extra special of exponent 3 and width $w \geq 2$. Let $\mathfrak{G}_0 = C_{\mathfrak{G}_1}(I)$ and let \mathfrak{G}_2 be the set of elements of \mathfrak{G}_1 inverted by I . Here I is an involution of \mathfrak{R} which does not centralize \mathfrak{Z} . Since $\mathfrak{Z} \subseteq \mathfrak{G}_2$, it follows that \mathfrak{G}_0 is abelian. Since I centralizes $[H, H']$ for all H, H' in \mathfrak{G}_2 , it follows that $\langle \mathfrak{G}_2 \rangle$ is abelian. Hence, $\mathfrak{G}_2 = \langle \mathfrak{G}_2 \rangle$. As is well known, $\mathfrak{G}_1 = \mathfrak{G}_0 \mathfrak{G}_2$ and $\mathfrak{G}_0 \cap \mathfrak{G}_2 = 1$. Hence, \mathfrak{G}_2 is elementary of order 3^{w+1} and \mathfrak{G}_0 is elementary of order 3^w .

By Lemmas 7.5 and 9.6, there is a subgroup \mathfrak{M} in $\mathcal{MS}(\mathfrak{G})$ with $\mathfrak{G}_0 \subseteq \mathfrak{M}$ such that \mathfrak{M} satisfies Hypothesis 7.2 and $p = 2$. Let w_1 be the width of $O_2(\mathfrak{M})$. Hence, $w \leq w_1 \leq 4$, the first inequality holding since \mathfrak{G}_0 is faithfully represented on $O_2(\mathfrak{M})$, the second inequality holding by Lemma 7.5.

Suppose $w \geq 3$. Hence, $w_1 \geq 3$. If $H \in \mathfrak{G}_0^*$ and $C(H) \cap O_2(\mathfrak{M})$ contains a four-subgroup \mathfrak{B} containing $l_1(Z(O_2(\mathfrak{M})))$, then by Lemma 7.2, both $\langle H, \mathfrak{Z} \rangle$ and \mathfrak{B} satisfy the hypothesis of Lemma 7.4, so $C(H)$ is nonsolvable. This is impossible, so H is not available. This implies that $w = 2$.

Suppose $\mathfrak{G} = \mathfrak{P}$. In this case, if H is any element of order 3 in \mathfrak{G} , then $\mathfrak{Z} \text{ char } C_{\mathfrak{G}}(H)$. This implies immediately that \mathfrak{Z} is weakly closed in \mathfrak{P} , which in turn implies that \mathfrak{R} contains the centralizer of each of its nonidentity 3-elements. This implies that $O_2(\mathfrak{M}) \subseteq \mathfrak{R}$, which

is not the case. Hence $\mathfrak{G} \subset \mathfrak{F}$.

Since every 3, 5-subgroup of $S_4(3)$ is either a 3-group or a 5-group, it follows from the preceding paragraph that \mathfrak{N} is a 2, 3-group, $S_4(3)$ being a 2, 3, 5-group. By Lemma 9.3, it then follows that $O^{3'}(\mathfrak{N})/\mathfrak{G} \cong SL(2, 3)$. Furthermore, if J is an involution of $O^{3'}(\mathfrak{N})$, then $C_{\mathfrak{G}}(J) \triangleleft \mathfrak{N}$. It follows that J inverts $\mathfrak{G}/Z(\mathfrak{G})$.

If I centralizes $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$, we conclude that I centralizes $O^{3'}(\mathfrak{N})/\mathfrak{G}$. But $C(I) \subseteq \mathfrak{M}$, so in particular, $C_{\mathfrak{N}}(I) \subseteq \mathfrak{M}$. Since I centralizes $O^{3'}(\mathfrak{N})/\mathfrak{G}$, it follows that I centralizes a S_2 -subgroup \mathfrak{Q} of $O^{3'}(\mathfrak{N})$. Hence, \mathfrak{Q} normalizes \mathfrak{G}_0 . Hence, $\mathfrak{Q}\mathfrak{G}_0$ is of index 3 in $C(I) \cap O^{3'}(\mathfrak{N})$. Let $\mathfrak{Q}' = \langle J \rangle$. By the preceding paragraph, \mathfrak{Q} is faithfully represented on \mathfrak{G}_0 . Thus, $\mathfrak{D} = C(I) \cap N(\mathfrak{Q}) \cap O^{3'}(\mathfrak{N}) \cong SL(2, 3)$ and \mathfrak{D} is faithfully represented on \mathfrak{G}_0 .

Since \mathfrak{G}_0 is faithfully represented on $O_2(\mathfrak{M})$, so is $\mathfrak{G}_0\mathfrak{D}$. Since $\mathfrak{G}_0 \cap \mathfrak{D} = 1$, S_3 -subgroups of $\mathfrak{G}_0\mathfrak{D}$ are of exponent 3. Since the four subgroups of \mathfrak{G}_0 of order 3 are permuted transitively by \mathfrak{D} , it follows that $w_1 \geq 4$. Hence, $w_1 = 4$ and $O_2(\mathfrak{M})$ is extra special. Let \mathfrak{F}_0 be a S_3 -subgroup of $\mathfrak{G}_0\mathfrak{D}$. We can choose P in $\mathfrak{F}_0 - \mathfrak{G}_0$ such that $C(P) \cap O_2(\mathfrak{M})$ contains a four-group. Since $C_{\mathfrak{N}}(P)$ clearly contains an element of $\mathcal{Z}(3)$, Lemma 7.4 is violated. We conclude that I does not centralize $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$.

Since $\text{Aut}(Z(\mathfrak{G}))$ is abelian, the preceding paragraph implies that $Z(\mathfrak{G}) = Z(\mathfrak{F})$.

Since $O_2(\mathfrak{M}) \not\subseteq \mathfrak{N}$, we can choose H in \mathfrak{G}_0^* such that $C(H) \not\subseteq \mathfrak{N}$.

Let $|Z(\mathfrak{G})| = 3^a$, and suppose $a \geq 2$. Let \mathfrak{F} be a S_3 -subgroup of $C_{\mathfrak{N}}(H)$. Thus, $Z(\mathfrak{G}) \subseteq Z(\mathfrak{F})$, and $\mathfrak{Z} = \sigma^{a-1}(Z(\mathfrak{F})) \text{ char } \mathfrak{F}$, whence \mathfrak{F} is a S_3 -subgroup of $C(H)$. By Lemma 7.2 applied to $\langle H, \mathfrak{Z} \rangle$, it follows that \mathfrak{Z} centralizes $O_3(C(H))$, and so

$$[O_3(C(H)), C_{\mathfrak{G}}(H)] \subseteq \mathfrak{G} \cap O_3(C(H)) = 1.$$

By Lemma 9.4, we have $O_3(C(H)) = 1$. Let $\mathfrak{F}_1 = O_3(C(H)) \subseteq \mathfrak{F}$. Thus, $Z(\mathfrak{G}) \subseteq Z(\mathfrak{F}_1)$, and we get $\mathfrak{Z} = \sigma^{a-1}(Z(\mathfrak{F}_1))$, whence $C(H) \subseteq \mathfrak{N}$. This contradiction forces $a = 1$, $Z(\mathfrak{G}) = \mathfrak{Z}$, $|\mathfrak{F}| = 3^6$.

Throughout the remainder of this lemma, the following notation is used: \mathfrak{Q} is a S_2 -subgroup of $O^{3'}(\mathfrak{N})$ normalized by I . Since \mathfrak{Q} is a quaternion group, our preceding information implies the existence of an element Q in \mathfrak{Q} of order 4 such that $IQI = Q^{-1}$. Let $J = Q^2$. Thus, J centralizes \mathfrak{Z} and inverts $\mathfrak{G}/\mathfrak{Z}$.

We argue that \mathfrak{G} is not 3-normal. Namely, for some H in \mathfrak{G}_0^* , we have $\mathfrak{C} = C(H) \not\subseteq \mathfrak{N}$. If $|\mathfrak{C}|_3 = |\mathfrak{G}|_3$, then $\langle H \rangle$ is a conjugate of \mathfrak{Z} contained in \mathfrak{G} , and we are done. Otherwise, it is clear that $\mathfrak{C} \cap \mathfrak{N}$ contains a S_3 -subgroup of \mathfrak{C} and since $\mathfrak{Z} \subseteq \mathfrak{C}$, $O_3(\mathfrak{C})$ contains at least two conjugates of \mathfrak{Z} . As $O_3(\mathfrak{C}) \subseteq \mathfrak{N}$, we again are done.

We next argue that \mathfrak{Z} is not weakly closed in \mathfrak{G} . Choose G in

\mathfrak{G} such that $\mathfrak{Z}_1 = \mathfrak{Z}^\sigma \subseteq \mathfrak{P}$ and $\mathfrak{Z} \neq \mathfrak{Z}_1$. If $\mathfrak{Z}_1 \subseteq \mathfrak{H}$, we are done. Otherwise, let $\mathfrak{P}_0 = C_{\mathfrak{P}}(\mathfrak{Z}_1) = \mathfrak{Z}_1 \times C_{\mathfrak{H}}(\mathfrak{Z}_1)$. Since \mathfrak{P}_0 is not a S_3 -subgroup of \mathfrak{G} but \mathfrak{P}_0 is a S_3 -subgroup of $C_{\mathfrak{H}}(\mathfrak{Z}_1)$, it follows that \mathfrak{Z} ch/ar \mathfrak{P}_0 . This implies that \mathfrak{P}_0 is elementary. Clearly, $27 \leq |\mathfrak{P}_0| \leq 81$, since $w = 2$. We assume without loss of generality that $\mathfrak{P} \cap \mathfrak{P}^\sigma = \mathfrak{P}_0$. If $\mathfrak{Z} \subseteq \mathfrak{H}^\sigma$, we are done, so we may assume that $\mathfrak{Z} \not\subseteq \mathfrak{H}^\sigma$, which yields $\mathfrak{P}_0 = \mathfrak{Z} \times (\mathfrak{P}_0 \cap \mathfrak{H}^\sigma)$. In particular, $\mathfrak{H} \cap \mathfrak{H}^\sigma \neq 1$; let $\mathfrak{B} = \mathfrak{H} \cap \mathfrak{H}^\sigma$, a group of order at least 3. Suppose $|\mathfrak{B}| = 3$. Let $\mathfrak{R} = C(B)$, so that $|\mathfrak{R} \cap \mathfrak{P}| = |\mathfrak{R} \cap \mathfrak{P}^\sigma| = 3^5$. If $|\mathfrak{R}|_3 = 3^6$, we are done, so we may assume that $|\mathfrak{R}|_3 = 3^5$. In this case, we see that $|O_3(\mathfrak{R})| = 3^4$, which implies that $|O_3(\mathfrak{R}) \cap \mathfrak{H}| \geq 27$, $|O_3(\mathfrak{R}) \cap \mathfrak{H}^\sigma| \geq 27$. Hence, $|\mathfrak{H} \cap \mathfrak{H}^\sigma| \geq 9$, contrary to assumption. Thus, we may assume that $|\mathfrak{B}| = 9$. We may also assume that \mathfrak{B} contains no conjugate of \mathfrak{Z} . We have $\mathfrak{P}_0 = \mathfrak{Z} \times \mathfrak{Z}_1 \times \mathfrak{B}$. We argue that $\mathfrak{P}_0 \triangleleft \langle \mathfrak{P}, \mathfrak{P}^\sigma \rangle$. Namely, let $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \mathfrak{P}$. If \mathfrak{P}_0 char \mathfrak{P}_1 , then clearly \mathfrak{P} normalizes \mathfrak{P}_0 . Suppose \mathfrak{P}_0 ch/ar \mathfrak{P}_1 . Then $\mathfrak{P}_1 = \mathfrak{P}_0 \mathfrak{P}_0^*$, where \mathfrak{P}_0^* is elementary of order 3^4 . Hence, $Z(\mathfrak{P}_1) = \mathfrak{P}_0 \cap \mathfrak{P}_0^*$ is of order 27. This implies that \mathfrak{P}_1 is of order 3. Since $\mathfrak{P}_1 \cap \mathfrak{H}$ is nonabelian, we have $\mathfrak{P}_1 = \mathfrak{Z}$. Thus, \mathfrak{Z}_1 centralizes $(\mathfrak{P}_1 \cap \mathfrak{H})/\mathfrak{Z}$. This is not the case, since involutions of $O^p(\mathfrak{R})$ invert $\mathfrak{H}/\mathfrak{Z}$, so that the action of \mathfrak{Z}_1 on $\mathfrak{H}/\mathfrak{Z}$ is given either by $J_3 \oplus J_1$ or by $J_2 \oplus J_2$. By symmetry, we have $\mathfrak{P}_0 \triangleleft \langle \mathfrak{P}, \mathfrak{P}^\sigma \rangle$. It is easy to verify that $O_3(N(\mathfrak{P}_0))$ is of order 3^5 , which implies that $|\mathfrak{H} \cap \mathfrak{H}^\sigma| \geq 27$, the desired contradiction.

Since all parts of Hypothesis 9.3 are satisfied, Lemma 9.5 is violated. The proof of the lemma is complete.

LEMMA 9.8. \mathfrak{R} is the only element of $\mathcal{MS}(\mathfrak{G})$ which contains \mathfrak{H} .

Proof. Suppose false. Choose $\mathfrak{R} \in \mathcal{Sol}(\mathfrak{G})$ so that $\mathfrak{H} \subseteq \mathfrak{R} \not\subseteq \mathfrak{R}$, and with this restriction, minimize $|\mathfrak{R}|$. Since $\mathcal{N}(\mathfrak{H})$ contains only 1, Lemma 0.7.6 implies that $l_3(\mathfrak{R}) \leq 2$. If $l_3(\mathfrak{R}) = 1$, then $\mathfrak{Z} \triangleleft \mathfrak{R}$, contrary to assumption. Hence, $l_3(\mathfrak{R}) = 2$; and \mathfrak{R} is a 3, p -group for some prime p . Furthermore, \mathfrak{H} acts irreducibly on $O_{3,p}(\mathfrak{R})/D(O_{3,p}(\mathfrak{R} \text{ mod } O_3(\mathfrak{R})))$. Let $\mathfrak{H}_0 = \mathfrak{H} \cap O_3(\mathfrak{R})$, $\mathfrak{B} = \Omega_1(Z(O_3(\mathfrak{R})))$. By Lemma 9.4, $\mathfrak{B} \subseteq \mathfrak{H}$, so $|\mathfrak{B}| \leq 9$. Thus, $O_{3,p}(\mathfrak{R})/O_3(\mathfrak{R})$ is a quaternion group whose involution inverts \mathfrak{B} . Since $\mathfrak{Z} \subset \mathfrak{B}$, Lemma 9.7 is violated. The proof is complete.

LEMMA 9.9. Every involution I of \mathfrak{R} inverts $\mathfrak{H}/Z(\mathfrak{H})$.

Proof. By Lemma 9.7, I centralizes \mathfrak{Z} , so centralizes $Z(\mathfrak{H})$. If the lemma is false, then $C_{\mathfrak{H}}(I)$ contains a subgroup \mathfrak{A} of type (3, 3) with $\mathfrak{A} \supset \mathfrak{Z}$. This violates Lemma 9.3 (iii).

LEMMA 9.10. If $\mathfrak{A} \in \mathcal{A}_4(\mathfrak{P})$, then \mathfrak{R} is the only element of $\mathcal{MS}(\mathfrak{G})$ which contains \mathfrak{A} .

Proof. As in O , let $\mathcal{A}_1 = \{\mathfrak{A} \mid \mathfrak{A} \text{ is a 3-subgroup of } \mathfrak{N}, \mathfrak{A} \text{ contains an element of } \mathcal{S}\mathcal{E}\mathcal{N}_3(\mathfrak{B}^N) \text{ for some } N \text{ in } \mathfrak{N}, \mathcal{A}_{n+1} = \{\mathfrak{A} \mid \mathfrak{A} \text{ is a 3-subgroup of } \mathfrak{N}, \mathfrak{A} \text{ contains a subgroup } \mathfrak{B} \text{ of type } (3, 3), C_{\mathfrak{N}}(B) \text{ contains an element of } \mathcal{A}_n \text{ for all } B \text{ in } \mathfrak{B}\}$. Among all $\mathfrak{A} \in \mathcal{A}_1$ which violate the conclusion of the lemma, maximize $|\mathfrak{A} \cap \mathfrak{G}|$, and with this restriction, maximize $|\mathfrak{A}|$. By Lemma 9.8, $\mathfrak{G} \not\subseteq \mathfrak{A}$. Let $\mathfrak{M} \in \mathcal{MS}(\mathfrak{G})$ with $\mathfrak{A} \subseteq \mathfrak{M}$, $\mathfrak{M} \neq \mathfrak{N}$. By maximality of $|\mathfrak{A}|$, it follows that \mathfrak{A} is a S_3 -subgroup of \mathfrak{M} . We can therefore choose a prime q and a q -subgroup \mathfrak{Q} of \mathfrak{M} permutable with \mathfrak{A} such that $\mathfrak{Z} = \mathfrak{A}\mathfrak{Q}$ is not contained in \mathfrak{N} . Let \mathfrak{Q} be minimal with these properties. By Lemma 0.7.6, $l_3(\mathfrak{Z}) \leq 2$.

We first show that $O_q(\mathfrak{Z}) \subseteq \mathfrak{N}$. Suppose $\mathfrak{A} \cap \mathfrak{G}$ is noncyclic. Let \mathfrak{B} be a subgroup of $\mathfrak{A} \cap \mathfrak{G}$ of type $(3, 3)$. It suffices to show that $C(B) \subseteq \mathfrak{N}$ for all $B \in \mathfrak{B}^*$. Suppose false. Then maximality of $|\mathfrak{A} \cap \mathfrak{G}|$ yields $|\mathfrak{G} : \mathfrak{A} \cap \mathfrak{G}| \leq 3$. In this case, $\mathcal{N}(\mathfrak{A} \cap \mathfrak{G}) = 1$, so $O_q(\mathfrak{Z}) = 1$. Thus, we may assume that $\mathfrak{A} \cap \mathfrak{G}$ is cyclic. Since $w \geq 2$, it follows that if P is any element of \mathfrak{B} of order 3, then $C_{\mathfrak{G}}(P)$ is noncyclic. Hence, every subgroup of \mathfrak{N} of type $(3, 3)$ is in \mathcal{A}_1 . Since \mathfrak{A} contains a subgroup of type $(3, 3)$, maximality of $|\mathfrak{A} \cap \mathfrak{G}|$ implies that $C(A) \subseteq \mathfrak{N}$ for all elements A of \mathfrak{A} of order 3. Thus, in all cases, we have $O_q(\mathfrak{Z}) \subseteq \mathfrak{N}$.

By minimality of \mathfrak{Q} , $O_{q,3}(\mathfrak{Z}) = O_q(\mathfrak{Z}) \times O_3(\mathfrak{Z})$. Since $l_3(\mathfrak{Z}) \leq 2$, it follows that $l_3(\mathfrak{Z}) = 2$, by maximality of $|\mathfrak{A}|$ and the structure of $O_{q,3}(\mathfrak{Z})$. Since $D(\mathfrak{Q})$ is permutable with \mathfrak{A} , we get $D(\mathfrak{Q}) \subseteq \mathfrak{N}$, by minimality of \mathfrak{Q} .

Clearly, \mathfrak{A} is a S_3 -subgroup of $N(O_3(\mathfrak{Z}))$. Hence, $\mathfrak{Z} \subseteq Z(O_3(\mathfrak{Z}))$. Since $\mathfrak{Q}O_3(\mathfrak{Z}) \triangleleft \mathfrak{Z}$, and since \mathfrak{A} is a S_3 -subgroup of $N(O_3(\mathfrak{Z}))$, and since $\mathfrak{G} \not\subseteq \mathfrak{A}$, it follows that $\mathfrak{G} \cap \mathfrak{A}$ acts nontrivially on $Q_3(\mathfrak{Z})$, but trivially on every proper \mathfrak{A} -invariant subgroup of $Q_3(\mathfrak{Z})$. Since $D(\mathfrak{Q})$ centralizes \mathfrak{Z} , it follows that $D(\mathfrak{Q})$ centralizes $\mathfrak{B} = \mathfrak{Z}^{\mathfrak{Z}}$.

If $q \geq 5$, then maximality of \mathfrak{A} and Theorem 1 of [39] imply that $\mathfrak{A} = \mathfrak{B}$, against Lemma 9.8. Hence, $q = 2$. We may apply Theorem 1 of [39] once again and conclude that $D(\mathfrak{Q}) \neq 1$. By Lemma 9.9, each element of $D(\mathfrak{Q})^*$ inverts $\mathfrak{G}/\mathfrak{Z}(\mathfrak{G})$. Since $Z(\mathfrak{G})$ is a normal cyclic subgroup of \mathfrak{B} , it follows that $\mathfrak{A} \cap Z(\mathfrak{G}) \subseteq O_3(\mathfrak{Z})$. Since $\mathfrak{A} \cap \mathfrak{G} \not\subseteq O_3(\mathfrak{Z})$, choose $H \in \mathfrak{A} \cap \mathfrak{G} - O_3(\mathfrak{Z})$. Let I be the element in $D(\mathfrak{Q})^*$. Then $H^I = H^{-1}H_0$ with H_0 in $Z(\mathfrak{G})$. Since $H_0 \in \mathfrak{A}$, it follows that $[H, I]$ is contained in $\mathfrak{A} \cap \mathfrak{G} \cap O_{3,4}(\mathfrak{Z}) \subseteq \mathfrak{A} \cap O_3(\mathfrak{Z})$. This violates the fact that $\mathfrak{A} \cap \mathfrak{G} \not\subseteq O_3(\mathfrak{Z})$. The proof is complete.

It is now easy to show that Hypothesis 9.2 is not satisfied. Otherwise, \mathfrak{N} contains a four-subgroup \mathfrak{T} . But by Lemma 9.9, each element of \mathfrak{T}^* inverts $\mathfrak{G}/Z(\mathfrak{G})$. This is not possible, since $\mathfrak{G} = \langle C_{\mathfrak{G}}(J) \mid J \in \mathfrak{T}^* \rangle$.

