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ON EXTENDING ALMOST PERIODIC FUNCTIONS

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Deleeuw and Glicksberg have shown that every weakly almost periodic function on a subgroup (respectively, a dense subgroup) of a locally compact abelian topological group (respectively, an abelian topological group) extends to the whole group as a weakly almost periodic function. By way of a theorem about extending strongly almost periodic functions on subsemigroups of semitopological semigroups, we show that every almost periodic function on a subgroup (respectively, a dense subgroup) of a locally compact abelian topological group (respectively a topological group) extends to the whole group as an almost periodic function. Furthermore, for the algebra of weakly almost periodic functions on a dense subsemigroup of a semitopological semigroup to consist of restrictions of weakly almost periodic functions on the larger semigroup, we need only require that each weakly almost periodic function on the subsemigroup extend continuously. Analogous statements hold for almost periodic and strongly almost periodic functions on dense subsemigroups of semitopological semigroups.

A *semitopological semigroup* is a semigroup S endowed with a Hausdorff topology such that the functions

$$s \to st\colon\ S \to S$$

and

$$s \to ts\colon\ S \to S$$

are continuous. (If the function

$$(s, t) \to st\colon\ S \times S \to S$$

is continuous, then S is a *topological semigroup*.)

Let S be a semitopological semigroup with identity 1. Let $C(S)$ be the C^*-algebra of bounded continuous complex valued functions on S. For $s \in S$, define a continuous linear operator $R_s\colon C(S) \to C(S)$ by

$$R_s f(t) = f(ts),\ t \in S,\ f \in C(S)\ .$$

If the closure of $R_S f = \{R_s f\colon s \in S\}$ in $C(S)$ is compact, then f is *almost periodic*; if the weak closure of $R_S f$ in $C(S)$ is weakly compact, then f is *weakly almost periodic*. The almost periodic functions form a sub-C^*-algebra of $C(S)$, which we denote by $A(S)$. Likewise, the weakly almost periodic functions form a sub-C^*-algebra of $C(S)$, which we denote by $W(S)$. Let $U(S)$ be the sub-C^*-algebra of $C(S)$ generated by the coefficients of continuous finite dimensional unitary representations of S. Functions in $U(S)$ are called *strongly almost periodic*. For groups the strongly almost periodic and the almost periodic functions coincide.

Regarding $R_S = \{R_s : s \in S\}$ as a set of linear operators on $W(S)$ and taking its weak operator closure, we obtain a compact semitopological semigroup ΩS called the *weakly almost periodic compactification* of S. Similarly, the strong operator closure of R_S on $A(S)$ is a compact topological semigroup AS called the *almost periodic compactification* of S; and the strong operator closure of R_S on $U(S)$ is a compact topological group ΥS called the *strongly* almost periodic compactification of S.[1] Let

$$\omega_S : S \longrightarrow \Omega S \ ,$$
$$\alpha_S : S \longrightarrow AS \ ,$$

and

$$\upsilon_S : S \longrightarrow \Upsilon S$$

denote the embedding maps. There is an isomorphism of C^*-algebras $W(\omega_S): C(\Omega S) \rightarrow W(S)$ given by

$$W(\omega_S)F = F \circ \omega_S,\ F \in C(\Omega S) \ .$$

Isomorphisms $A(\alpha_S): C(AS) \rightarrow A(S)$ and $U(\upsilon_S): C(\Upsilon S) \rightarrow U(S)$ are defined similarly. If $\varphi: S \rightarrow T$ is a morphism of semitopological semigroups, then there is a morphism $\Omega\varphi: \Omega S \rightarrow \Omega T$ of compact semitopological semigroups so that the diagram

$$\begin{array}{ccc} \Omega S & \xrightarrow{\Omega\varphi} & \Omega T \\ {\scriptstyle \omega_S}\uparrow & & \uparrow{\scriptstyle \omega_T} \\ S & \xrightarrow{\varphi} & T \end{array}$$

commutes. The dual diagram

$$\begin{array}{ccc} C(\Omega S) & \xleftarrow{C(\Omega\varphi)} & C(\Omega T) \\ {\scriptstyle W(\omega_S)}\downarrow & & \downarrow{\scriptstyle W(\omega_T)} \\ W(S) & \xleftarrow{W(\varphi)} & W(T) \end{array}$$

is defined naturally. There are similar morphisms $A\varphi: AS \rightarrow AT$ and $\Upsilon\varphi: \Upsilon S \rightarrow \Upsilon T$.

