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THEOREMS ON CESÀRO SUMMABILITY OF SERIES

S. MUKHOTI

1.1. We consider the Cesàro summability, for integral orders, of the series

(1.1) $$\sum_{\nu=0}^{\infty} a_\nu d_\nu .$$

In this paper we establish equivalence theorems for the series (1.1) which are valid for a substantial class of sequences d_ν including $e^{-\nu}$ and $\nu^{-\delta}$. Results of this character, but not overlapping with those in this paper, were given by Hardy and Littlewood and by Andersen. Andersen's result was extended by Bosanquet and Chow, and further extended by Bosanquet.

Notation. 1.2. We write $A_n^0 = A_n = a_0 + a_1 + \cdots + a_n$,

$$A_n^k = A_0^{k-1} + A_1^{k-1} + \cdots + A_n^{k-1}$$

and we get the identities: See Hardy [8].

(1.2) $$A_n^k = \sum_{\nu=0}^{n} B_{n-\nu}^{k-1} A_\nu ,$$

(1.3) $$A_n^k = \sum_{\nu=0}^{n} B_{n-\nu}^{k} a_\nu ,$$

where

(1.4) $$B_{n-\nu}^{k} = \binom{n-\nu+k}{k} ;$$

$E_n^k = A_n^k$ when $a_0 = 1$, $a_n = 0$, for $n > 0$, i.e., when $A_n = 1$, for all n. Hence

(1.5) $$E_n^k = \binom{n+k}{k} \sim \frac{n^k}{k!} .$$

If

(1.6) $$\frac{A_n^k}{E_n^k} \to A, \text{ when } n \to \infty,$$

or equivalently if

(1.7) $$\frac{k! A_n^k}{n^k} \to A, \text{ when } n \to \infty,$$

then we say that $\sum_{n=0}^{\infty} a_n$ is summable (C, k) to sum A and we write

$$\sum_{n=0}^{\infty} a_n = A(C, k) . \tag{1.8}$$

1.3. *Statement of lemma and identiy.* We write

$$\Delta d_n = d_n - d_{n-1},\ \Delta^k u_n = \Delta\Delta^{k-1} u_n \qquad (k \geqq 2)$$

and $\Delta^0 u_n = u_n$.

We shall use the following well-known identity:

$$\Delta^k(u_n v_n) = \sum_{\nu=0}^{k} \binom{k}{\nu} \Delta^\nu u_n \Delta^{k-\nu} v_{n-\nu} . \tag{1.9}$$

LEMMA A. *In order that*

$$t_m = \sum C_{m,n} S_n \to S\ (m \to \infty) , \qquad (m = 0, 1, 2, \cdots) \tag{1.10}$$

whenever

$$S_n \to S \qquad (n \to \infty) , \tag{1.11}$$

it is necessary and sufficient that

(1.12) (i) $$\sum |C_{m,n}| < H ,$$

where H is independent of m;

(1.13) (ii) $$C_{m,n} \to 0 ,$$

for each n, when $m \to \infty$;

(1.14) (iii) $$\sum C_{m,n} \to 1,\ \text{when } m \to \infty .$$

Lemma A is mentioned by Hardy [8, Th. 2], which is due to Toeplitz [12]. Toeplitz considers only *triangular* transformations, in which $C_{m,n} = 0$ for $n > m$. The extension to general transformations was made by Steinhaus [11].

2. Statement and proof of the theorem.

THEOREM (*the cases* $k = 1, 2, \cdots$). *Suppose that* $d_n >$ *for* $n \geqq 0$, *and*

(2.1) (i) $$d_{n+1}^k = o(n^k) \text{ as } n \to \infty ,$$

(2.2) (ii) $$(1/B_n^k) \sum_{m=0}^{n} B_m^k \left| \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \right\} \right| = O(1) ,$$

(Δ *operating on* m) .

Then necessary and sufficient conditions for

(2.3) (I) $\sum_{\nu=0}^{\infty} a_\nu d_\nu$ *to be summable* (C, k) *to* S

are that

(2.4) (II) $-\sum_{\nu=0}^{\infty} S_\nu \Delta d_{\nu+1}$ *should be summable* (C, k) *to* S

and

(2.5) (III) $S_n d_{n+1} = o(1)\,(C, k)$ *as* $n \to \infty$,

where

$$S_n = \sum_{\nu=0}^{n} a_\nu . \tag{2.6}$$

Proof. We have

$$\sum_{\nu=0}^{n} a_\nu d_\nu = S_n d_{n+1} - \sum_{\nu=0}^{n} S_\nu \Delta d_{\nu+1} \tag{2.7}$$

i.e.,

$$C_n = F_n - G_n ,$$

and hence

$$C_n^k = F_n^k - G_n^k . \tag{2.8}$$

The *sufficiency* follows immediately from (2.8).
Necessity. We are given that

$$C_n^k / B_n^k \to S \text{ as } n \to \infty , \tag{2.9}$$

and it will be enough to prove that

$$-G_n^k / B_n^k \to S \text{ as } n \to \infty . \tag{2.10}$$

From (2.7) we have

$$\begin{aligned} \frac{C_n \Delta d_{n+1}}{d_{n+1}} &= S_n \Delta d_{n+1} - \frac{\Delta d_{n+1}}{d_{n+1}} G_n , \\ &= \frac{d_n (d_{n+1} \Delta G_n - G_n \Delta d_{n+1})}{d_n d_{n+1}} = d_n \Delta\left(\frac{G_n}{d_{n+1}}\right) . \end{aligned} \tag{2.11}$$