The remaining lemmas in this section are proved on the hypothesis

that \mathfrak{G} contains a noncyclic characteristic abelian subgroup.

Among all noncyclic normal elementary subgroups of \mathfrak{N} , let \mathfrak{E} be minimal. Thus, $\mathfrak{E}/\mathfrak{Z}$ is a chief factor of \mathfrak{N} . Let $\mathcal{E} : \mathfrak{E} \supset \mathfrak{Z} \supset 1$. We will show that $A_{\mathfrak{G}}(\mathcal{E}) = A(\mathcal{E})$. First, suppose \mathfrak{E} is not 3-reducible in \mathfrak{N} . Let $\mathfrak{X} = O_3(\mathfrak{N} \text{ mod } C(\mathfrak{E}))$. Since $\mathfrak{E}/\mathfrak{Z}$ is a chief factor of \mathfrak{N} , we have $[\mathfrak{X}, \mathfrak{E}] = \mathfrak{Z}$, and $\mathfrak{Z} = C_{\mathfrak{E}}(\mathfrak{X})$. These equalities imply immediately that \mathfrak{X} maps onto $A(\mathcal{E})$. Suppose \mathfrak{E} is 3-reducible in \mathfrak{N} . Let $\mathfrak{X} = O_3(\mathfrak{N} \text{ mod } C(\mathfrak{E}))$. Then $\mathfrak{Z} = C_{\mathfrak{E}}(\mathfrak{X})$ and $[\mathfrak{X}, \mathfrak{E}]$ admits $N_{\mathfrak{N}}(\mathfrak{X}) = \mathfrak{N}$. Since $A_{\mathfrak{X}}(\mathfrak{E})$ is a 3'-group it follows that $[\mathfrak{X}, \mathfrak{E}]$ is a normal subgroup of \mathfrak{N} disjoint from \mathfrak{Z} . Hence, $[\mathfrak{X}, \mathfrak{E}] = 1$, since \mathfrak{Z} is the only minimal normal subgroup of \mathfrak{N} . Since \mathfrak{E} is 3-reducible in \mathfrak{N} and $O_3(\mathfrak{N} \text{ mod } C(\mathfrak{E}))$, it follows that $\mathfrak{E} \subseteq Z(\mathfrak{N})$. This is absurd since \mathfrak{Z} is the only minimal normal subgroup of \mathfrak{N} . Thus, $A_{\mathfrak{G}}(\mathcal{E}) = A(\mathcal{E})$.

Throughout the remainder of this section, the following notation is used: $\mathfrak{F}, \mathfrak{Z}, \mathfrak{N}$ are as before, and \mathfrak{E} is a noncyclic normal elementary subgroup of \mathfrak{N} such that $\mathfrak{E}/\mathfrak{Z}$ is a chief factor of \mathfrak{N} . Also, $\mathcal{E} : \mathfrak{E} \supset \mathfrak{Z} \supset 1$.

LEMMA 9.11. (i) *If \mathfrak{S} is a 2, 3-subgroup of \mathfrak{G} and \mathfrak{S} contains an element \mathfrak{A} of $\mathcal{E}(3)$, then $O_2(\mathfrak{S}) = 1$.*

(ii) *If $\mathfrak{A} \in \mathcal{E}(3)$, then $|C(\mathfrak{A})|$ is odd.*

Proof. (i) Suppose I is an involution in $O_2(\mathfrak{S})$. Since \mathfrak{A} centralizes $O_2(\mathfrak{S})$, Lemmas 7.4 and 5.38 imply that $C(I)$ is nonsolvable.

(ii) Suppose I is an involution of $C(\mathfrak{A})$. Then $\mathfrak{A} \times \langle I \rangle$ violates (i).

LEMMA 9.12. (i) *If I is an involution of $C(\mathfrak{Z})$, then I inverts $\mathfrak{E}/\mathfrak{Z}$.*

(ii) *$C(\mathfrak{Z})$ contains no four-group.*

(iii) *If \mathfrak{X} is an abelian 2-subgroup of \mathfrak{N} , then $A_{\mathfrak{N}}(\mathfrak{X})$ is a 2-group.*

Proof. (i) is a consequence of Lemmas 9.11 and 7.3, and (ii), (iii) are consequences of (i).

LEMMA 9.13. *\mathfrak{N} does not contain a noncyclic abelian subgroup of order 8.*

Proof. Suppose false. Let \mathfrak{D}_3^* be a S_2 -subgroup of \mathfrak{N} permutable with \mathfrak{F} , and let $\mathfrak{N}_0 = \mathfrak{F}\mathfrak{D}_3^*$. Let $\mathfrak{D}_0 = \mathfrak{D}_3^* \cap O^{3'}(\mathfrak{N}_0)$. Thus, \mathfrak{D}_0 is either a quaternion group or $\mathfrak{D}_0 = 1$. Let \mathfrak{D} be a subgroup of \mathfrak{D}_3^* which contains \mathfrak{D}_0 , is permutable with \mathfrak{F} , contains a noncyclic abelian subgroup of order 8, and is minimal with these properties. Let $\mathfrak{N}_1 = \mathfrak{F}\mathfrak{D}$. Thus, \mathfrak{D} is abelian of type (2, 4) if and only if every 2, 3-subgroup of \mathfrak{N} is 3-closed. If $\mathfrak{D}_0 \neq 1$, then $|\mathfrak{D}| = 2^4$ and \mathfrak{D} is either the direct product of a group of order 2 and \mathfrak{D}_0 or \mathfrak{D} is the central product of a cyclic group of order 4 and \mathfrak{D}_0 . Let $\mathfrak{F}/\mathfrak{Z}$ be a chief factor of \mathfrak{N}_1

with $\mathfrak{F} \cong \mathbb{C}$. Let $\mathfrak{P}_0 = O_3(\mathfrak{N}_1)$, $\mathfrak{P}_1 = C_{\mathfrak{P}_0}(\mathfrak{F})$. Since $A_{\mathfrak{P}_1}(\mathcal{E}) = A(\mathcal{E})$, so also $A_{\mathfrak{P}_1}(\mathcal{E}_0) = A(\mathcal{E}_0)$ where $\mathcal{E}_0: \mathfrak{F} \supset \mathfrak{Z} \supset 1$. Hence, $\mathfrak{P}_0/\mathfrak{P}_1$ is also a chief factor of \mathfrak{N}_1 with the same order as $\mathfrak{F}/\mathfrak{Z}$. If $\Omega' = 1$, then

$$(9.19) \quad \mathfrak{P} \triangleleft \mathfrak{N}_1, \Omega \text{ is of type } (4, 2), \text{ and } |\mathfrak{F}:\mathfrak{Z}| = 9.$$

Suppose $\Omega' \neq 1$. If $\Omega = \Omega_0 \times \Omega_1$, where $|\Omega_1| = 2$, then

$$(9.20) \quad \mathfrak{N}_1/\mathfrak{P}_0 \cong SL(2, 3) \times Z_2, \text{ and } |\mathfrak{F}:\mathfrak{Z}| = 9.$$

Suppose Ω is the central product of Ω_0 and a cyclic group of order 4. Then

$$(9.21) \quad \begin{aligned} \mathfrak{N}_1/\mathfrak{P}_0 \text{ is the central product of} \\ SL(2, 3) \text{ and } Z_4, \text{ and } |\mathfrak{F}:\mathfrak{Z}| = 3^4. \end{aligned}$$

By Lemmas 5.41 and 9.12, (9.19), (9.20), (9.21) exhaust all possibilities.

It is clear from Lemma 9.12 that

$$(9.22) \quad \text{if (9.19) holds, a } S_{2,3}\text{-subgroup of } \mathfrak{N} \text{ is 3-closed.}$$

We next will show that

$$(9.23) \quad \text{every subgroup of } \mathfrak{F} \text{ of order 9 is in } \mathcal{S}.$$

To see this, let \mathfrak{F}_0 be a subgroup of \mathfrak{F} of order 9. If $\mathfrak{Z} \subset \mathfrak{F}_0$, then $\mathfrak{F}_0 \in \mathcal{E}(3) \subseteq \mathcal{S}$. Thus, we may assume that $\mathfrak{F}_0 \cap \mathfrak{Z} = 1$. Let \mathfrak{X} be an abelian subgroup in $\mathfrak{N}(\mathfrak{F}_0; 2)$ and assume by way of contradiction that $[\mathfrak{X}, \mathfrak{F}_0] \neq 1$. We may assume that \mathfrak{X} is a four-group. Let $\mathfrak{F}_1 = \mathfrak{F}_0 \cap C(\mathfrak{X})$, a group of order 3. Let $\mathbb{C} = C(\mathfrak{F}_1) \cong \langle \mathfrak{F}, \mathfrak{X} \rangle$. Since $m(\mathfrak{F}) \geq 3$ and $\mathfrak{F} \triangleleft \mathfrak{P}$, there is $\mathfrak{A} \in \mathcal{S}\mathcal{C}\mathcal{N}_3(\mathfrak{P})$ with $\mathfrak{F} \subseteq \mathfrak{A}$. Hence, $\mathfrak{A} \subseteq \mathbb{C} = C(\mathfrak{F}_1)$ implies $O_3(\mathbb{C}) = 1$ by hypothesis (ii) of Theorem 9.1.

By Lemma 5.5, $\mathfrak{Z} \subseteq O_3(\mathbb{C})$. Let $\mathfrak{B} = \Omega_1(\mathbf{Z}(O_3(\mathbb{C})))$, and let \mathfrak{P}^* be a S_3 -subgroup of \mathbb{C} . Let \mathfrak{P}^g be a S_3 -subgroup of \mathbb{C} which contains \mathfrak{P}^* . Then $\mathfrak{Z}^g \subset \mathfrak{P}^*$, so $\mathfrak{Z}^g \subseteq \mathfrak{B}$. By Lemma 9.12 (iii), \mathfrak{X} is faithfully represented on \mathfrak{B} . Hence, if $F \in \mathfrak{F}_0 - \mathfrak{F} \cap C(\mathfrak{X})$, then the minimal polynomial of F on \mathfrak{B} is $(x - 1)^3$. On the other hand, \mathfrak{Z} centralizes \mathfrak{B} . Since $\mathbb{C} \triangleleft \mathfrak{N}$, the minimal polynomial of F on \mathfrak{B} is a divisor of $(x - 1)^2$. This contradiction establishes (9.23).

Since Ω contains an abelian subgroup of type (2, 4), we can choose an involution I of Ω such that $\mathfrak{F}_0 = C_{\mathfrak{F}}(I)$ is noncyclic. By Lemmas 7.4 and 5.38, $C(I)$ contains no element of $\mathcal{E}(3)$. Hence, I inverts \mathfrak{Z} . Thus, in cases (9.19), (9.20) respectively, we have

$$(9.19)'-(9.20)' \quad \mathfrak{F} = \mathfrak{F}_0 \times \mathfrak{Z}.$$

In case (9.21), we have

$$(9.21)' \quad |\mathfrak{F}_0| = |C_{\mathfrak{P}_0/\mathfrak{P}_1}(I)| = 9.$$

Thus, in case (9.21), we have $|C_{\mathfrak{F}_0}(I)| \geq 3^4$.

Let \mathfrak{Z} be a $S_{2,3}$ -subgroup of $C(I)$ which contains \mathfrak{F}_0 . Since $\mathfrak{F}_0 \in \mathcal{D}$ and since $C(I)$ contains an element of $\mathcal{Z}(2)$, there is an element \mathfrak{M} of $\mathcal{M}\mathcal{C}(G)$ which satisfies all the conclusions of Lemma 7.5, contains \mathfrak{F}_0 and contains a S_2 -subgroup of \mathfrak{Z} . By Lemma 7.5 (f), $I \in O_2(\mathfrak{M})$.

We will show that

$$(9.24) \quad C_{\mathfrak{M}_1}(I) \subseteq \mathfrak{M} .$$

By Lemma 7.5 (f), it suffices to show that \mathfrak{M} contains an S_2 -subgroup of $C(I)$. By construction, \mathfrak{M} contains an S_2 -subgroup of \mathfrak{Z} , which is an $S_{2,3}$ -subgroup of $C(I)$. This proves (9.24).

Suppose (9.19) holds. In this case, we have (9.22). Also, (9.19) implies that every element of \mathcal{G} of order 3 centralizes an element of $\mathcal{E}(3)$. Let \mathfrak{F}_1 be a subgroup of \mathfrak{F}_0 of order 3 such that

$$[O_2(\mathfrak{M}) \cap C(\mathfrak{F}_1), \mathfrak{F}_0] = \Omega^* \neq 1 .$$

Thus, Ω^* is a quaternion group and a $S_{2,3}$ -subgroup of $C(\mathfrak{F}_1)$ is not 3-closed. By (9.22), $\mathfrak{F}_1 \simeq \mathfrak{Z}$. Since $|C_{\mathfrak{M}_1}(\mathfrak{F}_1)|_3 = |\mathfrak{P}|/3$, it follows that $C_{\mathfrak{M}_1}(\mathfrak{F}_1)$ contains a S_3 -subgroup of $C(\mathfrak{F}_1)$. Let \mathfrak{P}^* be a S_3 -subgroup of $C_{\mathfrak{M}_1}(\mathfrak{F}_1)$. Since $C(\mathfrak{F}_1)$ contains an element of $\mathcal{L}_{ev_3}(\mathfrak{P})$, it follows that $O_3(\mathbb{C}) = 1$, where $\mathbb{C} = C(\mathfrak{F}_1)$. Let $\mathfrak{R} = O_3(\mathbb{C})/\mathfrak{F}_1$. Thus, $\Omega^* \langle F \rangle$ is faithfully represented on $Z(\mathfrak{R})$ for each F in $\mathfrak{F}_0 - \mathfrak{F}_1$. But $[\mathfrak{R}, F] \subseteq \langle \mathfrak{Z}, \mathfrak{F}_1 \rangle / \mathfrak{F}_1$, so Ω^* centralizes a subgroup of $O_3(\mathfrak{R})$ of index 9.

Suppose (9.19) holds and $O_3(\mathbb{C}) \cap C(I)$ is noncyclic, where I is the involution of Ω^* . In this case, since \mathfrak{F}_0 centralizes I and $O_3(\mathbb{C}) \cap \mathfrak{F}_0 = \mathfrak{F}_1$, it follows that $\mathbb{C} \cap \mathfrak{M}$ contains a subgroup of order 27 and exponent 3. Since every element of \mathcal{G} of order 3 centralizes an element of $\mathcal{E}(3)$, it follows that a S_3 -subgroup of \mathfrak{M} is nonabelian of order 27 and the width of $O_2(\mathfrak{M})$ is 3. Since $|O_3(\mathbb{C}) : O_3(\mathbb{C}) \cap C(I)| = 9$, it follows that $|O_3(\mathbb{C})| \leq 3^4$. Since $O_3(\mathbb{C}) \cap C(I)$ is assumed noncyclic, and since $m(Z(O_3(\mathbb{C}))) \geq 3$, it follows that $O_3(\mathbb{C})$ is elementary of order 3^4 . Since $\Omega^* \subseteq \mathbb{C}$, and since $\langle I \rangle = O_2(\mathfrak{M}) \cap C(\mathfrak{F}_0)$, it follows that $O_3(\mathbb{C})$ is of index 3 in \mathfrak{P}^* . Hence, $|\mathfrak{P}| = 3^6$, since \mathfrak{P}^* is of index 3 in some S_3 -subgroup of \mathcal{G} .

Since $|\mathfrak{P}^*| = 3^5$, we have $\mathfrak{P}^* = O_3(\mathbb{C})\mathfrak{F}_0$.

We argue that $C_{\mathfrak{P}^*}(F)$ is of index 9 in \mathfrak{P}^* for every F in $\mathfrak{F}_0 - \mathfrak{F}_1$. This assertion is equivalent to the assertion that $O_3(\mathbb{C}) \cap C(F)$ is of order 9, since $\mathfrak{P}^* = O_3(\mathbb{C}) \langle F \rangle$. Now $O_3(\mathbb{C}) = \mathfrak{U}_1 \times \mathfrak{U}_2$, where $\mathfrak{U}_1 = C(I) \cap O_3(\mathbb{C})$, \mathfrak{U}_2 is inverted by I , and $|\mathfrak{U}_1| = |\mathfrak{U}_2| = 9$. Since $\mathfrak{U}_i \triangleleft \mathfrak{P}^* \Omega^*$, $i = 1, 2$, we must show that F does not centralize either \mathfrak{U}_1 or \mathfrak{U}_2 . It is obvious that F does not centralize \mathfrak{U}_2 . If F centralizes \mathfrak{U}_1 , then $\langle \mathfrak{U}_1, F \rangle$ is elementary of order 27 and is contained in \mathfrak{M} , whereas we already know that S_3 -subgroups of \mathfrak{M} are nonabelian of order 27. So

$$|\mathfrak{P}^*: C_{\mathfrak{P}^*}(F)| = 9.$$

Since $C_{\mathfrak{P}^*}(F)$ is of index 9 in \mathfrak{P}^* for every F in $\mathfrak{F}_0 - \mathfrak{F}_1$, it follows that $O_3(\mathbb{C}) \text{ char } \mathfrak{P}^*$. Thus, $N(O_3(\mathbb{C}))$ contains a S_3 -subgroup of \mathbb{C} and $S_{2,3}$ -subgroup of $N(O_3(\mathbb{C}))$ are not 3-closed. This implies that if $\tilde{\mathfrak{P}}$ is a S_3 -subgroup of $N(O_3(\mathbb{C}))$, then $O_3(\mathbb{C})$ is not characteristic in $\tilde{\mathfrak{P}}$. More explicitly, $N(O_3(\mathbb{C})) \cap N(\tilde{\mathfrak{P}})$ does not contain a noncyclic abelian subgroup of order 8, while $N(\tilde{\mathfrak{P}})$ does. Let \mathfrak{A} be an elementary subgroup of \mathfrak{P} of order 3^4 with $\mathfrak{A} \neq O_3(\mathbb{C})$. If $\mathfrak{A} \cap O_3(\mathbb{C})$ is of order 9, then $\tilde{\mathfrak{P}} = \mathfrak{A}O_3(\mathbb{C})$ and $Z(\tilde{\mathfrak{P}})$ is not cyclic. Hence, $\mathfrak{A} \cap O_3(\mathbb{C})$ is of order 27. Hence, $\tilde{\mathfrak{P}} = \mathfrak{P}^*\mathfrak{A}$, and it follows that $N(O_3(\mathbb{C})) \cap C(I)$ contains S_3 -subgroups of order 3^4 . Furthermore, every subgroup of $\tilde{\mathfrak{P}}$ of order 3 centralizes an element of $\mathcal{E}(3)$. Since the width of $O_2(\mathfrak{M})$ is 3, it follows that a S_3 -subgroup of \mathfrak{M} is of the shape $Z_3 \wr Z_3$. But we have already shown that S_3 -subgroups of \mathfrak{M} are of order 27.

Suppose (9.19) holds and $O_3(\mathbb{C}) \cap C(I)$ is cyclic. Since S_3 -subgroups of \mathfrak{M} are of exponent 3 or 9, it follows that $|O_3(\mathbb{C}) \cap C(I)| = 3$ or 9. Hence, $|O_3(\mathbb{C})| \leq 3^4$, so $O_3(\mathbb{C})$ is abelian. Hence, $\mathfrak{P}^* = O_3(\mathbb{C})\mathfrak{F}_0$. Since elements of $\mathfrak{F}_0 - \mathfrak{F}_1$ have quadratic minimal polynomial of $O_3(\mathbb{C})$, it follows that $\sigma^{-1}(\mathfrak{P}^*) = \sigma^{-1}(O_3(\mathbb{C})) = \sigma^{-1}(O_3(\mathbb{C}) \cap C(I))$. Hence, $\sigma^{-1}(\mathfrak{P}^*) = 1$, since otherwise $\sigma^{-1}(\mathfrak{P}^*)$ is conjugate to \mathfrak{Z} , against (9.22). Hence, $O_3(\mathbb{C})$ is elementary of order 27.

Since $|O_3(\mathbb{C})| = 27$, we get $|\mathfrak{P}^*| = 3^4$, $|\mathfrak{P}| = 3^5$. Since (9.19) holds, Ω is of type (4, 2) and Ω normalizes \mathfrak{P} . Let $\tilde{\Omega} = \Omega \cap C(\mathfrak{Z})$. Thus, $\tilde{\Omega}$ is cyclic of order 4, by Lemma 9.12 (ii). Also, the involution Q of $\tilde{\Omega}$ inverts $\mathfrak{F}/\mathfrak{Z}$, so inverts $\mathfrak{P}/\mathfrak{Z}$. Hence, $\mathfrak{P}/\mathfrak{Z}$ is elementary of order 3^4 and is the direct sum of \mathbb{C}/\mathfrak{Z} and another irreducible Ω -module. This implies that \mathfrak{P} is of exponent 3 and is extra special. Thus, for each P in \mathfrak{P} , $\mathfrak{Z} \text{ char } C_{\mathfrak{P}}(P)$. This implies that \mathfrak{Z} is weakly closed in \mathfrak{P} . But turning back to \mathbb{C} , it follows that Ω^* does not normalize \mathfrak{Z} , so \mathfrak{Z} is not weakly closed in \mathfrak{P} . This contradiction shows that (9.19) does not hold.

Suppose (9.20) holds. By (9.24) it follows that $\Omega\mathfrak{F}_0 \subseteq \mathfrak{M}$. Hence, the width of $O_2(\mathfrak{M})$ is four. Hence, $C_{\mathfrak{P}_0}(I) = \mathfrak{F}_0$. Thus, $C_{\mathfrak{M}_1}(I)$ contains a S_3 -subgroup $\tilde{\mathfrak{P}}$ which is a nonabelian group of order 27 and exponent 3. This is not the case, since $C(P) \cap O_2(\mathfrak{M})$ contains no four-subgroup for any element P of $\tilde{\mathfrak{P}}^*$.