[1] One can obtain the compactifications ΩS, AS, and ΥS by way of the coadjoint existence theorem (see [**1**]).

If S is a subsemigroup of T, and if every function in a dense subset of the weakly almost periodic functions on S extends as a weakly almost periodic function on T, then the inclusion $\iota: S \to T$ induces an isomorphism of semitopological semigroups between ΩS and the closure of $\omega_T(S)$ in ΩT (the isomorphism being the corestriction of $\Omega\iota$). However, if $\tau \in \Omega S$, then τ is an operator on $W(S)$, whereas $\tau' = (\Omega\iota)\tau$ is an operator on $W(T)$. Nevertheless, we may identify ΩS with $\omega_T(S)^-$ even as semigroups of operators. This and the analogous statements for the almost periodic and strongly almost periodic cases follow from the following lemma:

LEMMA 1. *Let T be a semitopological semigroup with identity* 1, *and let S be a subsemigroup of T containing* 1. *Suppose that $f \in W(S)$ and that $f^e \in W(T)$ is an extension of f. Then, for every $\tau \in \Omega S$, we have that*

$$\tau f = \tau' f^e | S ,$$

where $\tau' = (\Omega\iota)\tau$.

Similar statements hold for almost periodic and strongly almost periodic functions.

Proof. We prove the weakly almost periodic case. The proofs of the other cases are completely analogous.

Consider the commutative diagram

$$\begin{array}{ccc} \Omega S & \xrightarrow{\Omega\iota} & \Omega T \\ {\scriptstyle \omega_S}\uparrow & & \uparrow{\scriptstyle \omega_T} \\ S & \xrightarrow{\iota} & T \end{array}$$

and its dual

$$\begin{array}{ccc} C(\Omega S) & \xleftarrow{C(\Omega\iota)} & C(\Omega T) \\ {\scriptstyle W(\omega_S)}\downarrow & & \downarrow{\scriptstyle W(\omega_T)} \\ W(S) & \xleftarrow{W(i)} & W(T) . \end{array}$$

We have

$$\begin{aligned} \tau' f^e | S &= W(\iota)\tau' f^e \\ &= \{W(\iota) W(\omega_T)\} W(\omega_T)^{-1}\tau' f^e \\ &= \{W(\omega_S) C(\Omega\iota)\} W(\omega_T)^{-1}\tau' f^e \\ &= \{W(\omega_S) C(\Omega\iota)\} R_{\tau'} W(\omega_T)^{-1} f^e \end{aligned}$$

$$
\begin{aligned}
&= \{W(\omega_S)R_\tau C(\Omega_{\varphi'})\}W(\omega_T)^{-1}f^e \\
&= \tau\{W(\omega_S)C(\Omega_{\varphi'})\}W(\omega_T)^{-1}f^e \\
&= \tau\{W(\varphi')W(\omega_T)\}W(\omega_T)^{-1}f^e \\
&= \tau W(\varphi')f^e \\
&= \tau f\,.
\end{aligned}
$$

The following lemma is immediate:

LEMMA 2. *Let T be a semitopological semigroup, and let S be a subsemigroup of T. Suppose that $\varphi\colon T \to S$ is a morphism of semitopological semigroups and a retract. Then $C(\varphi')$ maps $W(T)$ onto $W(S)$, $A(T)$ onto $A(S)$, and $U(T)$ onto $U(S)$.*

LEMMA 3. *Let E and F be Banach spaces, and let the linear transformation $\varphi\colon E \to F$ be an isometry. A necessary and sufficient condition that a set $K \subset E$ be weakly compact is that φK be weakly compact in F.*