Thus

$$\frac{G_n}{d_{n+1}} = \sum_{\nu=0}^{n} \frac{C_\nu}{d_\nu} \frac{\Delta d_{\nu+1}}{d_{\nu+1}} , \tag{2.12}$$

so

$$-G_n = d_{n+1}\sum_{\nu=0}^{n} C_\nu \Delta(1/d_{\nu+1}) ,$$

and hence

$$\begin{aligned} -G_n^k &= \sum_{\nu=0}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \sum_{m=0}^{\nu} C_m \Delta(1/d_{m+1}) \\ &= \sum_{m=0}^{n} C_m \Delta(1/d_{m+1}) \sum_{\nu=m}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \\ &= \sum_{m=0}^{n} (-1)^k C_m^k \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \right\} . \end{aligned} \tag{2.13}$$

It follows that

$$\begin{aligned} -\frac{G_n^k}{B_n^k} &= \frac{1}{B_n^k} \sum_{m=0}^{n} (-1)^k \frac{C_m^k}{B_m^k} \cdot B_m^k \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \right\} \\ &= \sum_{m=0}^{n} T_m \gamma_{n,m} , \end{aligned} \tag{2.14}$$

where

$$T_m = C_m^k / B_m^k , \tag{2.15}$$

and

$$\gamma_{n,m} = (-1)^k \frac{B_m^k}{B_n^k} \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \right\} . \tag{2.16}$$

Hence

$$\text{(i)} \quad \sum_{m=0}^{n} |\gamma_{n,m}| = (1/B_n^k) \sum_{m=0}^{n} B_m^k \left| \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \right\} \right| < H , \tag{2.17}$$

by hypothesis (ii).

Now, from (2.16), we have, for each m

$$\begin{aligned} \gamma_{n,m} = (-1)^k (B_m^k/B_n^k) \Big[& \Delta^{k+1}(1/d_{m+k+1}) \sum_{\nu=m+k}^{n} B_{n-\nu}^{k-1} d_{\nu+1} \\ & + \alpha_k \Delta^k (1/d_{m+k}) \binom{n-m}{k-1} d_{m+k} + \cdots \\ & + \alpha_1^1 \Delta(1/d_{m+1}) \binom{n-m}{k-1} \Delta^{k-1} d_{m+k} \\ & + \cdots + \alpha_{k-1}^1 \Delta(1/d_{m+1}) \Delta^{k-1} \binom{n-m}{k-1} d_{m+1} \Big] , \end{aligned} \tag{2.18}$$

(α various constants)

using the identity (1.9).

Then from (2.18) it follows that for each m

$$(2.19)\qquad \gamma_{n,m} = A_{n,m} + O\Big(\frac{1}{n}\Big) = A_{n,m} + o(1), \text{ as } n \to \infty ,$$

where

$$(2.20)\quad |A_{n,m}| < \frac{k!}{n^k} B_m^k\, |\Delta^{k+1}(1/d_{m+k+1})| \sum_{\nu=0}^{n} B_{n-\nu}^{k-1} d_{\nu+1} < \frac{K}{n^k} d_{n+1}^k = o(1)$$

for each m, as $n \to \infty$, by hypothesis (i).

Hence it follows from (2.19)-(2.20) that

$$(2.21)\quad \text{(ii)}\qquad \gamma_{n,m} \to 0 \text{ for each } m, \text{ as } n \to \infty .$$

Let us take

$$a_0 = 1,\ a_\nu = 0, \text{ for } \nu \geqq 1, \text{ and } d_0 = 1$$

in

$$(2.22)\qquad C_n = \sum_{\nu=0}^{n} a_\nu d_\nu .$$

Then we have, for $n \geqq 0$, $C_n = 1$, and hence

$$(2.23)\qquad C_n^k / B_n^k = 1 .$$

Next, since $C_\nu = 1$, $d_0 = 1$, we obtain from (2.12)

$$-G_n = d_{n+1} \sum_{\nu=0}^{n} \Delta(1/d_{\nu+1}) = 1 - d_{n+1} ,$$

and hence

$$(2.24)\qquad -G_n^k/B_n^k = 1 - d_{n+1}^k/B_n^k \to 1 \text{ as } n \to \infty ,$$

by hypothesis (i).

But this implies, from (2.14)-(2.15), that

$$(2.25)\quad \text{(iii)}\qquad -G_n^k/B_n^k = \sum_{m=0}^{n} \gamma_{n,m} \to 1 \text{ as } n \to \infty .$$

It follows that conditions (i), (ii) and (iii) of Lemma A are satisfied, and hence

$$(2.26)\qquad -G_n^k/B_n^k \to S \text{ as } n \to \infty .$$

Note. Hypotheses (i) *and* (ii) *of the Theorem are necessary.* For suppose that $-G_n^k/B_n^k \to S$ as $n \to \infty$, whenever $C_n^k/B_n^k \to S$ as $n \to \infty$.