Suppose (9.21) holds. By (9.24) and (9.21)', it follows that S_3 -subgroups of \mathfrak{M} are of order at least 3^4 . Hence, the width of $O_2(\mathfrak{M})$ is four, and $C(I) \cap \mathfrak{P}_0$ contains no subgroup of order 27 and exponent 3, and of course $C(I) \cap \mathfrak{P}_0$ is of exponent 9. This is absurd, since S_3 -subgroups of $\text{Aut}(O_2(\mathfrak{M}))$ contain subgroups of index and exponent 3. This completes the proof of this lemma.

LEMMA 9.14. $N(J(\mathfrak{P}))$ does not contain a noncyclic abelian subgroup

of order 8.

Proof. First, suppose that $J(\mathfrak{P})$ is not elementary. Then $\mathfrak{B} = Z(J(\mathfrak{P})) \cap D(J(\mathfrak{P})) \neq 1$. If \mathfrak{B} is cyclic, then $\Omega_1(\mathfrak{B}) = \mathfrak{Z} \text{ char } N(J(\mathfrak{P}))$, so $N(J(\mathfrak{P})) \subseteq \mathfrak{N}$, and this lemma follows from Lemma 9.13. We may assume that \mathfrak{B} is noncyclic. Let \mathfrak{B}_1 be a noncyclic elementary subgroup of order 9. We will show that $\mathfrak{B}_1 \in \mathcal{E}(3)$. Choose $\mathfrak{Q} \in \mathcal{N}(\mathfrak{B}_1; 3')$, minimal subject to $[\mathfrak{Q}, \mathfrak{B}_1] \neq 1$. Let $\mathfrak{B}_0 = C_{\mathfrak{B}_1}(\mathfrak{Q})$ so that $|\mathfrak{B}_0| = 3$. Let $\mathfrak{C} = C(\mathfrak{B}_0)$, and let \mathfrak{P}^* be a S_3 -subgroup of \mathfrak{C} which contains $J(\mathfrak{P})$. Hence, $J(\mathfrak{P}) = J(\mathfrak{P}^*)$. Let $\mathfrak{P} = \mathfrak{P}^g$ be a S_3 -subgroup of \mathfrak{G} which contains \mathfrak{P}^* .

Since \mathfrak{C} contains an element of $\mathcal{S}_{\text{non}}(\mathfrak{P})$, it follows that $O_3(\mathfrak{C}) = 1$. Since $\mathfrak{B}_1 \subseteq Z(J(\mathfrak{P})) = Z(J(\mathfrak{P}^*))$, we have $[O_3(\mathfrak{C}), \mathfrak{B}_1] \subseteq J(\mathfrak{P})$ and $[O_3(\mathfrak{C}), \mathfrak{B}_1, \mathfrak{B}_1] = 1$. It follows that \mathfrak{Q} is a quaternion group. Let $\mathfrak{B} = (\mathfrak{Z}^g)^{\mathfrak{C}}$. Thus, \mathfrak{B} is a normal elementary 3-subgroup of \mathfrak{C} , and by Lemma 5.10, \mathfrak{B} is 3-reducible in \mathfrak{C} . It is a straightforward consequence of Lemma 5.2 that $\mathfrak{B} \subseteq J(\mathfrak{P}^*)$. Thus, \mathfrak{Q} centralizes \mathfrak{B} , as \mathfrak{B}_1 centralizes \mathfrak{B} . Thus, it follows that $\mathfrak{B}_1 \not\subseteq O_3(\mathfrak{N}_1)$, where \mathfrak{N}_1 is a $S_{2,3}$ -subgroup of \mathfrak{N}^g which contains \mathfrak{P}^g . By Lemma 9.12, $|\mathfrak{P}^g : O_3(\mathfrak{N}_1)| \leq 3$. Thus, $\mathfrak{B}_1 \not\subseteq D(\mathfrak{P}^g)$. This is absurd, since $\mathfrak{B}_1 \subseteq D(J(\mathfrak{P}))$, and $J(\mathfrak{P}) = J(\mathfrak{P}^*) = J(\mathfrak{P}^g)$.

It is an immediate consequence of the preceding argument and Lemmas 7.4 and 5.38 that if $D(J(\mathfrak{P})) \neq 1$, then this lemma holds.

Assume now that $J(\mathfrak{P})$ is elementary. To complete the proof of the lemma, it suffices to show that each subgroup of $J(\mathfrak{P})$ of order 9 is in $\mathcal{E}(3)$. Suppose false, and $\mathfrak{B} \subseteq J(\mathfrak{P})$, $|\mathfrak{B}| = 9$, $\mathfrak{B} \notin \mathcal{E}(3)$. Let \mathfrak{X} be an element of $\mathcal{N}(\mathfrak{B}; 3')$ minimal subject to $[\mathfrak{B}, \mathfrak{X}] \neq 1$. Let $\mathfrak{B}_0 = \mathfrak{B} \cap C(\mathfrak{X})$, so that $|\mathfrak{B}_0| = 3$.

Let $\mathfrak{C} = C(\mathfrak{B}_0) \cong \langle J(\mathfrak{P}), \mathfrak{X} \rangle$. Let \mathfrak{P}^* be a S_3 -subgroup of \mathfrak{C} which contains $J(\mathfrak{P})$, and let \mathfrak{P}^g be a S_3 -subgroup of \mathfrak{G} which contains \mathfrak{P}^* . Hence, $J(\mathfrak{P}) = J(\mathfrak{P}^g)$. Since $J(\mathfrak{P}^g) = J(\mathfrak{P})^g$, we get that $G \in N(J(\mathfrak{P}))$. Replacing \mathfrak{B} by $\mathfrak{B}^{g^{-1}}$ and \mathfrak{X} by $\mathfrak{X}^{g^{-1}}$, we assume without loss of generality that $\mathfrak{P}^* \subseteq \mathfrak{P}$.

Since \mathfrak{P}^* contains an element of $\mathcal{S}\mathcal{E}\mathcal{N}_3(\mathfrak{P})$, it follows that $O_3(\mathfrak{C}) = 1$. Hence, $\mathfrak{Z} \subseteq O_3(\mathfrak{C})$. Let $\mathfrak{B} = \mathfrak{Z}^{\mathfrak{C}}$, so that \mathfrak{B} is a normal elementary subgroup of \mathfrak{C} . Since \mathfrak{B} is 3-reducible in \mathfrak{C} , it follows that $\mathfrak{B} \subseteq J(\mathfrak{P})$. Hence, \mathfrak{X} centralizes \mathfrak{B} . In particular, \mathfrak{X} centralizes \mathfrak{Z} .

Let $\tilde{\mathfrak{Z}} = J(\mathfrak{P}) \cap O_3(\mathfrak{C})$. Thus, $\mathfrak{B} \not\subseteq \tilde{\mathfrak{Z}}$, and $J(\mathfrak{P})/\tilde{\mathfrak{Z}}$ acts faithfully on $O_{3,3'}(\mathfrak{C})/O_3(\mathfrak{C})$. Let $\mathfrak{R} = [O_{3,3'}(\mathfrak{C}), J(\mathfrak{P})]O_3(\mathfrak{C})$. Since

$$[O_3(\mathfrak{C}), J(\mathfrak{P}), J(\mathfrak{P})] = 1,$$

it follows that $\bar{\mathfrak{R}} = \mathfrak{R}/O_3(\mathfrak{C})$ is a 2-group, and that $J(\mathfrak{P})$ centralizes every characteristic abelian subgroup of $\bar{\mathfrak{R}}$. Since $J(\mathfrak{P})$ centralizes \mathfrak{B} , so does \mathfrak{R} . By Lemma 9.12 (ii), \mathfrak{R} contains no four-group. So \mathfrak{R} is a quaternion group and $J(\mathfrak{P})/\tilde{\mathfrak{Z}}$ is of order 3, whence $J(\mathfrak{P}) = \tilde{\mathfrak{Z}}\mathfrak{B}$, and so $\mathfrak{R} =$

$[O_{3,3}(\mathbb{C}), \mathfrak{W}]O_3(\mathbb{C})$. Since $J(\mathfrak{P})O_3(\mathbb{C})/O_3(\mathbb{C})$ is of order 3, it follows that $J(\mathfrak{P}) \subseteq O_{3,3',3}(\mathbb{C})$, and so $\mathfrak{X} \subseteq O_{3,3'}(\mathbb{C})$, whence $\mathfrak{X} \subseteq \mathfrak{K}$, and so $\mathfrak{K} = \mathfrak{X}O_3(\mathbb{C})$, and $\mathfrak{X} \cong \bar{\mathfrak{K}}$ is a quaternion group. Note that \mathfrak{X} is permutable with \mathfrak{P}^* , as \mathfrak{P}^* normalizes $[O_{3,3'}(\mathbb{C}), J(\mathfrak{P})]$.

Enlarge $\mathfrak{P}^*\mathfrak{X}$ to a $S_{2,3}$ -subgroup \mathfrak{N}_0 of \mathfrak{N} , and let $\mathfrak{N}_1 = O_3(\mathfrak{N}_0)$. Thus, $\mathfrak{N}_1 = \mathfrak{X}\tilde{\mathfrak{P}}$, where $\tilde{\mathfrak{P}} = \mathfrak{P}^N$ for some N in \mathfrak{N} . Since $J(\mathfrak{P}) = J(\mathfrak{P}^N) = J(\mathfrak{P})^N$, replacing \mathfrak{W} by $\mathfrak{W}^{N^{-1}}$ and \mathfrak{X} by $\mathfrak{X}^{N^{-1}}$, we assume without loss of generality that $\tilde{\mathfrak{P}} = \mathfrak{P}$ is permutable with \mathfrak{X} .

Let \mathfrak{W}_1 be a subgroup of \mathfrak{W} of order 3 different from \mathfrak{W}_0 . Let $\mathfrak{P}_0 = O_3(\mathfrak{N}_1)$. Thus, $\mathfrak{P} = \mathfrak{P}_0\mathfrak{W}_1$ and $\mathfrak{W}_1\mathfrak{X}$ is a complement to \mathfrak{P}_0 in \mathfrak{N}_1 . Let I be the involution of \mathfrak{X} , let T be an element of \mathfrak{X} of order 4, let $\mathfrak{R} = J(\mathfrak{P}) \cap \mathfrak{P}_0$, and let $\mathfrak{Z} = \langle \mathfrak{R}, \mathfrak{R}^T \rangle$. Since $T^2 = I$ normalizes \mathfrak{P} , and $J(\mathfrak{P})$ char \mathfrak{P} , it follows that T^2 normalizes \mathfrak{R} . Hence, T normalizes \mathfrak{Z} . Of course, \mathfrak{W}_1 also normalizes \mathfrak{Z} , since $[\mathfrak{W}_1, \mathfrak{Z}] \subseteq \mathfrak{R} \subseteq \mathfrak{Z}$. Since $\mathfrak{R} \triangleleft \mathfrak{P}_0$, it follows that $\mathfrak{Z} \triangleleft \mathfrak{P}_0$, so $\mathfrak{Z} \triangleleft \mathfrak{N}_1$. Since

$$\mathfrak{Z}' = [\mathfrak{R}, \mathfrak{R}^T] \subseteq \mathfrak{R} \cap \mathfrak{R}^T \subseteq \mathfrak{R} \subseteq J(\mathfrak{P}),$$

$J(\mathfrak{P})$ centralizes \mathfrak{Z}' . Since $\mathfrak{W}_1 \subseteq J(\mathfrak{P})$, it follows that \mathfrak{X} centralizes \mathfrak{Z}' . Hence, $\mathfrak{Z}' \subseteq \mathfrak{Z}$, as otherwise I centralizes an element of $\mathcal{Z}(3)$.

Clearly, \mathfrak{Z} is of exponent 3, being of class at most 2 and being generated by its elementary subgroups. The definition of $J(\mathfrak{P})$ forces $\mathfrak{R} = C_{\mathfrak{Z}}(\mathfrak{R})$. Hence, $Z(\mathfrak{Z}) = \mathfrak{Z} = \mathfrak{Z}'$, so that \mathfrak{Z} is extra special, while $\mathfrak{R} \in \mathcal{L}_{ev}(\mathfrak{Z})$. The width of \mathfrak{Z} is at least 2, since otherwise, Hypothesis 9.1 would be satisfied.

Now I centralizes \mathfrak{Z} and normalizes \mathfrak{R} . We argue that I inverts $\mathfrak{R}/\mathfrak{Z}$. Suppose false and $\tilde{\mathfrak{F}}$ is a subgroup of \mathfrak{R} of order 9 which contains \mathfrak{Z} and is centralized by I . Since $A_{\mathfrak{N}_1}(\tilde{\mathcal{E}}) = A(\tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}}: \mathfrak{R} \supset \mathfrak{Z} \supset 1$, it follows from Lemma 5.5 that $\tilde{\mathfrak{F}} \in \mathcal{E}(3)$. Thus, $C(I)$ is nonsolvable by Lemmas 7.4 and 5.38. This contradiction shows that I inverts $\mathfrak{R}/\mathfrak{Z}$. Hence I inverts $\mathfrak{Z}/\mathfrak{Z}$.

We next show that \mathfrak{X} centralizes $\mathfrak{P}_0/\mathfrak{Z}$. This is clear, since $[\mathfrak{P}_0, \mathfrak{W}_1] \subseteq \mathfrak{R} \subseteq \mathfrak{Z}$, so that \mathfrak{W}_1 centralizes $\mathfrak{P}_0/\mathfrak{Z}$.

Since I inverts $\mathfrak{Z}/\mathfrak{Z}$, it follows that if $\tilde{\mathfrak{W}}$ is any subgroup of $J(\mathfrak{P})$ of order 9, and $\tilde{\mathfrak{X}}$ is any element of $\mathfrak{N}(\tilde{\mathfrak{W}}; 3')$ which is minimal subject to $[\tilde{\mathfrak{W}}, \tilde{\mathfrak{X}}] \neq 1$, then $\tilde{\mathfrak{X}}$ is a quaternion group and $\tilde{\mathfrak{W}} \cap C(\tilde{\mathfrak{X}}) \sim \mathfrak{Z}$. In particular, $\mathfrak{W} \in \mathcal{S}$.

Let \mathfrak{M} be the subgroup given by Lemma 7.5 which contains a S_2 -subgroup of $C(I)$ and contains \mathfrak{W} . Then Lemma 9.12 (ii) implies that $O_2(\mathfrak{M})$ is extra special, and that $\langle I \rangle = O_2(\mathfrak{M})'$. Hence, $C_{\mathfrak{N}_1}(I) \subseteq \mathfrak{M}$.

If the width of $O_2(\mathfrak{M})$ is 2, then $\mathfrak{W} = C_{\mathfrak{P}}(I)$ is a S_3 -subgroup of $C_{\mathfrak{N}_1}(I)$ so $\mathfrak{Z} = \mathfrak{P}_0$ is extra special. Since $\mathfrak{P}_0 = O_3(\mathfrak{N}_1)$, it follows that Hypothesis 9.2 is satisfied, an excluded case. Hence, the width of $O_2(\mathfrak{M})$ is 3 or 4. On the other hand, every element of $(C(I) \cap O_3(\mathfrak{N}_1))\mathfrak{W}$ of order 3

centralizes an element of $\mathcal{E}(3)$, so $C_{\mathfrak{M}_1}(I)$ contains no subgroup of exponent 3 and order 27. Hence, $C_{\mathfrak{M}_0}(I)$ is cyclic of order 3 or 9. If $C_{\mathfrak{M}_0}(I) = 3$, Hypothesis 9.2 is satisfied, an excluded case. Hence, $C_{\mathfrak{M}_0}(I)$ is cyclic of order 9. This means that if \mathfrak{W}_1 is any subgroup of \mathfrak{W} of order 3 such that $C(\mathfrak{W}_1) \cap O_2(\mathfrak{M}) \supset \langle I \rangle$, then $\mathfrak{W}_1 = \langle P^3 \rangle$ for some P in \mathfrak{M} . This is absurd, since we get that $\mathfrak{W} \cong \mathcal{O}^1(\tilde{\mathfrak{F}})$ for some S_3 -subgroup $\tilde{\mathfrak{F}}$ of \mathfrak{M} , while $\tilde{\mathfrak{F}}$ is isomorphic to a subgroup of $Z_3 \times (Z_3 \wr Z_3)$. The proof is complete.

LEMMA 9.15. *If $\mathfrak{A} \in \mathcal{E}(3)$ and \mathfrak{B} is a noncyclic abelian subgroup of order 8, then $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is nonsolvable.*

Proof. Suppose false. Let \mathcal{S} be the set of all 2, 3-subgroups \mathfrak{C} of \mathfrak{G} such that

- (i) \mathfrak{C} contains an element of $\mathcal{E}(3)$.
- (ii) $\mathfrak{C}/O_3(\mathfrak{C})$ satisfies the hypothesis of Lemma 5.41.

Thus, $\mathcal{S} \neq \emptyset$.

If \mathfrak{C}_1 and \mathfrak{C}_2 are elements of \mathcal{S} , we say that $\mathfrak{C}_1 \ll \mathfrak{C}_2$ if and only if either $|\mathfrak{C}_1|_3 < |\mathfrak{C}_2|_3$ or $\mathfrak{C}_1 = \mathfrak{C}_2$.

Let \mathfrak{C} be a maximal element of \mathcal{S} under \ll . Let \mathfrak{C}_p be a S_p -subgroup of \mathfrak{C} , $p = 2, 3$. Since \mathfrak{C} contains an element of $\mathcal{E}(3)$, it follows from Lemma 9.11 (ii) that $O_2(\mathfrak{C}) = 1$.

Replacing \mathfrak{C} by a conjugate if necessary, we assume that $\mathfrak{C}_3 \cong \mathfrak{F}$. By Lemma 5.41, \mathfrak{C} has 2-length 1. If $\mathfrak{C}'_2 = 1$, then

$$\mathfrak{C} = C_{\mathfrak{C}}(Z(\mathfrak{C}_3)) \cdot N_{\mathfrak{C}}(J(\mathfrak{C}_3))$$

by Theorem 1 of [43]. Since $Z(\mathfrak{F}) \cong Z(\mathfrak{C}_3)$, S_2 -subgroups of $C_{\mathfrak{C}}(Z(\mathfrak{C}_3))$ are cyclic. Thus, $C_{\mathfrak{C}}(Z(\mathfrak{C}_3)) \cong N_{\mathfrak{C}}(\mathfrak{C}_3) \cong N_{\mathfrak{C}}(J(\mathfrak{C}_3))$, so $J(\mathfrak{C}_3) \triangleleft \mathfrak{C}$. Maximality of \mathfrak{C} forces $\mathfrak{C}_3 = \mathfrak{F}$, against Lemma 9.14. Hence, $\mathfrak{C}'_2 \neq 1$.

Suppose \mathfrak{C}_2 is extra special of width at least 2. By Lemma 5.52, it follows that $\mathfrak{C} = C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))N_{\mathfrak{C}}(J(\mathfrak{C}_3))$. Thus, maximality of \mathfrak{C} together with Lemmas 9.13 and 9.14 imply that neither $C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))$ nor $N_{\mathfrak{C}}(J(\mathfrak{C}_3))$ contains a noncyclic abelian subgroup of order 8. Let $\mathfrak{X}_0 = C_{\mathfrak{C}}(Z(O_3(\mathfrak{C}))) \cap \mathfrak{C}_2$, $\mathfrak{X}_1 = N_{\mathfrak{C}}(J(\mathfrak{C}_3)) \cap \mathfrak{C}_2$.

Since $\mathfrak{C}_2 = \mathfrak{X}_0\mathfrak{X}_1$ and \mathfrak{X}_i has no noncyclic abelian subgroup of order 8, the width of \mathfrak{C}_2 is 2, and $4 \leq |\mathfrak{X}_i| \leq 8$, $i = 0, 1$.

Suppose $\mathfrak{C}_3 \cdot C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))$ is 3-closed. Then \mathfrak{X}_0 normalizes \mathfrak{C}_3 , so normalizes $J(\mathfrak{C}_3)$. This yields $\mathfrak{X}_0 \cong \mathfrak{X}_1$, which is not the case. Thus, $\mathfrak{C}_3 \cdot C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))$ is not 3-closed. Since $\mathfrak{X}_0 \triangleleft \mathfrak{C}_2$, it follows that \mathfrak{X}_0 is a quaternion group. Suppose $N_{\mathfrak{C}}(J(\mathfrak{C}_3))$ is 3-closed. Since $Z(\mathfrak{C}_3) \cong Z(O_3(\mathfrak{C}))$, it follows that $Z(\mathfrak{C}_3) \triangleleft \mathfrak{C}$. Maximality of $|\mathfrak{C}|_3$ forces $\mathfrak{C}_3 = \mathfrak{F}$. This violates Lemma 9.13. Thus, $N_{\mathfrak{C}}(J(\mathfrak{C}_3))$ is not 3-closed. Since $\mathfrak{C}'_2 \cong N_{\mathfrak{C}}(\mathfrak{C}_3) \cong N_{\mathfrak{C}}(J(\mathfrak{C}_3))$, it follows that \mathfrak{X}_1 is also a quaternion group. Since $l_2(\mathfrak{C}) = 1$, \mathfrak{C}_2 is the central product of \mathfrak{X}_0 and \mathfrak{X}_1 .

Let $\mathfrak{P} = \mathfrak{P}_1, \dots, \mathfrak{P}_r$ be all the S_3 -subgroups of \mathfrak{G} which contain \mathfrak{S}_3 , and let $\mathfrak{Z}_i = \Omega_1(\mathbf{Z}(\mathfrak{P}_i))$. Thus, $\mathfrak{Z}_i \subseteq \mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$ for all i , so that \mathfrak{X}_0 centralizes each \mathfrak{Z}_i . Let $\langle T \rangle = \mathfrak{X}_0 \cap \mathfrak{X}_1$ so that T is an involution which centralizes each \mathfrak{Z}_i . Also, $\mathfrak{X}_0 \subseteq \mathbf{C}(\mathfrak{Z}_i)$ for each i , so for each i , $\mathfrak{X}_1 \not\subseteq \mathbf{C}(\mathfrak{Z}_i)$.