Proof. The necessity follows from the fact that a continuous linear transformation of Banach spaces is weakly continuous. The sufficiency follows from the fact that the adjoint transformation $\varphi'\colon F' \to E'$ is surjective (Hahn-Banach): for if $\boldsymbol{x}$ is a net on K with $\varphi\boldsymbol{x} \to \varphi x$ weakly in F for some x in K, then if $\lambda \in E'$ and if $\theta \in F'$ is such that $\varphi'\theta = \lambda$, we have

$$
\begin{aligned}
\lim \langle \boldsymbol{x}, \lambda\rangle &= \lim \langle \boldsymbol{x}, \varphi'\theta\rangle \\
&= \lim \langle \varphi\boldsymbol{x}, \theta\rangle \\
&= \langle \varphi x, \theta\rangle \\
&= \langle x, \varphi'\theta\rangle \\
&= \langle x, \lambda\rangle\,.
\end{aligned}
$$

PROPOSITION 4. *Let T be a semitopological semigroup with identity* 1. *Suppose that $S \subseteq T$ is a dense subsemigroup containing* 1. *If every weakly almost periodic function $f \in W(S)$ extends as a continuous function to T, then $W(S) = W(T)|S$.*

Similarly, if $A(S) \subseteq C(T)|S$, then $A(S) = A(T)|S$; and if $U(S) \subseteq C(T)|S$, then $U(S) = U(T)|S$.

Proof. For $f \in W(S)$ (resp., $f \in A(S)$; resp., $f \in U(S)$), let f^e denote the (necessarily unique) continuous extension of f to T.

(A) We show, first of all, that if $f \in W(S)$ (resp., $f \in A(S)$), then for each $t \in T$, there is a function $g_t \in (R_S f)^-$ such that $R_t(f^e) = (g_t)^e$:

Let $\boldsymbol{s}$ be a net on S with $\boldsymbol{s} \to t$ and $R_{\boldsymbol{s}}f$ weakly convergent (resp., convergent). Let $g_t = \lim R_{\boldsymbol{s}}f$. Then for $s \in S$, we have

$$\begin{aligned}[R_t(f^e)](s) &= f^e(st) \\ &= \lim f(s\boldsymbol{s}) \\ &= \lim [R_{\boldsymbol{s}}f](s) \\ &= g_t(s) .\end{aligned}$$

Thus we must have $R_t(f^e) = (g_t)^e$ since S is dense in T.

(B) Now, if $f \in W(S)$, we have that $R_T(f^e)|S$ is relatively weakly compact in $C(S)$; so since restriction to a dense subspace is an isometry, Lemma 3 yields that $R_T(f^e)$ is relatively weakly compact in $C(T)$.

Using the fact that restriction to a dense subspace is an isometry, we obviously have that if $f \in A(S)$, then $f^e \in A(T)$.

(C) Let $\mathfrak{A}(n)$ be the unitary group on $\boldsymbol{C}^n$. Let $\xi: S \to \mathfrak{U}(n)$ be a morphism of semitopological semigroups. We show that we can extend ξ to T:

Suppose $t \in T$ and $\boldsymbol{s}$ is a net on S converging to t. For $x, y \in \boldsymbol{C}^n$, we have that the function $\varphi_{x,y}: S \to \boldsymbol{C}$ defined by

$$\varphi_{x,y}(s) = \langle \xi(s)x, y \rangle$$

can be extended as a continuous function $\varphi^e_{x,y}: T \to \boldsymbol{C}$. Now,

$$\begin{aligned}\varphi^e_{x,y}(t) &= \lim \varphi_{x,y}(\boldsymbol{s}) \\ &= \lim \langle \xi(\boldsymbol{s})x, y \rangle ,\end{aligned}$$

and this must hold for every coefficient $\varphi_{x,y}$ on ξ. Whence, $\xi(\boldsymbol{s})$ converges in the compact group $\mathfrak{U}(n)$. Define $\xi^e: T \to \mathfrak{A}(n)$ by $\xi^e(t) = \lim \xi(\boldsymbol{s})$. (This definition is independent of the particular net $\boldsymbol{s}$ converging to t). By definition ξ^e is a continuous function on T; since it is a morphism on a dense subsemigroup of T, ξ^e is a morphism [1, p. 64].

(D) Now if $f \in U(S)$, then f is the uniform limit of a sequence $k \to \varphi_k$ of coefficients of continuous finite dimensional unitary representations of S. By (C), each φ^e_k is a coefficient of a continuous finite dimensional unitary representation of T. It is readily seen that f^e is the uniform limit of the sequence $k \to \varphi^e_k$.