Then from (2.14)-(2.16), condition (i) of Lemma A must hold, but this implies (2.17) and hence hypothesis (ii) of Theorem 2.

Next, let us choose C_n so that (2.23) holds. Then (2.3) holds, with $S = 1$, and hence (2.24) holds. Hence it follows that $d_{n+1}^k/B_n^k =$

$o(1)$ as $n \to \infty$, and this implies hypothesis (i) of the theorem.

Further the summability (C, k) of (2.4) can be improved to the summability $(C, k-1)$, by the following Lemma.

LEMMA B. *If d_n is monotonically decreasing and*

$$\text{(2.27)} \quad \text{(i)} \qquad n^j \Delta^j d_{n+1} = O(d_{n+1}) \, ,$$

$$\text{(2.28)} \quad \text{(ii)} \qquad n^j \Delta^j t_{n+1} = O(t_{n+1}) \, ,$$

for $j = 1, 2, \cdots, k+1$, where

$$\text{(2.29)} \qquad t_n = 1/d_n \, ,$$

then

$$\text{(2.30)} \quad \text{(iii)} \quad S_n d_{n+1} = o(1)\,(C, k) \Rightarrow n S_n \Delta d_{n+1} = o(1)\,(C, k) \, .$$

Proof. We have

$$\text{(2.31)} \qquad H_n = S_n d_{n+1} = o(1)\,(C, k) \, .$$

We will prove that

$$\text{(2.32)} \qquad H_n g_n = o(1)\,(C, k) \, ,$$

where

$$\text{(2.33)} \qquad g_n = \frac{n \Delta d_{n+1}}{d_{n+1}} = n \Delta d_{n+1} t_{n+1} \, .$$

By a theorem of Bosanquet [6, Th. 1], which is an extension of another theorem of Bosanquet [4, Lemma 1], it will be enough to prove that

$$\text{(2.34)} \qquad \Delta^j g_n = O(n^{-j}) \, , \qquad j = 0, 1, \cdots k-1 \, ,$$

and

$$\text{(2.35)} \qquad \sum_{\nu=1}^{n} \nu^k \, |\Delta^k g_\nu| = O(n) \, .$$

Now

$$\text{(2.36)} \qquad g_n = n \Delta d_{n+1} t_{n+1} \leqq K d_{n+1} t_{n+1} \leqq K \, ,$$

by (2.44).

Next, using the identity (1.9), we have

$$\text{(2.37)} \quad \begin{aligned} |\Delta^{k-1} g_n| &= \alpha_1 n \, |\Delta d_{n+1}| \, |\Delta^{k-1} t_{n+1}| + \cdots + \alpha_{k-1} n \, |\Delta^k d_{n+1}| t_{n+1-k+1} \\ &\quad + \alpha^1 \, |\Delta d_{n+1}| \, |\Delta^{k-2} t_{n+1}| + \cdots + \alpha^{k-2} \, |\Delta^{k-1} d_{n+1}| \, t_{n-1-k+2} \\ &\leqq K/n^{k-1} \, . \qquad (\alpha \text{ various constant}) \end{aligned}$$

The other conditions in (2.34) are easily obtained similarly, but it is well known that the inequalities for $j = 0$ and $k - 1$ imply those for $j = 1, 2, \cdots, k - 2$: See Hardy and Littlewood [9].

Next we have

$$\begin{aligned} |\Delta^k g_n| &\leqq \alpha_1 n\,|\Delta d_{n+1}|\,|\Delta^k t_{n+1}| + \cdots + \alpha_k n\,|\Delta^{k+1} d_{n+1}|\,t_{n+1-k} \\ &\quad + \alpha^1 |\Delta d_{n+1}|\,|\Delta^{k-1} t_{n+1}| + \cdots + \alpha^{k-1} |\Delta^k d_{n+1}|\,t_{n+1-k+1} \\ &\leqq K/n^k\,. \end{aligned} \tag{2.38}$$

Hence

$$\sum_{\nu=1}^{n} \nu^k\,|\Delta^k g_\nu| \leqq \sum_{\nu=1}^{n} \nu^k \frac{K}{\nu^k} < Kn\,, \tag{2.39}$$

and this completes the proof the lemma.

Next we will consider the case $k = 0$.

Now we have

$$nS_n \Delta d_{n+1} \leqq K d_{n+1} S_n$$

by (2.27), and since

$$S_n d_{n+1} = o(1) \text{ as } n \to \infty\,, \tag{2.40}$$

it follows that

$$nS_n \Delta d_{n+1} = o(1) \text{ as } n \to \infty\,. \tag{2.41}$$

Next since

$$\sum_{\nu=0}^{\infty} S_\nu \Delta d_{\nu+1} \tag{2.42}$$

is convergent, it follows from the definition that (2.42) is summable $(C,-1)$.

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INDIAN INSTITUTE OF TECHNOLOGY
NEW DELHI