Suppose $\mathfrak{S}_3 = \mathfrak{P}$. Since $\mathfrak{S}'_2 = \mathfrak{X}'_0$ centralizes $\mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$, it follows that $\mathfrak{Z} = \Omega_1(\mathbf{Z}(\mathbf{O}_3(\mathfrak{S})))$; otherwise, \mathfrak{S}'_2 centralizes an element of $\mathcal{U}(\mathfrak{S})$. Hence, $\mathfrak{Z} \triangleleft \mathfrak{S}$, against Lemma 9.13. We conclude that $\mathfrak{S}_3 \subset \mathfrak{P}$.

Enlarge $\mathfrak{S}_3\mathfrak{X}_1$ to a $S_{2,3}$ -subgroup of $N(\mathbf{J}(\mathfrak{S}_3))$ and enlarge this subgroup to a maximal 2, 3-subgroup \mathfrak{Q} of \mathfrak{G} . Let $\mathfrak{X} = \mathbf{O}^{3'}(\mathfrak{Q})$. Since $|\mathfrak{X}|_3 > |\mathfrak{S}|_3$, it follows that \mathfrak{X} contains no noncyclic abelian subgroup of order 8. Since $\mathfrak{X} \cong \mathbf{O}^{3'}(\mathfrak{S}_3\mathfrak{X}_1) = \mathfrak{S}_3\mathfrak{X}_1$, it follows that \mathfrak{X}_1 is a S_2 -subgroup of \mathfrak{X} . Let \mathfrak{X}_3 be a S_3 -subgroup of \mathfrak{X} which contains \mathfrak{S}_3 . Thus, $\mathfrak{X}_3 \subseteq \mathfrak{P}_i$ for some i .

Let \mathfrak{W} be the normal closure of \mathfrak{Z}_i in \mathfrak{X} . Thus, $\mathbf{C}_{\mathfrak{Q}}(\mathfrak{W})$ contains T . Since $\mathfrak{X}_0 \subseteq \mathbf{C}(\mathfrak{Z}_i)$, it follows that $\mathbf{C}_{\mathfrak{Q}}(\mathfrak{W}) \cap \mathfrak{X}_1 = \langle T \rangle$, so S_2 -subgroups of $A_{\mathfrak{Q}}(\mathfrak{W})$ are four-groups. It follows that $\mathbf{J}(\mathfrak{X}_3) \triangleleft \mathfrak{X}$. Hence, $\mathfrak{X}_3 = \mathfrak{P}_i$ and so T centralizes an element of $\mathcal{Z}(\mathfrak{P}_i)$. Thus, by Lemmas 7.1 (i) and 7.4, $\mathbf{C}(T)$ is nonsolvable. This contradiction shows that \mathfrak{S}_2 is not extra special of width ≥ 2 .

Suppose \mathfrak{S}_2 is the central product of a quaternion group and a cyclic group of order 4. If $\mathbf{J}(\mathfrak{S}_3) \subseteq \mathbf{O}_3(\mathfrak{S})$, then again $\mathfrak{S}_3 = \mathfrak{P}$ and Lemma 9.14 is violated. Hence, $\mathbf{J}(\mathfrak{S}_3) \not\subseteq \mathbf{O}_3(\mathfrak{S})$. If \mathfrak{S}'_2 centralizes $\mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$, then we get $\mathfrak{S} = \mathbf{C}_{\mathfrak{S}}(\mathbf{Z}(\mathfrak{S}_3))N_{\mathfrak{S}}(\mathbf{J}(\mathfrak{S}_3))$, so that either $\mathbf{Z}(\mathfrak{S}_3)$ or $\mathbf{J}(\mathfrak{S}_3)$ is normal in \mathfrak{S} . Both these possibilities are excluded by Lemmas 9.13 and 9.14, so we may assume that $[\mathfrak{S}'_2, \mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))] = \mathfrak{W} \neq 1$. Let \mathfrak{X} be a minimal normal subgroup of \mathfrak{S} with $\mathfrak{X} \subseteq \mathfrak{W}$. Thus, \mathfrak{S}_2 is faithfully represented on \mathfrak{X} . Since $|\mathfrak{S}_3: \mathbf{O}_3(\mathfrak{S})| = 3$ and $\mathbf{J}(\mathfrak{S}_3) \not\subseteq \mathbf{O}_3(\mathfrak{S})$, it follows that elements of $\mathfrak{S}_3 - \mathbf{O}_3(\mathfrak{S})$ centralize a hyperplane of \mathfrak{X} . This is not the case, since $|\mathfrak{X}| = 3^4$. Thus, \mathfrak{S}_2 is not the central product of a quaternion group and a cyclic group of order 4.

By Lemma 5.41 and maximality of \mathfrak{S} under \ll , it follows that \mathfrak{S}_2 is either the direct product of a quaternion group and a group of order 2 or \mathfrak{S}_2 is special with $|\mathfrak{S}'_2| = 4$. Let $\mathfrak{W} = \mathbf{Z}(\mathfrak{S}_2)$, so that in both cases, \mathfrak{W} is a four-group. We will exploit \mathfrak{W} by showing that $\mathfrak{S}_3 = \mathfrak{P}$, that is, by showing that \mathfrak{S}_3 is a S_3 -subgroup of \mathfrak{G} . Suppose by way of contradiction that $\mathfrak{S}_3 \subset \mathfrak{P}$.

We argue that \mathfrak{W} normalizes $\mathbf{J}(\mathfrak{S}_3)$. For if this is not the case, then \mathfrak{W} centralizes $\mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$, against Lemma 9.12 (ii).

Since \mathfrak{W} normalizes $\mathbf{J}(\mathfrak{S}_3)$, we may enlarge $\mathfrak{S}_3\mathfrak{W}$ to a $S_{2,3}$ -subgroup \mathfrak{Q} of $N(\mathbf{J}(\mathfrak{S}_3))$. Since \mathfrak{S}_3 is not a S_3 -subgroup of \mathfrak{Q} , \mathfrak{Q} does not contain a noncyclic abelian subgroup of order 8.

Let \mathfrak{X}_p be a S_p -subgroup of \mathfrak{Q} , $p = 2, 3$, with $\mathfrak{W} \subseteq \mathfrak{X}_3$, $\mathfrak{S}_3 \subset \mathfrak{X}_3$.

Case 1. $O_3(\mathfrak{C})\mathfrak{B}/O_3(\mathfrak{C}) \cong Z(\mathfrak{C}/O_3(\mathfrak{C}))$.

Let $\tilde{\mathfrak{C}}_3$ be a maximal element of $\mathcal{N}_Q(\mathfrak{B}; 3)$ with $\mathfrak{C}_3 \subseteq \tilde{\mathfrak{C}}_3$. Suppose $\mathfrak{C}_3 \subset \tilde{\mathfrak{C}}_3$. Choose \mathfrak{C}_3^* in $\mathcal{N}_Q(\mathfrak{B}; 3)$ so that $|\mathfrak{C}_3^* : \mathfrak{C}_3| = 3$, and let $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{C}_3^*/\mathfrak{C}_3)$. Hence, $[\mathfrak{C}_3^*, \mathfrak{B}_0] = [\mathfrak{C}_3, \mathfrak{B}_0]$ is normal in \mathfrak{C} and in \mathfrak{C}_3^* . Maximality of \mathfrak{C} in \mathcal{S} forces $[\mathfrak{C}_3, \mathfrak{B}_0] = 1$, against $O_2(\mathfrak{C}) = 1$. Thus, \mathfrak{C}_3 is a maximal element of $\mathcal{N}_Q(\mathfrak{B}; 3)$. In particular, \mathfrak{B} is not 3-closed. Hence, $O_3(\mathfrak{B})$ is of index 3 in \mathfrak{B} and $O_3(\mathfrak{B}) \subseteq \mathfrak{C}_3$. Hence, $O_3(\mathfrak{B}) = \mathfrak{C}_3$. If $\mathfrak{B} = \mathfrak{L}_2$, then $[\mathfrak{B}, \mathfrak{C}_3] \triangleleft \mathfrak{B}$ so that $\langle \mathfrak{C}, \mathfrak{L} \rangle \cong N([\mathfrak{B}, \mathfrak{C}_3])$, since $[\mathfrak{B}, \mathfrak{C}_3] = [\mathfrak{B}, O_3(\mathfrak{C})] \neq 1$. This is impossible, so $\mathfrak{B} \subset \mathfrak{L}_2$.

Case 1a. \mathfrak{C}_2 is special.

Since $\mathfrak{B} \cong Z(\mathfrak{C}_2)$, it follows that $\mathfrak{C}_3\mathfrak{B}$ is a maximal subgroup of \mathfrak{C} . Thus, $O_3(\mathfrak{C})\mathfrak{C}_2/O_3(\mathfrak{C})\mathfrak{B}$ is a chief factor of \mathfrak{C} . Let \mathfrak{B}_0 be a subgroup of \mathfrak{B} of order 2 and let $\mathfrak{C}_2^0 = Z(\mathfrak{C}_2 \text{ mod } \mathfrak{B}_0)$. Since $O_3(S)\mathfrak{B}_0 \triangleleft \mathfrak{C}$, $\mathfrak{C}_3\mathfrak{C}_2^0$ is a group. Hence, $\mathfrak{C}_2^0 = \mathfrak{B}$ or $\mathfrak{C}_2^0 = \mathfrak{C}_2$. If $\mathfrak{C}_2^0 = \mathfrak{C}_2$, then $\mathfrak{C}_2^0 \subseteq \mathfrak{B}_0$, so that $\mathfrak{C}_2/\mathfrak{B}_0$ is abelian. This is not the case, since $\mathfrak{C}_2^0 = \mathfrak{B}$. Hence, $\mathfrak{C}_2^0 = \mathfrak{B}$, so that $\mathfrak{C}_2/\mathfrak{B}_0$ is extra special of width ≥ 2 . It follows from the proof of Lemma 5.52 that $J(\mathfrak{C}_3) \triangleleft \mathfrak{C}$. Maximality of \mathfrak{C} in \mathcal{S} guarantees that \mathfrak{C}_3 is a S_3 -subgroup of \mathfrak{G} , against our assumption that $\mathfrak{C}_3 \subset \mathfrak{B}$.

Case 1b. \mathfrak{C}_2 is the direct product of a quaternion group and a group of order 2.

Since \mathfrak{L}_2 has no noncyclic abelian subgroup of order 8, it follows that \mathfrak{B} is a self centralizing subgroup of \mathfrak{L}_2 . Hence, \mathfrak{L}_2 is of maximal class. Let $\tilde{\mathfrak{L}}_2 = \mathfrak{L}_2 \cap O_{3,2}(\mathfrak{L})$. Thus, $\tilde{\mathfrak{L}}_2$ has an automorphism of order 3. Being a subgroup of a group of maximal class, $\tilde{\mathfrak{L}}_2$ is either a quaternion group or a four-group.

Suppose $\tilde{\mathfrak{L}}_2$ is a quaternion group. In this case, $\mathfrak{B}/O_3(\mathfrak{B}) \cong GL(2, 3)$ and \mathfrak{B} normalizes a S_3 -subgroup of \mathfrak{L} , against our previous argument. we conclude that $\tilde{\mathfrak{L}}_2$ is a four-group.

If $\tilde{\mathfrak{L}}_2 = \mathfrak{B}$, then $[O_3(\mathfrak{L}), \mathfrak{B}] \triangleleft \mathfrak{L}$. But $O_3(\mathfrak{L}) = \mathfrak{C}_3$ and $[\mathfrak{C}_3, \mathfrak{B}] = [O_3(\mathfrak{C}), \mathfrak{B}] \triangleleft \mathfrak{C}$. Hence, $[O_3(\mathfrak{L}), \mathfrak{B}] \triangleleft \langle \mathfrak{L}, \mathfrak{C} \rangle$ against the maximality of \mathfrak{C} in \mathcal{S} . We conclude that $\tilde{\mathfrak{L}}_2 \neq \mathfrak{B}$, so that \mathfrak{L}_2 is a dihedral group of order 8 whose two four-subgroups are \mathfrak{B} and $\tilde{\mathfrak{L}}_2$.

Since $\tilde{\mathfrak{L}}_2$ does not centralize $Z(O_3(\mathfrak{L}))$, it follows that $J(\mathfrak{L}_3) \triangleleft \mathfrak{L}_0$. Hence, $J(\mathfrak{L}_3) = J(\mathfrak{C}_3)$, so by construction of \mathfrak{L} , we conclude that \mathfrak{L}_3 is a S_3 -subgroup of \mathfrak{G} . We may therefore assume without loss of generality that $\mathfrak{B} = \mathfrak{L}_3$. Hence, $|\mathfrak{B} : \mathfrak{C}_3| = 3$.

Let $\mathfrak{X} = \Omega_1(Z(\mathfrak{C}_3))$. Recalling that $\mathfrak{C}_3 = O_3(\mathfrak{L})$, we get $\mathfrak{X} \triangleleft \mathfrak{L}$. Since $\tilde{\mathfrak{L}}_2$ does not centralize \mathfrak{B} , we get $\mathfrak{B} \subset \mathfrak{X}$. Since $Z(\mathfrak{B})$ is cyclic and $|\mathfrak{B} : \mathfrak{C}_3| = 3$, we get $|\mathfrak{X}| \leq 27$. Hence, $|\mathfrak{X}| = 27$ and \mathfrak{X} is the only minimal normal subgroup of \mathfrak{L} .

Since $J(\mathfrak{B}) \triangleleft \mathfrak{L}$, we get $\mathfrak{X}_0 \triangleleft \mathfrak{L}$, where $\mathfrak{X}_0 = \Omega_1(Z(J(\mathfrak{B})) \cap D(J(\mathfrak{B})))$, if

$D(J(\mathfrak{P})) \neq 1$, and $\mathfrak{x}_0 = J(\mathfrak{P})$ if $D(J(\mathfrak{P})) = 1$. If $|\mathfrak{x}_0| > 27$, then some element of $\mathfrak{P}^\#$ centralizes a noncyclic subgroup of \mathfrak{x}_0 . This was shown to be impossible in the proof of Lemma 9.14. Hence, $|\mathfrak{x}_0| \leq 27$. This implies that $\mathfrak{x}_0 = \mathfrak{x}$.

Suppose $A, B \in \mathfrak{x}$, and $A = B^B$ for some G in \mathfrak{G} . Thus, $\langle J(\mathfrak{P}), J(\mathfrak{P}^{G^{-1}}) \rangle \subseteq C(B)$, and we can choose C in $C(B)$ such that $J(\mathfrak{P})^C = J(\mathfrak{P}^{G^{-1}})$. Hence, $CG = N \in N(J(\mathfrak{P}))$ and $A = B^G = B^{G^{-1}N} = B^N$. Thus, elements of \mathfrak{x} are \mathfrak{G} -conjugate only if they are $N(\mathfrak{x})$ -conjugate.

Let $\mathfrak{x} = \mathfrak{x}_1 \times \mathfrak{x}_2 \times \mathfrak{x}_3$, where $|\mathfrak{x}_i| = 3$ and where \mathfrak{x}_i admits \mathfrak{B} , $i = 1, 2, 3$. Let $\mathfrak{B}_i = C(\mathfrak{x}_i) \cap \mathfrak{B} = \langle V_i \rangle$. Thus, $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3$ are the only subgroups of \mathfrak{x} of order 3 which admit \mathfrak{B} . Let Z be a generator for \mathfrak{B} . Then $Z = X_1 X_2 X_3$ with $X_i \in \mathfrak{x}_i$.

We argue that $\mathfrak{B} \not\sim \mathfrak{x}_i$ for $i = 1, 2, 3$. Namely, if $\mathfrak{B} \sim \mathfrak{x}_i$, there is $N \in N(\mathfrak{x})$ such that $\mathfrak{x}_i = \mathfrak{B}^N$. Let $\mathfrak{A} = A_{\mathfrak{G}}(\mathfrak{x})$. Thus, $|\mathfrak{A}|_3 = 3$, \mathfrak{A} is solvable, and $\mathfrak{A} \supseteq \mathfrak{B} = A_{\mathfrak{Q}}(\mathfrak{x}) \cong \Sigma_4$. So $\mathfrak{A} = \mathfrak{B}$ or $\mathfrak{A} = \mathfrak{B} \times \mathfrak{A}_0$, where $\mathfrak{A}_0 = \langle A \rangle$ and A inverts \mathfrak{x} . In neither case are \mathfrak{B} and \mathfrak{x}_i in the same orbit under \mathfrak{A} .

We now return to our study of \mathfrak{S} . Let $\mathfrak{S}_2 = \mathfrak{Q} \times \langle V \rangle$, where \mathfrak{Q} is a quaternion group and $V \in \mathfrak{B}$. Let $\langle V_0 \rangle = \mathfrak{Q}'$. Since V_0 does not centralize $Z(O_3(\mathfrak{S}))$, there is a minimal normal subgroup \mathfrak{Y} of \mathfrak{S} such that \mathfrak{Q} is represented faithfully on \mathfrak{Y} . Let $\mathfrak{B}^* = C_{\mathfrak{B}}(\mathfrak{Y})$ so that $|\mathfrak{B}^*| = 2$, $\mathfrak{S}_2 = \mathfrak{Q} \times \mathfrak{B}^*$. We see that $|\mathfrak{Y}| = 9$ and that $\mathfrak{Y}_0 = \mathfrak{Y} \cap Z(\mathfrak{S}_2)$ is of order 3 and admits \mathfrak{B} . Thus, $\mathfrak{Y}_0 \subset \mathfrak{x}$, so $\mathfrak{Y}_0 = \mathfrak{x}_i$ for some $i = 1, 2, 3$. Since $\mathfrak{x}_i \not\sim \mathfrak{B}$, it follows that \mathfrak{S}_3 is a S_3 -subgroup of $C(\mathfrak{Y}_0)$. Since $[\mathfrak{Y}, \mathfrak{S}_3] \subseteq \mathfrak{Y}_0$, it follows that $\mathfrak{Y} \subseteq O_3(C(\mathfrak{Y}_0))$. Since \mathfrak{Q} permutes transitively the subgroups of \mathfrak{Y} of order 3, it follows that $\mathfrak{Y} \subseteq O_3(C(\mathfrak{Y}^*))$ for every subgroup \mathfrak{Y}^* of \mathfrak{Y} of order 3. This implies that $\mathfrak{Y} \in \mathcal{E}(3)$. Now $C(\mathfrak{B}^*)$ contains \mathfrak{Y} and also contains an element of $\mathcal{Z}(2)$, so $C(\mathfrak{B}^*)$ is nonsolvable. This contradiction shows that this case does not arise.

Case 2. $O_3(\mathfrak{S})\mathfrak{B}/O_3(\mathfrak{S}) \not\subseteq Z(\mathfrak{S}/O_3(\mathfrak{S}))$.

We conclude that \mathfrak{S}_2 is special and that $\mathfrak{B} = \mathfrak{S}_2$. Since $\mathfrak{B} = Z(\mathfrak{S}_2)$, we get that $\mathfrak{S}_3\mathfrak{B}$ is a maximal subgroup of \mathfrak{S} . That is, $O_3(\mathfrak{S})\mathfrak{S}_2/O_3(\mathfrak{S})\mathfrak{B}$ is a chief factor of \mathfrak{S} .

Let \mathfrak{P}_0 be a maximal element of $\mathfrak{N}_{\mathfrak{S}}(\mathfrak{B}; 3)$ with $\mathfrak{P}_0 \subset \mathfrak{S}_3$. Hence, \mathfrak{P}_0 is of index 3 in \mathfrak{S}_3 , and all involutions of \mathfrak{B} are fused in $\mathfrak{S}_3\mathfrak{B}$. Also, $[\mathfrak{B}_0, \mathfrak{P}_0] = [O_3(\mathfrak{S}), \mathfrak{B}_0]$ for every subgroup \mathfrak{B}_0 of \mathfrak{B} .

Suppose \mathfrak{P}_0 is not a maximal element of $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{B}; 3)$. Choose \mathfrak{P}_1 in $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{B}; 3)$ so that $|\mathfrak{P}_1 : \mathfrak{P}_0| = 3$, and let $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{P}_1/\mathfrak{P}_0)$. Then \mathfrak{P}_1 and \mathfrak{S}_2 both normalize $[\mathfrak{B}_0, \mathfrak{P}_1]$. Let \mathfrak{R}^* be a $S_{2,3}$ -subgroup of $N([\mathfrak{B}_0, \mathfrak{P}_1])$ which contains $\mathfrak{B}\mathfrak{P}_1$, and let \mathfrak{R}_p^* be a S_p -subgroup of \mathfrak{R}^* with $\mathfrak{P}_1 \subseteq \mathfrak{R}_p^*$, $\mathfrak{B} \subseteq \mathfrak{R}_p^*$. Note that \mathfrak{R}^* contains a conjugate of \mathfrak{S}_2 .

By maximality, $\mathfrak{S}_3 = N_{\mathfrak{P}}(\Omega_1(Z(O_3(\mathfrak{S}))))$. Hence, $\mathfrak{B} \subset \mathfrak{S}_3$, and so $\mathfrak{B} \subseteq \Omega_1(Z(O_3(\mathfrak{S})))$, since $O_2(\mathfrak{S}) = 1$. If $u \in \mathcal{Z}(\mathfrak{B})$, then $[\mathfrak{B}, u] \subseteq \mathfrak{B}$, and so

$\mathbb{U} \subseteq \mathfrak{C}_3$. Since \mathfrak{B} is a 4-group and does not centralize $\Omega_1(\mathbf{Z}(\mathfrak{O}_3(\mathfrak{C})))$, we conclude from $[\mathfrak{B}, \mathbb{U}, \mathbb{U}] = 1$ that \mathbb{U} centralizes $\mathfrak{B}\mathfrak{O}_3(\mathfrak{C})/\mathfrak{O}_3(\mathfrak{C})$. That is, $\mathbb{U} \subseteq \mathfrak{F}_0 \subseteq \mathfrak{X}^*$. Hence, \mathfrak{X}^* contains an element of \mathcal{S} so maximality of \mathfrak{C} guarantees that $\mathfrak{F}_1 = \mathfrak{X}_3^*$, since \mathfrak{F}_1 and \mathfrak{C}_3 are of the same order.