PROPOSITION 5. *Let S be a semitopological semigroup with identity* 1. *Let e be an idempotent in the minimal ideal $M(\Omega S)$ of ΩS, and let G be the compact topological group $e\Omega Se$* ([1], p. 67). *If $W(S)$ has a left invariant mean, then G is a set of uniqueness for $\mathscr{U} = W(\omega_S)^{-1}(U(S))$; that is, if $F_1, F_2 \in \mathscr{U}$, and if $F_1|G = F_2|G$, then $F_1 = F_2$. Moreover, $F(\tau e) = F(\tau)$ for every $\tau \in \Omega S$ and every $F \in \mathscr{U}$.*

Proof. Since $W(S)$ has a left invariant mean, $M(\Omega S)$ is a minimal right ideal ([1], p. 77). Hence, $M(\Omega S) = e\Omega S$ and $\Omega Se = e\Omega Se = G$. Therefore, $\tau \to \tau e\colon \Omega S \to G$ is a morphism of semitopological semigroups. By Lemma 2, if $f \in U(G) = C(G)$, then f may be extended to a function $F \in U(\Omega S)$ by setting $F(\tau) = F(\tau e) = f(\tau e)$ for every $\tau \in \Omega S$. From Proposition 4 one gets that $\mathscr{U} = U(\Omega S)$, so we need only observe that $R_e | \mathscr{U}$ is injective. But that is merely the observation that e is a projection of $W(S)$ onto $U(S)$ ([1], p. 83) for if $F \in \mathscr{U}$, and $f = W(\omega_S)F$, then $F = W(\omega_S)^{-1}f = W(\omega_S)^{-1}ef = R_e W(\omega_S)^{-1}f = R_e F$.

Quite simple examples of semitopological semigroups (or even topological semigroups) show that the requirement in Proposition 4 that the functions in $W(S)$ (resp., $A(S)$; resp., $U(S)$) extend as continuous functions to T is a necessary hypothesis. However, the following example if so pathological that it may be independently interesting.

EXAMPLE 6. Let S be a compact semitopological semigroup. Define the semigroup T to be the set $S \times \{0, 1\}$. Topologize T as follows: Take neighborhoods of points $(s, 0)$ to be of the form

$$V \times \{0\} \cup V \times \{1\} \backslash \{(s, 1)\} ,$$

where V is a neighborhood of s in S; take the relative topology of $S \times \{1\}$ to be discrete. Under coordinatewise multiplication, T is a compact semitopological semigroup with a dense discrete subsemigroup $S \times \{1\}$. Moreover, if S has an identity 1, then $(1, 1)$ is an identity for T; and $M(T) = M(S) \times \{0\}$.

Now suppose that S is a compact abelian group and that χ is a discontinuous character of S. We may consider χ to be strongly almost periodic on the discrete subsemigroup $S \times \{1\}$ of T; but by Proposition 5, it is clear that χ cannot be extended to all of T.

LEMMA 7. *If H is a topological group, and if G is a dense subgroup of H, then every coefficient of a continuous finite dimensional unitary representation of G extends to H as a coefficient of a continuous finite dimensional unitary representation of H.*

Proof. Let $\mathfrak{U}(n)$ be the unitary group on C^n. It is clearly sufficient to show that every morphism $\xi\colon G \to \mathfrak{U}(n)$ of topological groups can be extended to H. Observing that ξ is uniformly continuous and using the fact that $\mathfrak{U}(n)$ is a complete uniform space, we get that ξ extends as a continuous function. That the extension is a morphism now follows as in 4(C).

We observe in passing that the requirement that G be dense is necessary for consider the special unitary group $G = \mathfrak{SU}(2)$ as a subgroup of the special linear group $\mathfrak{SL}(2)$.