Let $\mathfrak{X}_2^{**} = \mathfrak{X}_2^* \cap \mathfrak{O}_{3,2}(\mathfrak{X}^*)$. Thus, \mathfrak{X}_2^{**} contains a noncyclic abelian subgroup of order 8, since \mathfrak{X}^* contains a conjugate of \mathfrak{C}_2 .

Suppose every subgroup of \mathfrak{X}_2^{**} which is characteristic and abelian is also cyclic. Let \mathfrak{X}_2^{***} be a subgroup of \mathfrak{X}_2^{**} which is minimal subject to (a) containing a noncyclic abelian subgroup of order 8 and (b) being permutable with \mathfrak{X}_3^* . Then since $\mathbf{D}(\mathfrak{X}_2^{***}) \subseteq \mathbf{D}(\mathfrak{X}_2^{**})$, it follows that \mathfrak{X}_2^{***} is not a special group with center of order 4. Since $\mathfrak{X}_3^*\mathfrak{X}_2^{***} \in \mathcal{S}$, our previous reduction excludes this possibility.

Let $\tilde{\mathfrak{X}}_2$ be a noncyclic characteristic abelian subgroup of \mathfrak{X}_2^{**} . If $|\tilde{\mathfrak{X}}_2| > 4$, then $\mathfrak{F}_1\tilde{\mathfrak{X}}_2$ contains an element of \mathcal{S} , against our previous reduction. We may assume that $\tilde{\mathfrak{X}}_2$ is a four-group. If $\tilde{\mathfrak{X}}_2 \cap \mathfrak{B} \neq 1$, then $\mathfrak{O}_3(\mathfrak{X}^*)\tilde{\mathfrak{X}}_2/\mathfrak{O}_3(\mathfrak{X}^*)$ is centralized by \mathfrak{F}_1 , since \mathfrak{B} normalizes \mathfrak{F}_1 . But in this case, there are maximal elements of \mathcal{S} which do not satisfy our previous reduction. If $\tilde{\mathfrak{X}}_2 \cap \mathfrak{B} = 1$, then $\mathfrak{F}_1\mathfrak{B}\tilde{\mathfrak{X}}_2$ contains an element of \mathfrak{C} which also violates our previous reduction. Hence, \mathfrak{F}_0 is a maximal element of $\mathbf{N}_2(\mathfrak{B}; 3)$.

Since $\mathfrak{O}_3(\mathfrak{B}) \in \mathbf{N}_2(\mathfrak{B}; 3)$, we have $\mathfrak{O}_3(\mathfrak{B}) \subseteq \mathfrak{F}_0$. Maximality of \mathfrak{C} in \mathcal{S} implies that \mathfrak{B} contains no noncyclic abelian subgroup of order 8. Since the involutions of \mathfrak{B} are fused in $\mathfrak{C}_3\mathfrak{B}$, it follows that $\mathfrak{O}_{3,2}(\mathfrak{B}) = \mathfrak{O}_3(\mathfrak{B})\mathfrak{B}$. Hence, $\mathfrak{F}_0 = \mathfrak{O}_3(\mathfrak{B})$ is of index 3 in \mathfrak{X}_3 . This violates $|\mathfrak{X}_3| > |\mathfrak{C}_3| = 3|\mathfrak{F}_0|$.

Thus, in all cases, we have shown that $\mathfrak{C}_3 = \mathfrak{B}$.

Suppose that \mathfrak{B} normalizes \mathfrak{C}_3 . Let \mathfrak{W} be a minimal normal subgroup of \mathfrak{C} . Clearly, $\mathfrak{W} \supset \mathfrak{B}$. Since $\mathfrak{B}\mathfrak{O}_3(\mathfrak{C})/\mathfrak{O}_3(\mathfrak{C})$ is a central factor of \mathfrak{C} , some involution of \mathfrak{B} centralizes \mathfrak{W} . But \mathfrak{W} contains an element of $\mathcal{Z}(\mathfrak{B})$, so Lemmas 7.4 and 5.38 imply that $\mathbf{C}(V)$ is nonsolvable for some involution V of \mathfrak{B} . Thus, \mathfrak{B} does not normalize \mathfrak{B} . In particular, \mathfrak{C}_2 is special.

Let \mathfrak{F}_0 be the largest subgroup of \mathfrak{B} normalized by \mathfrak{B} . Thus, $|\mathfrak{B}:\mathfrak{F}_0| = 3$ and $N_{\mathfrak{B}}(\mathfrak{C}_2)$ permutes transitively the involutions of \mathfrak{B} .

Let \mathfrak{W} be a minimal normal subgroup of \mathfrak{C} . Clearly, $\mathfrak{W} \supset \mathfrak{B}$, so \mathfrak{B} is faithfully represented on \mathfrak{W} . Hence, \mathfrak{C}_2 is faithfully represented on \mathfrak{W} , so $\mathbf{C}_{\mathfrak{C}}(\mathfrak{W}) = \mathfrak{O}_3(\mathfrak{W})$. Let $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2 \times \mathfrak{W}_3$, where $\mathfrak{W}_i = \mathbf{C}_{\mathfrak{W}}(V_i)$ and V_1, V_2, V_3 are the involutions of \mathfrak{B} . Thus, $\mathfrak{C}_2\mathfrak{F}_0$ normalizes each \mathfrak{W}_i , and \mathfrak{C} permutes $\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3$ transitively. Obviously each \mathfrak{W}_i is an irreducible $\mathfrak{F}_0\mathfrak{C}_2$ -module.

Let $\mathfrak{K}_i = \mathbf{C}_{\mathfrak{F}_0\mathfrak{C}_2}(\mathfrak{W}_i)$, and $\mathfrak{C}_{2i} = \{S \in \mathfrak{C}_2 \mid [S, \mathfrak{C}_2] \subseteq \langle V_i \rangle\}$, for $i = 1, 2, 3$. Then $N_{\mathfrak{C}}(\mathfrak{C}_2)$ permutes $\{\mathfrak{C}_{21}, \mathfrak{C}_{22}, \mathfrak{C}_{23}\}$ transitively, and so one of the following holds:

- (a) $\mathfrak{C}_{2i} = \mathfrak{B}$, $i = 1, 2, 3$,
- (b) $\mathfrak{C}_{2i} \supset \mathfrak{B}$, $i = 1, 2, 3$.

If (a) holds, then $\mathfrak{S}_2/\langle V_i \rangle$ is extra special, and since V_j inverts \mathfrak{B}_i for $j \neq i$, it follows that $\mathfrak{R}_i \cap \mathfrak{S}_2 = \langle V_i \rangle$, whence $\mathfrak{R}_i = O_3(\mathfrak{S})\langle V_i \rangle$. The proof of Lemma 5.52 now shows that $O_3(\mathfrak{S}) \cong J(\mathfrak{S}_3)$, the desired contradiction.

Suppose (b) holds. The group $O_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}\mathfrak{S}_{23}$ is clearly \mathfrak{S}_3 -invariant, and since $O_3(\mathfrak{S})\mathfrak{S}_2/O_3(\mathfrak{S})\mathfrak{B}$ is a chief factor of \mathfrak{S} , we have $\mathfrak{S}_2 = \mathfrak{S}_{21}\mathfrak{S}_{22}\mathfrak{S}_{23}$. Obviously, $[\mathfrak{S}_{2i}, \mathfrak{S}_{2j}] \cong \langle V_i \rangle \cap \langle V_j \rangle = 1$ for $i \neq j$. Because \mathfrak{S}_2 is special, we conclude that $\mathfrak{S}_{2i'} = \langle V_i \rangle$ and hence that $\mathfrak{S}_{2j}/\langle V_i \rangle$ is extra special for $i \neq j$. If the width of $\mathfrak{S}_{2j}/\langle V_i \rangle$ is greater than 1, then since $J(\mathfrak{S}_3) \not\subseteq O_3(\mathfrak{S})$, and since $\mathfrak{S}_{2j}/\langle V_i \rangle$ acts faithfully on \mathfrak{B}_i , it follows that $J(\mathfrak{S}_3)$ centralizes $O_3(\mathfrak{S})\mathfrak{S}_{2j}/O_3(\mathfrak{S})\langle V_i \rangle$. But $J(\mathfrak{S}_3) \triangleleft \mathfrak{S}_3$, and so $J(\mathfrak{S}_3)$ centralizes $O_3(\mathfrak{S})\mathfrak{S}_2/O_3(\mathfrak{S})\mathfrak{B}$, that is, $J(\mathfrak{S}_3) \subseteq O_3(\mathfrak{S})$. We may assume that if $i \neq j$, then $\mathfrak{S}_{2i}/\langle V_i \rangle$ is of width 1. But then $\mathfrak{S}_{21}\mathfrak{S}_{22}/\langle V_3 \rangle$ is the central product of $\mathfrak{S}_{21}/\langle V_3 \rangle$ and $\mathfrak{S}_{22}/\langle V_3 \rangle$, so is extra special of width 2, acts faithfully on \mathfrak{B}_3 , and $O_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}$ admits $J(\mathfrak{S}_3)$. By Lemma 5.52, $J(\mathfrak{S}_3)$ centralizes $O_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}/O_3(\mathfrak{S})$, so again we get the contradiction $J(\mathfrak{S}_3) \subseteq O_3(\mathfrak{S})$. The proof is complete.

LEMMA 9.16. *If \mathfrak{A} is a subgroup of \mathfrak{G} of type (3, 3) and each element of \mathfrak{A} centralizes an element of $\mathcal{Z}(3)$, then*

- (i) $\mathfrak{A} \in \mathcal{D}$.
- (ii) $4 \nmid |C(\mathfrak{A})|$.

Proof. (i) Suppose false, and \mathfrak{X} is a four-group normalized but not centralized by \mathfrak{A} . Let $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{X})$, so that $|\mathfrak{A}_0| = 3$. Let \mathfrak{Q}_0 be a maximal 2, 3-subgroup of \mathfrak{G} which contains a $S_{2,3}$ -subgroup of $C(\mathfrak{A}_0)$ containing $\mathfrak{A}\mathfrak{X}$. Let $\mathfrak{Q} = O^{\mathfrak{X}}(\mathfrak{Q}_0)$. Since \mathfrak{Q} contains an element of $\mathcal{Z}(3)$, Lemma 9.15 implies that \mathfrak{Q} contains no noncyclic abelian subgroup of order 8. Hence, \mathfrak{X} is a S_2 -subgroup of \mathfrak{Q} and $\mathfrak{Q}/O_3(\mathfrak{Q}) \cong A_4$. Let \mathfrak{Q}_3 be a S_3 -subgroup of \mathfrak{Q} . Since \mathfrak{X} does not centralize $Z(O_3(\mathfrak{Q}))$, it follows that $J(\mathfrak{Q}_3) \triangleleft \mathfrak{Q}$. Hence, \mathfrak{Q}_3 is a S_3 -subgroup of \mathfrak{G} , and we may assume that $\mathfrak{Q}_3 = \mathfrak{B}$.

Let \mathfrak{X} be a minimal normal subgroup of \mathfrak{Q} with $\mathfrak{X} \subseteq Z(J(\mathfrak{B}))$. Thus, \mathfrak{X} is elementary of order 27 and $C_{\mathfrak{X}}(\mathfrak{X}) = 1$. Choose T in $\mathfrak{X}^{\#}$ so that $\mathfrak{X}_1 = C_{\mathfrak{X}}(T)$ is of order 3 and is inverted by the generator of $\mathfrak{X}/\langle T \rangle$. Hence, $\langle \mathfrak{A}_0, \mathfrak{X}_1 \rangle = \mathfrak{A}^*$ is elementary of order 9 and is normalized by \mathfrak{X} , and every element of \mathfrak{A}^* centralizes an element of $\mathcal{Z}(\mathfrak{B})$.

Let \mathfrak{C} be a $S_{2,3}$ -subgroup of $C(T)$ which contains $\mathfrak{A}^*\mathfrak{X}$. By Lemma 5.38, \mathfrak{C} contains an element of $\mathcal{Z}(2)$, so $|O_3(\mathfrak{C})| \leq 3$, and $O_3(\mathfrak{C}) \cap \mathfrak{A}^* = 1$. Hence, \mathfrak{A}^* is faithfully represented on $O_2(\mathfrak{C})$. Let \mathfrak{C}_0 be a characteristic abelian subgroup of $O_2(\mathfrak{C})$. Suppose \mathfrak{A}^* does not centralize \mathfrak{C}_0 . Hence, there is an element A in $\mathfrak{A}^{*\#}$ such that $C(A)$ contains an elementary subgroup of order 8. This is not the case, so \mathfrak{A}^* centralizes \mathfrak{C}_0 . If $|\mathfrak{C}_0| > 2$, then some element A of $\mathfrak{A}^{*\#}$ centralizes a noncyclic

abelian subgroup of $O_2(\mathbb{C})$ of order 8. This is not the case, by Lemma 9.15. Hence, $O_2(\mathbb{C})$ is extra special of width at least 2 and $\langle T \rangle = O_2(\mathbb{C})'$. Hence, \mathbb{C} contains a S_2 -subgroup of \mathbb{G} .

By Lemma 9.15, no element of \mathfrak{A}^{**} centralizes any noncyclic abelian subgroup of order 8. Hence, $\langle T \rangle = O_2(\mathbb{C}) \cap C(\mathfrak{A}^*)$. For each A in \mathfrak{A}^{**} , $O_2(\mathbb{C}) \cap C(A)$ is either $\langle T \rangle$ or is extra special. Thus, $O_2(\mathbb{C}) \cap C(A)$ is either $\langle T \rangle$ or is a quaternion group, so no element of \mathfrak{A}^{**} centralizes any four-subgroup of $O_2(\mathbb{C})$. Thus, the width of $O_2(\mathbb{C})$ is at most 4. Since $O_2(\mathbb{C}) \cap C(\mathfrak{A}_0)$ is centralized by \mathfrak{A}^* , it follows that $O_2(\mathbb{C}) \cap C(\mathfrak{A}_0) = \langle T \rangle$, and so the width of $O_2(\mathbb{C})$ is at most 3.

Consider $C^*(\mathfrak{x}_1) = \{G \in \mathbb{G}, G \text{ either centralizes or inverts } \mathfrak{x}_1\}$. By construction, $|C_3(\mathfrak{x}_1)|_3 = |\mathfrak{P}|/3$. Also, $\mathfrak{X} \subseteq C^*(\mathfrak{x}_1)$, and $C^*(\mathfrak{x}_1)$ contains no noncyclic abelian subgroup of order 8. Suppose $\mathfrak{A}_0 \not\subseteq O_3(\tilde{\mathfrak{X}})$, where $\tilde{\mathfrak{X}}$ is a $S_{2,3}$ -subgroup of $C^*(\mathfrak{x}_1)$ which contains $C_3^*(\mathfrak{x}_1)$. Then $\tilde{\mathfrak{X}}/O_3(\tilde{\mathfrak{X}})$ contains a subgroup isomorphic to $\mathfrak{A}_0 \times \mathfrak{X}$. This is obviously impossible, since S_2 -subgroups of $\tilde{\mathfrak{X}}/O_3(\tilde{\mathfrak{X}})$ are of maximal class. Hence, $\mathfrak{A}_0 \subseteq O_3(\tilde{\mathfrak{X}})$. This implies that \mathfrak{A}_0 centralizes $O_2(\mathbb{C}) \cap C(\mathfrak{x}_1)$, so the width of $O_2(\mathbb{C})$ is 2. Hence, $\mathfrak{x}_1 \times \mathfrak{A}_0$ is a S_3 -subgroup of $C(T)$, so $\mathfrak{x}_1 \times \mathfrak{A}_0$ is a S_3 -subgroup of $C_3(T)$. By a formula of Wielandt [40],

$$|O_3(\mathfrak{L})| = |O_3(\mathfrak{L}) \cap C(T)|^3 / |O_3(\mathfrak{L}) \cap C(\mathfrak{X})|^2.$$

Hence, $|O_3(\mathfrak{L})| = 3^6/3^2 = 3^4$, so that $O_3(L) = \mathfrak{A}_0 \times \mathfrak{x}$. This implies that $Z(\mathfrak{P})$ is noncyclic, since $|\mathfrak{P}:O_3(\mathfrak{L})| = 3$. The proof of (i) is complete.

As for (ii), suppose \mathfrak{X} is a subgroup of $C(\mathfrak{A})$ of order 4. Then $C(T)$ contains an element of $\mathcal{U}(2)$, T being an involution of \mathfrak{X} . Thus, by Lemma 7.5, there is a subgroup \mathfrak{M} in $\mathcal{MS}(\mathbb{G})$ which contains $\mathfrak{A}\mathfrak{X}$ and satisfies $O_2(\mathfrak{M}) = 1$, while $O_2(\mathfrak{M})$ is of symplectic type. Since \mathfrak{A} acts faithfully on $O_2(\mathfrak{M}) \cap C(\mathfrak{X})$, we can therefore choose A in $\mathfrak{A}^\#$ such that \mathfrak{A} does not centralize $O_2(\mathbb{C}) \cap C(\mathfrak{X}) \cap C(\mathfrak{A})$. Thus, $C(A)$ contains a noncyclic abelian subgroup of order 8, against Lemma 9.15. The proof of (ii) is complete.

LEMMA 9.17. *Suppose*

- (a) \mathfrak{R} is a maximal 2, 3-subgroup of \mathbb{G} .
- (b) \mathfrak{R} contains an element of \mathcal{D} .
- (c) \mathfrak{R} contains a noncyclic abelian subgroup of order 8.

Then $O_2(\mathfrak{R}) \neq 1$.

Proof. Let \mathfrak{R}_p be a S_p -subgroup of \mathfrak{R} , $p = 2, 3$. We assume without loss of generality that $\mathfrak{R}_3 \subseteq \mathfrak{P}$. Suppose by way of contradiction that $O_2(\mathfrak{R}) = 1$. Then $O_3(\mathfrak{R}) \neq 1$, so by maximality of \mathfrak{R} , $\mathfrak{R}_3 = N_{\mathfrak{P}}(O_3(\mathfrak{R}))$. Hence, $\mathfrak{P} \subseteq \mathfrak{R}_3$. Since $O_2(\mathfrak{R}) = 1$, we get $\mathfrak{P} \subseteq Z(O_3(\mathfrak{R}))$. Hence, \mathfrak{R}_3 contains every element of $\mathcal{U}(\mathfrak{P})$. This contradicts Lemma 9.15 and completes the proof.

We now begin the construction of the final configuration.

By hypothesis, $2 \sim 3$. Let \mathfrak{A} be a noncyclic abelian subgroup of \mathfrak{G} of order 8 and let \mathfrak{B} be an elementary subgroup of order 9 each of whose elements centralizes an element of $\mathcal{U}(3)$, chosen so that $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is solvable. We assume without loss of generality that $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is a 2, 3-group. Let \mathfrak{R} be a maximal 2, 3-subgroup of \mathfrak{G} which contains $\langle \mathfrak{A}, \mathfrak{B} \rangle$.

Let \mathfrak{R}_p be a S_p -subgroup of \mathfrak{R} , $p = 2, 3$, with $\mathfrak{B} \subseteq \mathfrak{R}_3$. By Lemma 9.17, $O_2(\mathfrak{R}) \neq 1$.

Let J be an involution in $Z(\mathfrak{R}_2) \cap O_2(\mathfrak{R})$. Since $\mathfrak{B} \in \mathcal{D}$, by Lemma 9.16, \mathfrak{B} centralizes $Z(O_2(\mathfrak{R}))$. Hence, $C(J)$ is a solvable subgroup containing $\mathfrak{B}, \mathfrak{R}_2$, and an element of $\mathcal{U}(2)$.

Since $\mathfrak{B} \in \mathcal{D}$, we may apply Lemma 7.5. Let \mathfrak{M} be an element of $\mathcal{MS}(\mathfrak{G})$ which contains \mathfrak{B} and \mathfrak{R}_2 and which satisfies all the conclusions of Lemma 7.5. Let \mathfrak{R} be a $S_{2,3}$ -subgroup of \mathfrak{M} and let $\mathfrak{R}_0 = O_2(\mathfrak{R})$. Since $O_2(\mathfrak{M}) = 1$, so also $O_3(\mathfrak{R}) = 1$. Since no element of \mathfrak{B}^* centralizes any noncyclic abelian subgroup of order 8, it follows that \mathfrak{R}_0 is extra special of width 2, 3 or 4, and $C_{\mathfrak{R}_0}(\mathfrak{B}) = \mathfrak{R}'_0 = \langle I \rangle$, the last equality serving to define I . Hence, $\mathfrak{R}_0 = O_2(\mathfrak{M})$. Let \mathfrak{R}_p be a S_p -subgroup of \mathfrak{R} , $p = 2, 3$, with $\mathfrak{B} \subseteq \mathfrak{R}_3$. Let $\mathfrak{R}_3^* = \mathfrak{R}_3 \cap O_{2,3}(\mathfrak{R})$, $\mathfrak{R}^* = N_{\mathfrak{R}}(\mathfrak{R}_3^*)$. Thus, $\mathfrak{R} = \mathfrak{R}_0\mathfrak{R}^*$ and $\mathfrak{R}_0 \cap \mathfrak{R}^* = C_{\mathfrak{R}_0}(\mathfrak{R}_3^*)$. Let $\mathfrak{R}_2^* = \mathfrak{R}^* \cap \mathfrak{R}_2$ so that $\mathfrak{R}^* = \mathfrak{R}_3\mathfrak{R}_2^*$. We assume without loss of generality that $\mathfrak{R}_3 \subseteq \mathfrak{R}_2^*$.

We argue that

$$(9.25) \quad \mathfrak{R}_0 \cap \mathfrak{R}^* = \langle I \rangle .$$

Namely, choose \mathfrak{U} in $\mathcal{U}(\mathfrak{B})$ and suppose $C(\mathfrak{U}) \cap \mathfrak{R}_3^*$ is noncyclic. Then by Lemma 9.16 (ii), no noncyclic abelian subgroup of $C(\mathfrak{U}) \cap \mathfrak{R}_3^*$ centralizes any subgroup of order 4, so (9.25) is clear. Suppose $C(\mathfrak{U}) \cap \mathfrak{R}_3^*$ is cyclic. Hence, \mathfrak{R}_3^* has a cyclic subgroup of index 3. Assume that (9.25) does not hold. Then $\mathfrak{B} \not\subseteq \mathfrak{R}_3^*$, so the 3-length of \mathfrak{R} is at least 2. Hence, \mathfrak{R}_3^* is elementary of order 9 and all elements of $\mathfrak{R}_3^{*\#}$ are fused in \mathfrak{R} . But then every element of \mathfrak{R}_3^* centralizes an element of $\mathcal{U}(3)$, so again (9.25) holds. Thus, (9.25) holds.

LEMMA 9.18. *If \mathfrak{R} is any 2, 3-subgroup of \mathfrak{G} which contains \mathfrak{B}, I , and also contains a noncyclic abelian subgroup of order 8, then $\mathfrak{R} \subseteq \mathfrak{M}$.*

Proof. We may assume that \mathfrak{R} is a maximal 2, 3-subgroup of \mathfrak{G} . By Lemma 9.17, we have $O_2(\mathfrak{R}) \neq 1$. Since $\mathfrak{B} \in \mathcal{D}$, \mathfrak{B} centralizes $Z(O_2(\mathfrak{R}))$. Since $\langle I \rangle$ is a S_2 -subgroup of $C(\mathfrak{B})$, by Lemma 9.16 (ii), it follows that $\langle I \rangle = Z(O_2(\mathfrak{R}))$, so $\mathfrak{R} \subseteq C(I) = \mathfrak{M}$.

LEMMA 9.19. *\mathfrak{R}_2^* contains no noncyclic abelian subgroup of order 8.*

Proof. Suppose false. In this case, \mathfrak{R}^* is a $S_{2,3}$ -subgroup of $N(\mathfrak{R}_3^*)$, by the preceding paragraph. Hence, the 3-length of \mathfrak{R}^* is at least 2. But in this case, $\mathfrak{Z} \subset \mathfrak{R}_3^*$, so \mathfrak{R}_3 contains every element of $\mathcal{Z}(\mathfrak{P})$, against Lemma 7.4. The proof is complete.

LEMMA 9.20. *If $\tilde{\mathfrak{R}}_3$ is a S_3 -subgroup of $N(\mathfrak{R}_3)$, then*

$$\tilde{\mathfrak{R}}_3 = \mathfrak{R}_3 \cdot C_{\tilde{\mathfrak{R}}_3}(\mathfrak{R}_3).$$

Proof. Let $\mathfrak{C}_1 = C(\mathfrak{R}_2)\mathfrak{R}_3$, $\mathfrak{N}_1 = N(\mathfrak{R}_3)$. Since $\mathfrak{B} \subseteq \mathfrak{R}_3$, it follows that $\langle I \rangle$ is a S_2 -subgroup of \mathfrak{C}_1 , by Lemma 9.16 (ii). Hence, \mathfrak{M} covers $\mathfrak{N}_1/\mathfrak{C}_1$, so \mathfrak{C}_1 contains \mathfrak{R}_3 , which is equivalent to our assertion.

LEMMA 9.21.

(a) *If $\tilde{\mathfrak{P}}$ is any 3-subgroup of \mathfrak{M} , then no S_3 -subgroup of $N(\tilde{\mathfrak{P}})$ is contained in any conjugate of \mathfrak{M} .*

(b) *If P is an element of \mathfrak{M} of order 3, then $C(P)$ contains a subgroup \mathfrak{A}^* of type (3, 3) such that $C(A)$ contains an element of $\mathcal{Z}(3)$ for each A in \mathfrak{A}^* .*

(c) *If $\tilde{\mathfrak{P}}$ is any nonidentity 3-subgroup of \mathfrak{M} , then $N(\tilde{\mathfrak{P}})$ contains no noncyclic abelian group of order 8.*

(d) \mathfrak{R}_3 contains no abelian subgroup of order 27.

(e) \mathfrak{R}_3 is isomorphic to one of the following groups:

(i) an elementary group of order 9.

(ii) a nonabelian group of order 27.

Proof. Let \mathfrak{P}^* be a S_3 -subgroup of $N(\tilde{\mathfrak{P}})$. Suppose $\mathfrak{P}^* \subseteq \mathfrak{M}^g$. Since $\tilde{\mathfrak{P}} \subseteq \mathfrak{P}^*$, we get $\tilde{\mathfrak{P}} \subseteq \mathfrak{M}^g$. Let $\mathfrak{P}_0 = \tilde{\mathfrak{P}}^{g^{-1}}$, $\mathfrak{P}_1 = \mathfrak{P}^{*g^{-1}}$. Then \mathfrak{P}_0 is a 3-subgroup of \mathfrak{M} and \mathfrak{P}_1 is a S_3 -subgroup of $N(\mathfrak{P}_0)$ which is contained in \mathfrak{M} . This violates Lemma 9.20, since $\mathfrak{R}_3 \subset \tilde{\mathfrak{R}}_3$. Hence, (a) holds.

Since $\mathfrak{B} \subseteq \mathfrak{R}_3$, it follows from Lemma 9.16 (ii) that $\langle I \rangle$ is a S_2 -subgroup of $\mathfrak{R}_3 C(\mathfrak{R}_3)$. Thus, $\mathfrak{R}_3 C(\mathfrak{R}_3)$ has a normal 2-complement. We assume without loss of generality that I normalizes $\tilde{\mathfrak{R}}_3$. Let $\hat{\mathfrak{R}}_3/\mathfrak{R}_3$ be a chief factor of $\tilde{\mathfrak{R}}_3 \langle I \rangle$. Hence, $\hat{\mathfrak{R}}_3 = \mathfrak{R}_3 \times \bar{\mathfrak{R}}_3$, where $|\bar{\mathfrak{R}}_3| = 3$. This implies that $C_{\hat{\mathfrak{R}}_3}(P)$ contains an elementary subgroup of order 27. Let \mathfrak{P}^g be a S_3 -subgroup of \mathfrak{G} containing $\tilde{\mathfrak{R}}_3$, and let $\mathfrak{U} \in \mathcal{Z}(\mathfrak{P}^g)$. Then $C(\mathfrak{U}) \cap C_{\tilde{\mathfrak{R}}_3}(P)$ is noncyclic, and any noncyclic subgroup of $C(\mathfrak{U}) \cap C_{\tilde{\mathfrak{R}}_3}(P)$ of order 9 may play the role of \mathfrak{A}^* in (b).

Let \mathfrak{R} be a maximal 2, 3-subgroup of \mathfrak{G} which contains a $S_{2,3}$ -subgroup of $N(\tilde{\mathfrak{P}})$. By (b), \mathfrak{R} contains an element of \mathcal{S} . Assume that \mathfrak{R} contains a noncyclic abelian subgroup of order 8. Then by Lemma 9.17, $O_2(\mathfrak{R}) \neq 1$. By Lemma 9.16 (ii), we get $|Z(O_2(\mathfrak{R}))| = 2$. Clearly, \mathfrak{R} is a $S_{2,3}$ -subgroup of $C(Z(O_2(\mathfrak{R})))$, and so it contains an

element \mathfrak{D} of \mathcal{S} and one of $\mathcal{Z}(2)$. Applying Lemma 7.5, we get a conjugate \mathfrak{M}^g of \mathfrak{M} containing \mathfrak{D} and a S_2 -subgroup of \mathfrak{R} . By Lemma 7.5 (f), $Z(\mathcal{O}_2(\mathfrak{R})) \subseteq C(\mathfrak{D}) \cap \mathcal{O}_2(\mathfrak{M}^g)$. By Lemma 9.16 (ii), the last group is of order 2, and so equals $Z(\mathcal{O}_2(\mathfrak{R}))$. Hence, $Z(\mathcal{O}_2(\mathfrak{R})) = Z(\mathcal{O}_2(\mathfrak{M}^g))$, and so $\mathfrak{R} \subseteq \mathfrak{M}^g$. This violates (a).

Suppose \mathfrak{C} is an abelian subgroup of \mathfrak{R}_3 of order 27. Then there is an element E in \mathfrak{C}^* of order 3 such that $C(E) \cap \mathcal{O}_2(\mathfrak{R})$ contains a noncyclic abelian subgroup of order 8. This violates (c) and establishes (d).

(e) is an immediate consequence of (d).

LEMMA 9.22. $\mathfrak{R}_2 - \mathfrak{R}_0$ contains an involution.

Proof. Suppose false. By a result of Glauberman [16], \mathfrak{R}_2 contains an involution J such that $J = I^g \neq I$. Since the lemma is false, $J \in \mathfrak{R}_0$. Let $\mathfrak{X} = C_{\mathfrak{R}_0}(J)$. Then \mathfrak{X} is generated by involutions, and $\mathfrak{X} \subseteq \mathfrak{M}^g = C(J)$. Since the lemma is false, $\mathfrak{X} \subseteq \mathfrak{R}_0^g$. In particular, $I \in (\mathfrak{R}_0^g)'$. Hence, $(\mathfrak{R}_0^g)' = \langle J \rangle = \langle I \rangle$, a contradiction.

LEMMA 9.23. The 3-length of \mathfrak{R} is 1.

Proof. Suppose false. By Lemma 9.21 (e), it follows that \mathfrak{R}_3^* is elementary of order 9. Consider $\mathfrak{R}^*/\langle I \rangle$. Since $\mathfrak{R}^* \cap \mathfrak{R}_0 = \langle I \rangle$ by (9.25), it follows that $\mathfrak{R}_3^*\langle I \rangle/\langle I \rangle = F(\mathfrak{R}^*/\langle I \rangle)$. This implies that $\mathfrak{R}^*/\langle I \rangle$ contains a quaternion subgroup $\mathfrak{Q}/\langle I \rangle$. Thus, \mathfrak{Q} is not of maximal class, since no group of maximal class and order 16 has a quaternion factor group. Hence, \mathfrak{Q} contains a noncyclic abelian subgroup of order 8. This violates Lemma 9.21 (c) with \mathfrak{R}_3^* in the role of \mathfrak{F} .

LEMMA 9.24. Each involution J of $\mathfrak{R}_2 - \mathfrak{R}_0$ normalizes a S_3 -subgroup of \mathfrak{R} .

Proof. Since $J \in \mathfrak{R}_0$, Lemma 5.36 implies that J inverts an element P of \mathfrak{R} of order 3. Let $\mathfrak{C} = C_{\mathfrak{R}}(P)$. Suppose $\mathfrak{C} \cap \mathfrak{R}_0 = \langle I \rangle$. Then since $\mathfrak{R}_3\mathfrak{R}_0 \triangleleft \mathfrak{R}$, it follows that \mathfrak{C} is 3-closed. Let \mathfrak{C}_3 be the S_3 -subgroup of \mathfrak{C} . Thus, \mathfrak{C}_3 is noncyclic. Since $N(\mathfrak{C}_3) \cap \mathfrak{R}_0 = \langle I \rangle$, $N_{\mathfrak{R}}(\mathfrak{C}_3)$ contains a S_3 -subgroup of \mathfrak{R} as a normal subgroup. Since $J \in N(\mathfrak{C}_3)$, we are done.

We may assume that $\mathfrak{C} \cap \mathfrak{R}_0 \supset \langle I \rangle$. Hence, $\mathfrak{C} \cap \mathfrak{R}_0 = \mathfrak{Q}$ is a quaternion group. Let \mathfrak{F} be a S_3 -subgroup of \mathfrak{C} . Thus, $\mathfrak{F}\mathfrak{Q} = \mathcal{O}^3(\mathfrak{C})$ char \mathfrak{C} , so $\langle J \rangle\mathfrak{F}\mathfrak{Q}$ is a group. Let $\mathfrak{F}_0 = \mathcal{O}_3(\mathfrak{F}\mathfrak{Q})$. Thus, $\mathfrak{F}\mathfrak{Q}/\mathfrak{F}_0 \cong SL(2, 3)$, and J stabilizes $\mathfrak{F}\mathfrak{Q}/\mathfrak{F}_0$. If J does not centralize $\mathfrak{Q}\mathfrak{F}/\mathfrak{Q}\mathfrak{F}_0$, it follows from Lemma 5.36 that J normalizes a S_3 -subgroup of $\mathfrak{Q}\mathfrak{F}$. Suppose J centralizes $\mathfrak{F}\mathfrak{Q}/\mathfrak{Q}\mathfrak{F}_0$. Then $\langle J \rangle\mathfrak{F}\mathfrak{Q}/\mathfrak{Q}\mathfrak{F}_0$ is a cyclic group of order 6, so J centralizes $\mathfrak{Q}/\mathfrak{Q}'$. This implies that $|C(J) \cap \mathfrak{Q}| \geq 4$, so that $\langle J, \mathfrak{Q} \rangle$ contains a noncyclic abelian subgroup of order 8. Since $\langle J, \mathfrak{Q} \rangle \subseteq N(\langle P \rangle)$, Lemma 9.21 (c) is violated. We conclude that J

normalizes a S_3 -subgroup of $\tilde{\mathfrak{B}}\Omega$. We assume without loss of generality that J normalizes $\tilde{\mathfrak{B}}$. Since $N(\tilde{\mathfrak{B}}) \cap \mathfrak{R}_0 = \langle I \rangle$, it follows that $N_{\mathfrak{R}}(\tilde{\mathfrak{B}})$ contains a S_3 -subgroup of \mathfrak{R} as a normal subgroup. The proof is complete.

LEMMA 9.25. (a) *If T is an involution of \mathfrak{G} , $C(T)$ contains a noncyclic abelian subgroup of order 8.*

(b) *If the width of \mathfrak{R}_0 is 2, then for each involution T of \mathfrak{G} , $|C(T)|_3 \leq 9$.*

Proof. (a) By Lemma 5.38, $C(T)$ contains an element u of $\mathcal{Z}(2)$. If $T \notin u$, then $\langle u, T \rangle$ is a noncyclic abelian subgroup of order 8 which is contained in $C(T)$. Suppose $T \in u$. Since $\mathcal{S}_{ev_3}(2) \neq \emptyset$, $C(T)$ contains an element of $\mathcal{S}_{ev_3}(2)$ by Lemma 0.8.9.

Suppose (b) is false, and T is an involution of \mathfrak{G} with $|C(T)|_3 \geq 27$. Let \mathfrak{C} be a maximal 2, 3-subgroup of \mathfrak{G} which contains a $S_{2,3}$ -subgroup of $C(T)$. By Lemma 5.38, \mathfrak{C} contains an element u of $\mathcal{Z}(2)$. Let \mathfrak{C}_p be a S_p -subgroup of \mathfrak{C} , $p = 2, 3$. We assume without loss of generality that $\mathfrak{C}_2 \subseteq \mathfrak{R}_2$.

Case 1. $O_3(\mathfrak{C}) \neq 1$.

Since $u \in \mathcal{Z}(2)$, u centralizes $O_3(\mathfrak{C})$. Since u contains a conjugate of I , it follows that $|O_3(\mathfrak{C})| \leq 9$. Suppose $|O_3(\mathfrak{C})| = 9$. Then $O_3(\mathfrak{C})$ is conjugate to \mathfrak{B} , since \mathfrak{B} is a S_3 -subgroup of \mathfrak{M} . But then Lemma 9.16 (ii) is violated. Hence, $|O_3(\mathfrak{C})| = 3$.

Since u centralizes $O_3(\mathfrak{C})$, $O_3(\mathfrak{C})$ is conjugate to a subgroup of \mathfrak{B} . By Lemma 9.21 (b), $C(O_3(\mathfrak{C}))$ contains an elementary subgroup \mathfrak{A}^* such that $C(A)$ contains an element of $\mathcal{Z}(3)$ for each A in \mathfrak{A}^* . Since \mathfrak{C} is a $S_{2,3}$ -subgroup of $N(O_3(\mathfrak{C}))$, we assume without loss of generality that $\mathfrak{A}^* \subseteq \mathfrak{C}$. By Lemma 9.16 (i), $\mathfrak{A}^* \in \mathcal{S}$. Now Lemma 9.17 yields $O_2(\mathfrak{C}) \neq 1$. Since $\mathfrak{A}^* \subseteq \mathfrak{C}$, Lemma 9.16 (ii) forces $|Z(O_2(\mathfrak{C}))| = 2$, and forces $Z(O_2(\mathfrak{C}))$ to be a maximal characteristic abelian subgroup of $O_2(\mathfrak{C})$. Since $|O_3(\mathfrak{C})| = 3$, it follows that $|O_2(\mathfrak{C})| > 2$. Hence, $O_2(\mathfrak{C})$ is extra special. Thus, $O_2(\mathfrak{C})'$ is of order 2 and is normalized by every element of $\mathcal{Z}(\mathfrak{R}_2)$. Hence, every element of $\mathcal{Z}(\mathfrak{R}_2)$ is contained in \mathfrak{C}_2 . Thus, I centralizes $O_3(\mathfrak{C})$. Since $I \in Z(\mathfrak{C}_2)$, we get $I \in O_2(\mathfrak{C})$, so that $\langle I \rangle = O_2(\mathfrak{C})'$. Hence, $\mathfrak{C} \subseteq \mathfrak{M}$, against $|\mathfrak{C}|_3 \geq 27$ and $|\mathfrak{M}|_3 = 9$.

Case 2. $O_3(\mathfrak{C}) = 1$.

Since $|\mathfrak{C}|_3 \geq 27$, it follows that $m(O_2(\mathfrak{C})) \geq 6$. Since the width of \mathfrak{R}_0 is 2, it follows that \mathfrak{R}_2 has no elementary subgroup of order 2^6 . Thus, $O_2(\mathfrak{C})$ is not elementary.

Now \mathfrak{C} is clearly not contained in any conjugate of \mathfrak{M} , since $|\mathfrak{C}|_3 > |\mathfrak{M}|_3$. Since $\langle I \rangle = Z(\mathfrak{R}_2)$, it follows that \mathfrak{C} is not 2-closed.

Since $|\mathfrak{R}_2| \leq 2^8$, we get $|O_2(\mathfrak{S})| = 2^7$. Hence, $|O_2(\mathfrak{S})| = 2^7$, so that $D(O_2(\mathfrak{S}))$ is a subgroup of order 2 and \mathfrak{S}_2 is of order 2^8 . Hence, $\mathfrak{S}_2 = \mathfrak{R}_2$ and $D(O_2(\mathfrak{S})) = \langle I \rangle$. This shows that $\mathfrak{S} \subseteq \mathfrak{M}$. This contradiction completes the proof.

- LEMMA 9.26. *If $\tilde{\mathfrak{F}}$ is any subgroup of \mathfrak{R}_3 of order 3, then*
- (a) $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$ is either $\langle I \rangle$ or a quaternion group;
 - (b) if $\tilde{\mathfrak{F}} \not\subseteq Z(\mathfrak{R}_3)$, then $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$ is quaternion.

Proof. (a) Suppose $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}}) \supset \langle I \rangle$. Then $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$ is extra special and does not contain a noncyclic abelian subgroup of order 8. Thus, $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$ is either dihedral or quaternion. Now $C_{\mathfrak{R}_3}(\tilde{\mathfrak{F}})$ contains an elementary subgroup \mathfrak{E} of order 9 with $\tilde{\mathfrak{F}} \subset \mathfrak{E}$. Hence, $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$ admits \mathfrak{E} . Since no element of $\mathfrak{E}^\#$ centralizes a noncyclic abelian subgroup of \mathfrak{R}_0 of order 8, $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$ is quaternion.

(b) Let $\mathfrak{E} = \langle \tilde{\mathfrak{F}}, Z(\mathfrak{R}_3) \rangle$, so that by Lemma 9.21 (e), \mathfrak{E} is elementary of order 9 and $\mathfrak{E} \triangleleft \mathfrak{R}_3$. It follows that the three subgroups of \mathfrak{E} of order 3 which are distinct from $Z(\mathfrak{R}_3)$ are conjugate in \mathfrak{R}_3 . We can choose E in $\mathfrak{E}^\#$ such that $\mathfrak{R}_0 \cap C(E)$ is not centralized by $Z(\mathfrak{R}_3)$. By (a), $\mathfrak{R}_0 \cap C(E)$ is a quaternion group; so (b) holds.

LEMMA 9.27. \mathfrak{R}_2^* is a four-group.

Proof. Suppose false. By Lemma 9.22, $\mathfrak{R}_2 - \mathfrak{R}_0$ contains an involution J . By Lemma 9.24, J normalizes a S_3 -subgroup of \mathfrak{R} . Thus, we can choose M in \mathfrak{M} such that $J^M = T_0$ normalizes \mathfrak{R}_3 . Since \mathfrak{R}_3 permutes transitively by conjugation the S_2 -subgroups of \mathfrak{R} , we may choose K in \mathfrak{R}_3 such that $T = T_0^K$ lies in \mathfrak{R}_2 . Thus, $T \in N(\mathfrak{R}_3) \cap \mathfrak{R}_2$.

By Lemma 9.23, $\mathfrak{R}_3^* = \mathfrak{R}_3$. Thus, $T \in \mathfrak{R}_2^*$. If $\langle T, I \rangle = \mathfrak{R}_2^*$, we are done, so suppose $\langle T, I \rangle \subset \mathfrak{R}_2^*$. Let \mathfrak{F} be a subgroup of \mathfrak{R}_2^* of order 8 which contains $\langle T, I \rangle$. By Lemma 9.19, \mathfrak{F} is dihedral of order 8. Let $\mathfrak{F}_0, \mathfrak{F}_1$ be the four-subgroups in \mathfrak{F} .

Suppose \mathfrak{X} is a subgroup of \mathfrak{R}_3 of order 3 which admits \mathfrak{F} and that $C(\mathfrak{X}) \cap \mathfrak{R}_0$ is a quaternion group. Hence, $C_{\mathfrak{M}}(\mathfrak{X})$ contains a normal quaternion subgroup and S_2 -subgroups of $N_{\mathfrak{M}}(\mathfrak{X})$ are of order at least 2^5 . Thus, $N_{\mathfrak{M}}(\mathfrak{X})$ contains a noncyclic abelian subgroup of order 8. This is impossible, by Lemma 9.21 (c). Hence, $C_{\mathfrak{R}_0}(\mathfrak{X}) = \langle I \rangle$, by Lemma 9.26 (a).

By Lemma 9.26 (b) and the preceding paragraph, it follows that \mathfrak{F} normalizes no noncentral subgroup of \mathfrak{R}_3 of order 3.

Suppose \mathfrak{R}_3 is nonabelian. Then \mathfrak{F} normalizes $\Omega_1(\mathfrak{R}_3)$, a group of exponent 3. Since $\langle I \rangle$ centralizes \mathfrak{R}_3 , it follows that $\Omega_1(\mathfrak{R}_3)\mathfrak{F}$ is supersolvable. Thus, $\Omega_1(\mathfrak{R}_3)\mathfrak{F}$ contains a normal subgroup of order 9, so \mathfrak{F} normalizes a noncentral subgroup of \mathfrak{R}_3 of order 3. This contradicts the preceding paragraph, so we conclude that \mathfrak{R}_3 is abelian, $\mathfrak{R}_3 = \mathfrak{B}$.

Let $\mathfrak{F}_i = \langle J_i, I \rangle$, $i = 0, 1$. If both J_0 and J_1 invert \mathfrak{R}_3 , then J_0J_1 centralizes \mathfrak{R}_3 , so $J_0J_1 \in \langle I \rangle$. This is not the case, since $\mathfrak{G}_0\mathfrak{G}_1$ is of order 8. Thus, we may assume notation is chosen so that J_0 centralizes \mathfrak{x}_0 and inverts \mathfrak{x}_1 . Here, $|\mathfrak{x}_i| = 3$, and $\mathfrak{R}_3 = \mathfrak{x}_0 \times \mathfrak{x}_1$. Since $\mathfrak{R}_0 \cap C(\mathfrak{x}_i) = \langle I \rangle$, for $i = 0, 1$, the width of \mathfrak{R}_0 is 2.

Let \mathfrak{G} be a $S_{2,3}$ -subgroup of $N(\mathfrak{R}_3)$ which contains \mathfrak{R}_2^* . Since $\mathfrak{R}_3 \subset \mathfrak{B}$, we get $|\mathfrak{R}_3| = 9 < |\mathfrak{G}|_3$. By Lemma 9.16 (ii), $|C(\mathfrak{R}_3)|_2 = 2$. By Lemma 9.20, we get that \mathfrak{G} is 3-closed. Let $\mathfrak{G}_3 = O_3(\mathfrak{G}) \supset \mathfrak{R}_3$.

Let F_0, F_1, F_2 be the three involutions of \mathfrak{F}_0 , and set

$$3^{f_i} = |\mathfrak{G}_3 \cap C(F_i)|, \quad i = 0, 1, 2.$$

By Lemma 9.25 (b), we have $f_i \leq 2$. Since $\mathfrak{G}_3 \cap C(\mathfrak{F}_0) = \mathfrak{x}_0$, a formula of Wielandt [44] yields

$$|\mathfrak{G}_3| = 3^{f_0+f_1+f_2-2} \leq 3^4.$$

Since the dihedral group \mathfrak{F} is faithfully represented on $\mathfrak{G}_3/\mathfrak{R}_3$, it follows that $|\mathfrak{G}_3| = 3^4$.

Let \mathfrak{D} be a $S_{2,3}$ -subgroup of $N(\mathfrak{G}_3)$. Let \mathfrak{D}_p be a S_p -subgroup of \mathfrak{D} , with $\mathfrak{R}_2^* \subseteq \mathfrak{D}_2$. By the formula of Wielandt [44] applied to \mathfrak{F}_0 acting on $O_3(\mathfrak{D})$, we get $O_3(\mathfrak{D}) = \mathfrak{G}_3$. If $\mathfrak{D}_3 = \mathfrak{G}_3$, then \mathfrak{G}_3 is a S_3 -subgroup of \mathfrak{G} . But the center of \mathfrak{G}_3 contains $\mathfrak{R}_3 = \mathfrak{B}$ by Lemma 9.20, and hence is noncyclic. This contradicts hypothesis (iii) of Theorem 9.1. Therefore $\mathfrak{D}_3 \supset \mathfrak{G}_3$ and \mathfrak{D} is not 3-closed.

By Lemma 9.25 (b), we get $O_2(\mathfrak{D}) = 1$. Since \mathfrak{D} is quite obviously contained in no conjugate of \mathfrak{M} , Lemma 9.18 implies that \mathfrak{D} contains no noncyclic abelian subgroups of order 8. Thus, \mathfrak{D}_2 is of maximal class. Hence, $\langle I \rangle = Z(\mathfrak{D}_2)$, so $\mathfrak{D}_2 = \mathfrak{R}_2^*$ is of order at most 16. Suppose $|\mathfrak{D}_2| = 16$. Since $O_2(\mathfrak{D}) = 1$, and \mathfrak{D} is not 3-closed, it follows that $O_{3,2}(\mathfrak{D}) \cap \mathfrak{D}_2$ is a quaternion group. But then \mathfrak{M} covers $\mathfrak{D}/O_3(\mathfrak{D})$. This is not the case, since $O_3(\mathfrak{D})$ contains a S_3 -subgroup of \mathfrak{M} , and since $3 \mid |\mathfrak{D} : O_3(\mathfrak{D})|$. Hence, $\mathfrak{D}_2 = \mathfrak{F}$ is dihedral of order 8. Let $\tilde{\mathfrak{D}}_2 = O_{3,2}(\mathfrak{D}) \cap \mathfrak{D}_2$. Thus, $\tilde{\mathfrak{D}}_2$ is a four-group and $\mathfrak{D}/O_3(\mathfrak{D}) \cong \Sigma_4$.

Since $\mathfrak{D}/O_3(\mathfrak{D}) \cong \Sigma_4$, some chief factor of \mathfrak{D} is of order 3^3 . Thus, $O_3(\mathfrak{D})$ is necessarily elementary, and elements of $\mathfrak{D}_3 - O_3(\mathfrak{D})$ induce automorphisms of $O_3(\mathfrak{D})$ with minimal polynomial $(x - 1)^3$. Hence, $O_3(\mathfrak{D}) = J(\mathfrak{D}_3)$, $O_3(\mathfrak{D}) = J(\mathfrak{D}_3)$ char \mathfrak{D}_3 , so \mathfrak{D}_3 is a S_3 -subgroup of \mathfrak{G} . This is not the case, since $Z(\mathfrak{D}_3)$ is noncyclic. The proof is complete.

LEMMA 9.28. *If the width of \mathfrak{R}_0 exceeds 2, then I is the only conjugate of I in \mathfrak{R}_0 .*

Proof. Suppose $T = I^g \neq I$, $T \in \mathfrak{R}_0$. Then $C(T) \cap \mathfrak{R}_0 \subseteq C(T) = \mathfrak{M}^g$. By Lemma 9.27, $C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g$ is of index at most 2 in $C(T) \cap \mathfrak{R}_0$.

Since $C_{\mathfrak{R}_0}(T)$ is of index 2 in \mathfrak{R}_0 , we get $|\mathfrak{R}_0 : C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g| \leq 4$. Since the width of \mathfrak{R}_0 is at least 3, it follows that $C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g$ is nonabelian. Hence, $\langle I \rangle = (C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g)' = \langle T \rangle$. This contradiction completes the proof.

LEMMA 9.29. $\mathfrak{R}_3 = \mathfrak{B}$ is of order 9.

Proof. Suppose false. By Lemma 9.21 (e), \mathfrak{R}_3 is nonabelian of order 27. Since \mathfrak{R}_3 is faithfully represented on \mathfrak{R}_0 , the width of \mathfrak{R}_0 is at least 3. By a result of Glauberman [14], \mathfrak{R}_2 contains a conjugate T of I distinct from I , $T = I^g \neq I$. By Lemma 9.28, $T \in \mathfrak{R}_2 - \mathfrak{R}_0$, so by Lemma 9.24, we may assume that $T \in \mathfrak{R}_2^*$. Thus, by Lemma 9.27, $\mathfrak{R}_2^* = \langle I, T \rangle$.

Since \mathfrak{R}_3 is nonabelian, it follows that $\mathfrak{X}_1 = \mathfrak{R}_3 \cap C(T)$ is of order 3. By Lemma 1.3 of [17], \mathfrak{R}_3 has a subgroup \mathfrak{X}_0 of order 3 which centralizes \mathfrak{X}_1 and is inverted by T . Let $\mathfrak{X} = \mathfrak{X}_0 \times \mathfrak{X}_1$ and let $\mathfrak{X}_2, \mathfrak{X}_3$ be the remaining subgroups of \mathfrak{X} of order 3.

Suppose $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 \supset \langle I \rangle$. By Lemma 9.26 (a), $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 = \mathfrak{Q}$ is a quaternion group. Since $C(\mathfrak{X}) \cap \mathfrak{R}_0 = \langle I \rangle$, it follows that \mathfrak{X}_1 is faithfully represented on \mathfrak{Q} . Since $\text{Aut}(\mathfrak{Q})$ has no element of order 6, T centralizes \mathfrak{Q} . But then $\langle \mathfrak{Q}, T \rangle = \mathfrak{Q} \times \langle T \rangle$ contains a noncyclic abelian subgroup of order 8. This violates Lemma 9.21 (c) with \mathfrak{X}_0 in the role of \mathfrak{B} . Hence, $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 = \langle I \rangle$.

Since $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 = \langle I \rangle$, the width of \mathfrak{R}_0 is at most 3. By Lemma 9.26 (b), we get $\mathfrak{X}_0 = Z(\mathfrak{R}_3)$. Thus, if we set $\mathfrak{Q}_i = \mathfrak{R}_0 \cap C(\mathfrak{X}_i)$, $i = 1, 2, 3$, then by Lemma 9.26, it follows that each \mathfrak{Q}_i is quaternion. Hence, \mathfrak{R}_0 is the central product of $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3$. Since T centralizes \mathfrak{X}_1 and interchanges \mathfrak{X}_2 and \mathfrak{X}_3 , it follows that T normalizes \mathfrak{Q}_1 and interchanges \mathfrak{Q}_2 and \mathfrak{Q}_3 . Since \mathfrak{X}_0 is faithfully represented on \mathfrak{Q}_1 , it follows that $\mathfrak{Q}_1 \mathfrak{X}_0 \langle T \rangle \cong GL(2, 3)$. Thus, we can choose generators A_i, B_i for \mathfrak{Q}_i such that $A_1^T = B_1, A_2^T = A_3, B_2^T = B_3$. It follows that $C(T) \cap \mathfrak{R}_0 = \langle A_2 A_3, B_2 B_3, I \rangle$, an elementary group of order 8. Let $\mathfrak{F} = C(T) \cap \mathfrak{R}_2 = \langle T \rangle \times C(T) \cap \mathfrak{R}_0$. It now follows that $\tilde{\mathfrak{R}}_2 = N_{\mathfrak{R}_2}(\mathfrak{F}) = \langle \mathfrak{Q}_2, \mathfrak{Q}_3, A_1 B_1, T \rangle$, a group of index 2 in \mathfrak{R}_2 . Since $Z(\tilde{\mathfrak{R}}_2) = \langle I \rangle$, it follows that $\tilde{\mathfrak{R}}_2$ is a S_3 -subgroup of $N(\mathfrak{F})$.

Now $T = I^g$, so $\mathfrak{F} \cong \mathfrak{M}^g$. By symmetry, $N(\mathfrak{F}) \cap \mathfrak{M}^g$ contains a S_2 -subgroup of $N(\mathfrak{F})$. This implies that $O_2(N(\mathfrak{F}))$ centralizes both T and I . Hence, $O_2(N(\mathfrak{F})) = \mathfrak{F}$.

Now $\tilde{\mathfrak{R}}_2$ permutes transitively the elements of $(\mathfrak{F} \cap \mathfrak{R}_0)T$, so $\mathfrak{S} = \{(\mathfrak{F} \cap \mathfrak{R}_0)T, I\}$ is the set of all the elements of \mathfrak{F} which are conjugate to I in \mathfrak{G} . Since $N(\mathfrak{F}) \cap \mathfrak{M}^g$ normalizes \mathfrak{S} but does not centralize I , it follows that $N(\mathfrak{F})$ permutes \mathfrak{S} transitively.

Since $N(\mathfrak{F})$ is transitive on \mathfrak{S} , it follows that $N(\mathfrak{F}) = 9 \cdot |N(\mathfrak{F}) \cap \mathfrak{M}|$. Since $C(\mathfrak{F}) = C_{\mathfrak{M}}(\mathfrak{F})$, it follows that $\mathfrak{F} = C(\mathfrak{F})$. Since T centralizes \mathfrak{X}_1 ,

it follows that \mathfrak{X}_1 normalizes $\mathfrak{R}_0 \cap C(T)$, so normalizes $\mathfrak{F} = \langle T \rangle \times \mathfrak{R}_0 \cap C(T)$. But it now follows that $27 \mid |N(\mathfrak{F})|$. Since \mathfrak{F} is elementary of order 2^4 , $\text{Aut}(\mathfrak{F})$ has no subgroup of order 27. This violates the equality $\mathfrak{F} = C(\mathfrak{F})$, and the proof is complete.

LEMMA 9.30. *If T is any involution of \mathfrak{G} , then $|C(T)|_3 \leq 9$.*

Proof. Since $|\mathfrak{R}_2: \mathfrak{R}_0| = 2$, Lemma 5.38 implies that every involution of \mathfrak{G} is conjugate to an involution of \mathfrak{R}_0 . Thus, we may assume that $T \in \mathfrak{R}_0$. If $T \sim I$, we are done by Lemma 9.29, so from now on we suppose $T \not\sim I$.

Let $\mathfrak{U} \in \mathcal{Z}(\mathfrak{R}_2)$, and let $\tilde{\mathfrak{R}}_0 = \mathfrak{R}_0 \cap C(\mathfrak{U})$. Thus, $\tilde{\mathfrak{R}}_0$ is of index 2 in \mathfrak{R}_0 . Since \mathfrak{R}_3 has no fixed points on $\mathfrak{R}_0/\langle I \rangle$, Lemma 5.38 implies that for some X in \mathfrak{R}_3 , $T^X \in \tilde{\mathfrak{R}}_0$. Thus, we assume without loss of generality that $T \in \tilde{\mathfrak{R}}_0$.

We argue that $C_{\mathfrak{M}}(T)$ contains a S_2 -subgroup of $C(T)$. This is clear if $|\mathfrak{M}: C_{\mathfrak{M}}(T)|_2 = 2$, since $T \not\sim I$. So suppose $|\mathfrak{M}: C_{\mathfrak{M}}(T)|_2 = 4$. In this case, $C_{\mathfrak{R}_0}(T)$ is a S_2 -subgroup of $C_{\mathfrak{M}}(T)$. Since $\langle I \rangle = C_{\mathfrak{R}_0}(T)'$ char $C_{\mathfrak{R}_0}(T)$, it follows that $C_{\mathfrak{R}_0}(T)$ is a S_2 -subgroup of $C(T)$.

Let \mathfrak{R} be a $S_{2,3}$ -subgroup of $C(T)$ which contains $C_{\mathfrak{R}}(T)$. Suppose $O_3(\mathfrak{R}) \neq 1$. Since $\mathfrak{U} \subseteq \mathfrak{R}$, \mathfrak{U} centralizes $O_3(\mathfrak{R})$, so $O_3(\mathfrak{R}) \subseteq \mathfrak{M}$. Since no element of \mathfrak{R}_3^* centralizes a four-subgroup of \mathfrak{R}_0 by Lemma 9.26 (a), we conclude that $O_3(\mathfrak{R}) = 1$.

Since $O_3(\mathfrak{R}) = 1$ and since $\mathfrak{R} \cap \mathfrak{M}$ contains a S_2 -subgroup of $C(T)$, it follows that $I \in Z(O_2(\mathfrak{R}))$. Suppose X is a 3-element of \mathfrak{R} and \mathfrak{X} centralizes I . Then $X \in C(\langle T, I \rangle)$, so $X = 1$ by Lemma 9.26 (a). Thus, a S_3 -subgroup \mathfrak{R}_3 of \mathfrak{R} is faithfully represented on $Z(O_2(\mathfrak{R}))$.

Let $\Omega_1(Z(O_2(\mathfrak{R}))) = \mathfrak{Y}_1 \times \mathfrak{Y}_2$, where $\mathfrak{Y}_1 = \Omega_1(Z(O_2(\mathfrak{R}))) \cap C(\mathfrak{R}_3)$, and $\mathfrak{Y}_2 = [\Omega_1(Z(O_2(\mathfrak{R}))), \mathfrak{R}_3]$. Thus, $T \in \mathfrak{Y}_1$ and \mathfrak{R}_3 is faithfully represented on \mathfrak{Y}_2 . Hence, $m(Z(O_2(\mathfrak{R}))) = m(\mathfrak{Y}_1) + m(\mathfrak{Y}_2) \geq 7$. Thus, \mathfrak{R}_0 has an elementary subgroup of order 2^6 , by Lemma 9.27. This is impossible, since the width of \mathfrak{R}_0 is at most 4. The proof is complete.

LEMMA 9.31. $|\mathfrak{G}|_3 > 3^4$.

Proof. Let \mathfrak{X} be a subgroup of \mathfrak{R}_3 of order 3 such that $\mathfrak{R}_0 \cap C(\mathfrak{X}) = \mathfrak{Q}$ is quaternion. Let \mathfrak{C} be a $S_{2,3}$ -subgroup of $C(\mathfrak{X})$ which contains $\mathfrak{R}_3\mathfrak{Q}$. Since $\mathfrak{R}_3 = \mathfrak{B} \in \mathcal{S}$, it follows that \mathfrak{R}_3 centralizes $Z(O_2(\mathfrak{C}))$. Since $\langle I \rangle$ is a S_2 -subgroup of $C(\mathfrak{R}_3)$, it follows that $O_2(\mathfrak{C}) = 1$, by Lemma 9.21(a).

Since $O_2(\mathfrak{C}) = 1$, \mathfrak{Q} is faithfully represented on $O_3(\mathfrak{C})$, so is faithfully represented on $O_3(\mathfrak{C})/\mathfrak{X}$. Hence, $|O_3(\mathfrak{C}): \mathfrak{X}| \geq 9$. Since $[\mathfrak{R}_3 \cap O_3(\mathfrak{C}), \mathfrak{Q}] \subseteq O_3(\mathfrak{C}) \cap \mathfrak{Q} = 1$, it follows that $\mathfrak{R}_3 \cap O_3(\mathfrak{C}) = \mathfrak{X}$. Hence, $|\mathfrak{C}|_3 \geq 3^4$. Suppose the lemma is false. Then \mathfrak{C} contains a S_3 -subgroup of \mathfrak{G} , and $O_3(\mathfrak{C})$ is of order 3^3 , while $\mathfrak{X} \sim \mathfrak{Z}$. If $O_3(\mathfrak{C})$ is nonabelian, then Hypothesis 9.1 is satisfied. This is not the case, so $O_3(\mathfrak{C})$ is elementary. Hence, $O_3(\mathfrak{C}) = \mathfrak{X} \times [O_3(\mathfrak{C}), \mathfrak{Q}]$. Hence, the center of a S_3 -subgroup of

\mathfrak{G} is noncyclic. This is not the case. The proof is complete.

LEMMA 9.32. *Choose J in $\mathfrak{R}_2^* - \langle I \rangle$. If J inverts \mathfrak{R}_3 , then $A_{\mathfrak{G}}(\mathfrak{R}_2^*) = \text{Aut}(\mathfrak{R}_2^*)$.*

Proof. Let \mathfrak{x} be any four-subgroup of \mathfrak{M} which contains I . We will show that

$$(9.25) \quad |A_{\mathfrak{M}}(\mathfrak{x})| = 2.$$

This is clear if $\mathfrak{x} \subseteq \mathfrak{R}_0$. If $\mathfrak{x} \not\subseteq \mathfrak{R}_0$, then by Lemmas 9.27 and 9.24, we see that \mathfrak{x} is conjugate to \mathfrak{R}_2^* in \mathfrak{M} . Let \mathfrak{Y} be a subgroup of \mathfrak{R}_3 such that $\mathfrak{Q} = \mathfrak{R}_0 \cap C(\mathfrak{Y})$ is quaternion. Since J inverts \mathfrak{R}_3 , \mathfrak{Q} admits $\langle J \rangle \mathfrak{R}_3 / \mathfrak{Y}$ as a group of automorphisms. Hence, J inverts an element Q of \mathfrak{Q} of order 4. Then $JQJ = Q^{-1}$, that is, $Q^{-1}JQ = JI$, so $Q \in N_{\mathfrak{M}}(\mathfrak{R}_2^*)$. Thus, (9.25) holds.

Suppose that $\mathfrak{R}_2^* - \langle I \rangle$ contains a conjugate J of I . By (9.25), we can choose M in $\mathfrak{M} \cap N(\mathfrak{R}_2^*)$ such that $M^{-1}JM = JI$. By (9.25) again, this time applied to the group $C(J)$, we can choose M_0 in $C(J)$ with $M_0^{-1}IM_0 = IJ$. Thus, the lemma follows in this case.

We may now assume that

$$(9.26) \quad I \text{ is the only conjugate of } I \text{ in } \mathfrak{R}_2^* .$$

By a result of Glauberman [16], \mathfrak{R}_2 contains a conjugate T of I with $T \neq I$. If the width of \mathfrak{R}_0 exceeds 2, then by Lemma 9.28, $T \in \mathfrak{R}_0$, so by Lemma 9.24, (9.26) is violated. So suppose the width of \mathfrak{R}_0 is 2. In this case, \mathfrak{R}_0 has exactly 18 noncentral involutions and they are permuted transitively in \mathfrak{M} . Since T lies in no \mathfrak{M} -conjugate of \mathfrak{R}_2^* , Lemma 9.24 implies that $T \in \mathfrak{R}_0$. Thus, every involution of \mathfrak{R}_0 is conjugate to I in \mathfrak{G} . But by Lemma 5.38, every involution of \mathfrak{G} is conjugate to an element of \mathfrak{R}_0 . The proof is complete.

LEMMA 9.33. *There is a S_3 -subgroup of \mathfrak{G} which contains \mathfrak{R}_3 and is normalized by \mathfrak{R}_2^* .*

Proof. Let $\tilde{\mathfrak{F}}$ be a maximal element of $N(\mathfrak{R}_2^*; 3)$ which contains \mathfrak{R}_3 . Suppose by way of contradiction that $|\tilde{\mathfrak{F}}| < |\mathfrak{G}|_3$. Let \mathfrak{C} be a $S_{2,3}$ -subgroup of $N(\tilde{\mathfrak{F}})$ which contains \mathfrak{R}_2^* . Let \mathfrak{C}_p be a S_p -subgroup of \mathfrak{C} , $p = 2, 3$, with $\mathfrak{R}_2^* \subseteq \mathfrak{C}_2$.

Suppose $O_2(\mathfrak{C}) \neq 1$. Then since $\mathfrak{R}_3 \in \mathcal{S}$, we get $Z(O_2(\mathfrak{C})) \sim \langle I \rangle$, so \mathfrak{C} is in a conjugate of \mathfrak{M} . This is not the case, by Lemma 9.21(a). Clearly, the maximality of $\tilde{\mathfrak{F}}$ forces $\tilde{\mathfrak{F}} = O_3(\mathfrak{C})$. Since $O_2(\mathfrak{C}) = 1$, the proof of Lemma 9.17 implies that \mathfrak{C} has no noncyclic abelian subgroup of order 8. Thus, $\bar{\mathfrak{C}} = \mathfrak{C}/O_3(\mathfrak{C})$ is a 2, 3-group of order divisible by 3

such that

- (a) $O_3(\bar{\mathbb{C}}) = 1$.
- (b) $\bar{\mathbb{C}}$ contains a four-group.
- (c) $\bar{\mathbb{C}}$ contains no noncyclic abelian subgroup of order 8.

It is routine to verify that $\bar{\mathbb{C}} \cong GL(2, 3)$ or $\bar{\mathbb{C}} \cong \Sigma_4$ or $\bar{\mathbb{C}} \cong A_4$. If $GL(2, 3) \cong \bar{\mathbb{C}}$, then every four-subgroup of $\bar{\mathbb{C}}$ normalizes a S_3 -subgroup of $\bar{\mathbb{C}}$, against the maximality of $\bar{\mathfrak{F}}$. Hence,

$$\mathbb{C}/O_3(\mathbb{C}) \cong \Sigma_4 \text{ or } A_4.$$

Let $\mathfrak{R}_2^* = \langle I, J \rangle$.

Case 1. J does not invert \mathfrak{R}_3 . Let $\mathfrak{X} = C(J) \cap \mathfrak{R}_3$, so that $\mathfrak{X} = C(\mathfrak{R}_2^*) \cap \mathfrak{R}_3 = O_3(\mathbb{C}) \cap C(\mathfrak{R}_2^*)$ is of order 3. By a formula of Wielandt [40], together with Lemma 9.30, we get $|O_3(\mathbb{C})| \leq 3^4$. Since $O_3(\mathbb{C}) = F(\mathbb{C})$, it follows from (B) that $m(O_3(\mathbb{C})) \geq 3$. Hence, $O_3(\mathbb{C})$ is elementary of order 3^3 or 3^4 . If $|O_3(\mathbb{C})| = 3^3$, then $O_3(\mathbb{C})$ char \mathbb{C}_3 , and so \mathbb{C}_3 is a S_3 -subgroup of \mathbb{G} , against Lemma 9.31. Hence, $O_3(\mathbb{C})$ is elementary of order 3^4 . This implies that $O_3(\mathbb{C})$ char \mathbb{C}_3 . Hence, \mathbb{C}_3 is a S_3 -subgroup of \mathbb{G} . This is not the case, since $Z(\mathbb{C}_3)$ is noncyclic.

Case 2. J inverts \mathfrak{R}_3 and $\bar{\mathbb{C}} \cong \Sigma_4$.

Let $\mathfrak{B} = \mathbb{C}_2 \cap O_{3,2}(\mathbb{C})$. Thus, \mathfrak{B} is a four-group. Suppose $\mathfrak{B} = \mathfrak{R}_2^*$. Let $\mathfrak{S} = \mathfrak{M} \cap \mathbb{C}$. Then $|\mathfrak{S}| = 8.9$, and $\mathfrak{R}_3 \triangle \mathfrak{S}$. This is not the case, since \mathfrak{R}_2^* is a four-group, by Lemma 9.27.

Since $\mathfrak{B} \neq \mathfrak{R}_2^*$, it follows that \mathfrak{B} and \mathfrak{R}_2^* are the four-subgroups of \mathbb{C}_2 . By Lemma 9.32, $A_{\mathbb{G}}(\mathfrak{R}_2^*) = \text{Aut}(\mathfrak{R}_2^*)$. Thus, $\mathfrak{B} \cap \mathfrak{R}_2^* = \langle V \rangle$ with $V \sim I$. Hence, all involutions of \mathfrak{B} are conjugate to I in \mathbb{G} .

Choose V in \mathfrak{B}^* . Suppose $|C(V) \cap O_3(\mathbb{C})| > 3$. Then $C(V) \cap O_3(\mathbb{C})$ is a S_3 -subgroup of $C(V)$, by Lemma 9.29, together with $V \sim I$. Hence, $|C_{\mathbb{G}}(V)| = 8.9$, and $C_{\mathbb{G}}(V)$ is 3-closed. This violates Lemma 9.27 applied to $C(V)$. Hence, $|O_3(\mathbb{C}) \cap (V)| \leq 3$.

Since $N(\mathfrak{B}) \cap \mathbb{C}$ permutes transitively the involutions of \mathfrak{B} , we get $|C(V) \cap O_3(\mathbb{C})| = 3$ for all V in \mathfrak{B}^* . Hence, $|O_3(\mathfrak{B})| = 27$ and $O_3(\mathbb{C}) = J(\mathbb{C}_3)$ char \mathbb{C}_3 . But then $|\mathbb{C}|_3 = |\mathbb{G}|_3$, against Lemma 9.31.

Case 3. J inverts \mathfrak{R}_3 and $\bar{\mathbb{C}} \cong A_4$.

Let $\mathbb{C}_0 = O_3(\mathbb{C})$. Then $\mathfrak{R}_3 = C_{\mathbb{C}_0}(I)$ is elementary of order 3^2 and inverts by each element of $\mathfrak{R}_2^* - \langle I \rangle$. Since the involutions of \mathfrak{R}_2^* are fused in \mathbb{C} , we conclude

- (a) for each $K \in (\mathfrak{R}_2^*)^*$, the group $C_{\mathbb{C}_0}(K)$ is elementary of order 3^2 and is inverted by each element of $\mathfrak{R}_2^* - \langle K \rangle$.

It follows that

- (b) \mathbb{C}_0 contains two chief factors of \mathbb{C} , each of order 3^3 .

Suppose that \mathfrak{C}_0 is abelian. By (a) and (b), it is elementary of order 3^6 and each element of $\mathfrak{C}_3 - \mathfrak{C}_0$ has minimal polynomial $(x - 1)^3$ on \mathfrak{C}_0 . Hence, $\mathfrak{C}_0 \text{ char } \mathfrak{C}_3$. So \mathfrak{C}_3 is a S_3 -subgroup of \mathfrak{G} . But $Z(\mathfrak{C}_3) = [\mathfrak{C}_0, \mathfrak{C}_3, \mathfrak{C}_3]$ is not cyclic, against hypothesis (iii) of Theorem 9.1. Therefore, \mathfrak{C}_0 is not abelian. So (a) and (b) imply:

- (c1) \mathfrak{C}_0 is special of order 3^6 and exponent 3,
- (c2) $D(\mathfrak{C}_0) = Z(\mathfrak{C}_0)$ is a chief factor of \mathfrak{C} of order 3^3 ,
- (c3) $\mathfrak{C}_0/D(\mathfrak{C}_0)$ is a chief factor of \mathfrak{C} of order 3^3 ,
- (c4) every element of $\mathfrak{C}_3 - \mathfrak{C}_0$ has minimal polynomial $(x - 1)^3$ on both $D(\mathfrak{C}_0)$ and $\mathfrak{C}_0/D(\mathfrak{C}_0)$,
- (c5) if $P \in \mathfrak{C}_3 - \mathfrak{C}_0$, then $|C_{\mathfrak{C}_3}(P)| \leq 3^3$.

This implies

- (d) $\mathfrak{C}_0 \text{ char } \mathfrak{C}_3$.

Indeed, if \mathfrak{C}_1 is any subgroup of index 3 in \mathfrak{C}_3 different from \mathfrak{C}_0 , then $\mathfrak{C}_1 \cap \mathfrak{C}_0 \cong D(\mathfrak{C}_0)$. Hence, (c4) implies that the exponent of \mathfrak{C}_1 is 9. This proves (d), and gives

- (e) \mathfrak{C}_3 is a S_3 -subgroup of \mathfrak{G} .

Now let \mathfrak{X}_0 be a subgroup of \mathfrak{R}_3 of order 3 such that $C_{\mathfrak{R}_0}(\mathfrak{X}_0) = \mathfrak{Q} \supset \langle I \rangle$. Let $\mathfrak{C}_3 = A_0 \times A_1$. Thus, \mathfrak{Q} is a quaternion group and $\mathfrak{Q}\mathfrak{X}_1 \langle J \rangle \cong GL(2, 3)$. Let \mathfrak{Y} be a $S_{2,3}$ -subgroup of $C_{\mathfrak{G}}^*(\mathfrak{X}_0)$ with $\mathfrak{Q}\mathfrak{R}_3 \langle J \rangle \subseteq L$. Let $\mathfrak{Y}_0 = O_3(\mathfrak{Y})$. Since $D(\mathfrak{C}_0)\mathfrak{R}_3$ is elementary of order 3^4 and contains an element of $\mathcal{Z}(3)$, it follows that $O_3(\mathfrak{Y}) = 1$. Since \mathfrak{Y} contains no noncyclic abelian subgroup of order 8, we get that $\mathfrak{Q}\mathfrak{X}_1 \langle J \rangle$ is a complement to \mathfrak{Y}_0 in \mathfrak{Y} . Since I inverts $\mathfrak{Y}_0/\mathfrak{X}_0$, it follows that $|\mathfrak{Y}_0: \mathfrak{X}_0| = 3^{2d}$ for some integer $d \geq 1$. If $d = 1$, then $D(\mathfrak{C}_0)\mathfrak{R}_3$ is a S_3 -subgroup of $C(\mathfrak{X}_0)$ and so S_3 -subgroups of \mathfrak{Y} are abelian. This is absurd, so $d \geq 2$. Since $|\mathfrak{G}|_3 = 3^7$ by (e), and since $|\mathfrak{Y}|_3 = 3^{2d+2}$, we get $d = 2$.

Let \mathfrak{Y}_3 be a S_3 -subgroup of \mathfrak{Y} containing \mathfrak{R}_3 . Since \mathfrak{Y}_3 is not a S_3 -subgroup of \mathfrak{G} , and since $\mathfrak{X}_0 \triangleleft \mathfrak{Y}$, it follows that $Z(\mathfrak{Y}_3)$ is noncyclic. In particular, $Z(\mathfrak{Y}_0) \cong Z(\mathfrak{Y}_3)$, so that $Z(\mathfrak{Y}_0)$ is not cyclic. Hence, $|Z(\mathfrak{Y}_0)| \geq 3^3$. This implies that if $L \in \mathfrak{Y}_0$, then $|C_{\mathfrak{Y}_0}(L)| \geq 3^4$. Choose G in \mathfrak{G} so that $\mathfrak{Y}_3 \subseteq \mathfrak{C}_3^G$, which is possible by (e). By (c5), we get $\mathfrak{Y}_0 \subseteq \mathfrak{C}_0^G$. Hence, $\mathfrak{Y}_0 = \mathfrak{Y}_{00} \times \mathfrak{Y}_{01}$, where $\mathfrak{Y}_{00}, \mathfrak{Y}_{01}$ admit \mathfrak{Q} , \mathfrak{Y}_{00} is nonabelian of exponent 3 and order 3^3 and \mathfrak{Y}_{01} is elementary of order 3^2 . Now $\mathfrak{X}_1 \subseteq \mathfrak{Y}_3$ and $|C_{\mathfrak{Y}_3}(\mathfrak{X}_1)| \geq 3^4$, so we get that $\mathfrak{X}_1 \subseteq \mathfrak{C}_0^G$. Hence, $\mathfrak{Y}_3 = \mathfrak{Y}_0\mathfrak{X}_1 \subseteq \mathfrak{C}_0^G$, and so $\mathfrak{Y}_3 = \mathfrak{C}_0^G$. This is impossible, since $|Z(\mathfrak{Y}_3)| = 3^2, |Z(\mathfrak{C}_0)| = 3^3$.

LEMMA 9.34. *Each involution of $\mathfrak{R}_2^* - \langle I \rangle$ inverts \mathfrak{R}_3 .*

Proof. Let \mathfrak{P}^* be a S_2 -subgroup of \mathfrak{G} which contains \mathfrak{R}_3 and is normalized by \mathfrak{R}_2^* , set $\mathfrak{X} = \mathfrak{R}_3 \cap C(\mathfrak{R}_2^*)$. Suppose $\mathfrak{X} \neq 1$. Then $|\mathfrak{X}| = 3$, so by a formula of Wielandt [44], $|\mathfrak{P}^*| \leq 3^4$. This contradicts Lemma 9.31. Hence, $\mathfrak{X} = 1$. As $\mathfrak{R}_2^* = \langle I, J \rangle$ for some involution J , the proof is complete.

LEMMA 9.35. (a) *If \mathfrak{X} is a subgroup of \mathfrak{R}_3 of order 3 and $C(\mathfrak{X}) \cap \mathfrak{R}_3$ is quaternion, then $|C(\mathfrak{X})|_3 = 3^4$.*

(b) $|C(\mathfrak{R})|_3 = 3^3$.

Proof. (a) Set $\mathfrak{Q} = C(\mathfrak{X}) \cap \mathfrak{R}_3$, and let \mathfrak{Y} be a subgroup of \mathfrak{R}_3 of order 3 distinct from \mathfrak{X} . Let J be an involution of $\mathfrak{R}_2^* - \langle I \rangle$. Thus, J inverts \mathfrak{R}_3 by Lemma 9.34. Also, $\langle J \rangle \mathfrak{Y} \mathfrak{Q} \cong GL(2, 3)$.

Let \mathfrak{C} be a $S_{2,3}$ -subgroup of $N(\mathfrak{X})$ which contains $\mathfrak{R}_3 \mathfrak{Q} \mathfrak{R}_2^*$. Thus, $\mathfrak{Q} \langle J \rangle$ is a S_2 -subgroup of \mathfrak{C} and $O_2(\mathfrak{C}) = 1$. Since

$$[O_3(\mathfrak{C}) \cap \mathfrak{R}_3, \mathfrak{Q}] \subseteq O_3(\mathfrak{C}) \cap \mathfrak{Q} = 1,$$

it follows that $O_3(\mathfrak{C}) \cap \mathfrak{R}_3 = \mathfrak{X}$. Hence, I inverts $O_3(\mathfrak{C})/\mathfrak{X}$. Hence, $O_3(\mathfrak{C})/\mathfrak{X}$ is the direct sum of a certain number, say k , of modules each isomorphic to the faithful irreducible $F_3 \mathfrak{Q}$ -module, so that $|O_3(\mathfrak{C}) : \mathfrak{X}| = 3^{2k}$. Hence, $|\mathfrak{C}|_3 = |C(\mathfrak{R})|_3 = 3^{2(k+1)}$. Suppose $k \geq 2$. Then by Lemma 9.33, we get $|\mathfrak{C}|_3 = |\mathfrak{G}|_3 = 3^6$.

We argue that $Z(O_3(\mathfrak{C})) = \mathfrak{X}$. Suppose false. We get $Z(O_3(\mathfrak{C})) = (Z(O_3(\mathfrak{C})) \cap C(I)) \times [Z(O_3(\mathfrak{C})), I]$. Since $\mathfrak{R}_3 \cap O_3(\mathfrak{C}) = \mathfrak{X}$, we get

$$Z(O_3(\mathfrak{C})) \cap C(I) = \mathfrak{X};$$

also $[Z(O_3(\mathfrak{C})), I]$ is normalized by \mathfrak{Y} , so if $[Z(O_3(\mathfrak{C})), I] \neq 1$, then a S_3 -subgroup of \mathfrak{G} has a noncyclic center. We conclude that $\mathfrak{X} = Z(O_3(\mathfrak{C}))$. This implies that $O_3(\mathfrak{C})$ is extra special of width 2. Since $\mathfrak{Q} \langle J \rangle$ is a S_2 -subgroup of $N(\mathfrak{X})$, it follows that $O_3(\mathfrak{C}) = O_3(N(\mathfrak{X}))$. Thus, Hypothesis 9.2 is satisfied. Since this is not the case, we get $k = 1$. Thus (a) holds.

By Lemma 9.20, we have $|C(\mathfrak{R}_3)|_3 \geq 27$. Since \mathfrak{R}_3 is not central in a S_3 -subgroup of \mathfrak{C} , (b) follows.

LEMMA 9.36. *Let \mathfrak{B} be a S_3 -subgroup of \mathfrak{G} . Then*

(a) $|\mathfrak{B}| = 3^5$.

(b) $\mathfrak{B}/Z(\mathfrak{B})$ is of maximal class and order 3^4 .

Proof. By Lemma 9.33, there is a conjugate \mathfrak{B} of \mathfrak{R}_2^* which normalizes \mathfrak{B} . By Lemma 9.32, all involutions of \mathfrak{B} are conjugate to I . Let V_1, V_2, V_3 be the involutions of \mathfrak{B} . By Lemma 9.34, $C_{\mathfrak{B}}(\mathfrak{B}) = 1$. By Lemma 9.29, $|C_{\mathfrak{B}}(V_i)| \leq 9$ for $i = 1, 2, 3$. Then by Wielandt [44], $|\mathfrak{B}| \leq 3^6$.

Set $\mathfrak{Z} = Z(\mathfrak{B})$. Since \mathfrak{Z} is cyclic, $C(\mathfrak{Z}) \cap \mathfrak{B} \neq 1$. We may assume notation is chosen so that V_1 is a generator for $C(\mathfrak{Z}) \cap \mathfrak{B}$. Thus, $|\mathfrak{Z}| = 3$. Suppose V_1 inverts $\mathfrak{B}/\mathfrak{Z}$. Then, $|\mathfrak{B}| \leq 3^5$, so by Lemma 9.31, $|\mathfrak{B}| = 3^5$. In this case, since \mathfrak{B} is generated by elements of order 3, we get that $\mathfrak{Z} = \mathfrak{B}' = D(\mathfrak{B})$. Since $O_3(N(\mathfrak{Z})) = 1$, so also $O_3(\mathfrak{Z})/\mathfrak{Z} = 1$.

Hence, $\mathfrak{P} \triangleleft N(\mathfrak{Z})$. Thus, Hypothesis 9.2 is satisfied. Since this is not the case, we conclude that V_1 does not invert $\mathfrak{P}/\mathfrak{Z}$.

Let \mathfrak{U} be a subgroup of $C_{\mathfrak{P}}(V_1)$ of order 3 distinct from \mathfrak{Z} . Thus, $C_{\mathfrak{P}}(V_1) = \mathfrak{Z}\mathfrak{U}$.

By Lemma 9.35(b), we get $|C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})| \leq 27$. Since $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$, we have $|C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})| = 27$. Again, since $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$, it follows that $|N_{\mathfrak{P}/\mathfrak{Z}}(\mathfrak{Z}\mathfrak{U}/\mathfrak{Z})| = 9$. Thus, $\mathfrak{P}/\mathfrak{Z}$ is of maximal class, and $\mathfrak{U}\mathfrak{Z}/\mathfrak{Z} \not\subseteq (\mathfrak{P}/\mathfrak{Z})'$. Since $\mathfrak{U}\mathfrak{Z}/\mathfrak{Z}$ is the set of fixed points of V_1 on $\mathfrak{P}/\mathfrak{Z}$, it follows that $\mathfrak{P}/\mathfrak{Z}$ has a subgroup $\mathfrak{P}_0/\mathfrak{Z}$ of index 3 which is inverted by \mathfrak{V}_1 . Since $\mathfrak{P}_0/\mathfrak{Z}$ is generated by elements of order 3, $\mathfrak{P}_0/\mathfrak{Z}$ is elementary. If $|\mathfrak{P}_0/\mathfrak{Z}| \geq 3^4$, then $\mathfrak{P}/\mathfrak{Z}$ is not of maximal class. Hence, $|\mathfrak{P}_0/\mathfrak{Z}| \leq 27$, so by Lemma 9.31, we have $|\mathfrak{P}_0:\mathfrak{Z}| = 27$. This establishes both (a) and (b).

We may now complete the proof of Theorem 9.1. Let $\mathfrak{P}, \mathfrak{P}_0, \mathfrak{Z}, \mathfrak{U}, \mathfrak{V}, V_1$ be as above. Thus, $|\mathfrak{P}_0| = 3^4$, $\mathfrak{P}_0/\mathfrak{Z}$ is elementary of order 27 and is inverted by V_1 . Being generated by elements of order 3, \mathfrak{P}_0 is of exponent 3. It follows that $Z(\mathfrak{P}_0)$ is not cyclic. Hence, we can choose a subgroup \mathfrak{W} of $Z(\mathfrak{P}_0)$ of order 9 which is normal in $\mathfrak{P}\mathfrak{W}$. Set $\mathfrak{Y} = \mathfrak{W}\mathfrak{U}$. Thus, \mathfrak{Y} is of order 27 and \mathfrak{Y} admits V_1 . Thus, $\mathfrak{Z}\mathfrak{U} \trianglelefteq \mathfrak{Y}$, so \mathfrak{Y} is abelian, since $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$. This implies that $\mathfrak{W} \subset Z(\mathfrak{P})$, since $\mathfrak{P} = \mathfrak{P}_0\mathfrak{U}$. This contradiction completes the proof of Theorem 9.1.

Theorems 8.1 and 9.1 provide a proof of Theorem ES.

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