LEMMA 8. *Let T be a semitopological semigroup with identity 1. Suppose $S \subseteqq T$ is a subsemigroup containing 1. Further suppose that $W(T)$ and $W(S)$ have left invariant means, and that every coefficient of a continuous finite dimensional unitary representation of S extends as a coefficient of a continuous finite dimensional unitary representational of T. If G is a maximal subgroup of $M(\Omega S)$ and H is a maximal subgroup of $M(\Omega T)$, then there is a natural injection of G into H.*

Proof. Since $W(T)$ and $W(S)$ have left invariant means, the minimal ideals $M(\Omega T)$ and $M(\Omega S)$ are minimal right ideals. Let e_T and e_S be idempotents in $M(\Omega T)$ and $M(\Omega_S)$, respectively. Let $\iota: S \to T$ be the inclusion map. For $\sigma \in \Omega S$, let $\sigma' = (\Omega\iota)\sigma$. We want to show that the map $\sigma \to \sigma' e_T: \Omega S \to H$, where $H = e_T \Omega T e_T = \Omega T e_T$, is injective when restricted to $G = e_S \Omega S e_S = \Omega S e_S$:

Suppose that $\sigma, \tau \in G$ with $\sigma \neq \tau$. By Proposition 5, there is a function $f \in U(S)$ with

$$\{W(\omega_S)^{-1}f\}(\sigma) \neq \{W(\omega_S)^{-1}f\}(\tau) .$$

Since f is uniformly approximated by linear combinations of coefficients φ_S of continuous finite dimensional unitary representations of S, there must be one such φ_S with

$$\{W(\omega_S)^{-1}\varphi_S\}(\sigma) \neq \{W(\omega_S)^{-1}\varphi_S\}(\tau) .$$

Now φ_S extends to T as a coefficient φ_T of a continuous finite dimensional unitary representation of T. Since $e_T\varphi_T = \varphi_T$, we have that $W(\omega_T)^{-1}\varphi_T = R_{e_T} W(\omega_T)^{-1}\varphi_T$. So,

$$\begin{aligned} \{W(\omega_T)^{-1}\varphi_T\}(\sigma' e_T) &= \{W(\omega_T)^{-1}\varphi_T\}(\sigma') \\ &= \{W(\omega_S)^{-1}\varphi_S\}(\sigma) \\ &\neq \{W(\omega_S)^{-1}\varphi_S\}(\tau) \\ &= \{W(\omega_T)^{-1}\varphi_T\}(\tau') \\ &= \{W(\omega_T)^{-1}\varphi_T\}(\tau' e_T) . \end{aligned}$$

Hence, $\sigma' e_T \neq \tau' e_T$.

THEOREM 9. *Let T be a semitopological semigroup with identity 1. Suppose $S \subseteqq T$ is a subsemigroup containing 1. Further suppose that $W(T)$ and $W(S)$ have left invariant means, and that every coefficient of a continuous finite dimensional unitary representation* *of S extends as a coefficient of a continuous finite dimensional unitary representation of T. Then every strongly almost periodic function on S extends as a strongly almost periodic function on T.*

Proof. We continue with the notation of the proof of Lemma 8.

Let $f \in U(S)$, and let $F = W(\omega_S)^{-1}f$. By Proposition 5, $F(\sigma) = F(\sigma e_S)$ for every $\sigma \in \Omega S$. Define $F' \in C(\Omega T)$ as follows: Let

$$F'(\sigma' e_T) = F'(\sigma') = F(\sigma)$$

for $\sigma \in \Omega S$, and extend F' continuously to all of ΩT by way of the Tietze Extension Theorem. That F' is well-defined follows from Lemma 8 because if $\sigma, \tau \in \Omega S$ and if $\sigma' e_T = \tau' e_T$, then $\sigma e_S = \tau e_S$, and we have $F(\sigma) = F(\tau)$ by Proposition 5. Let $f' = W(\omega_T)F' \in W(T)$, and let f^e be the strongly almost periodic function $e_T f' \in U(T)$. We show that f^e is the desired extension to T of the given function $f \in U(S)$:

Let $s \in S$; then

$$\begin{aligned} f^e(s) &= \{R_s e_T f'\}(1) \\ &= \{R_s e_T W(\omega_T)F'\}(1) \\ &= \{W(\omega_T)R_{\omega_T(s)e_T}F'\}(1) \\ &= \{R_{\omega_T(s)e_T}F'\}(\omega_T(1)) \\ &= F'(\omega_T(s)e_T) \\ &= F'(\omega_T(s)) \\ &= F(\omega_S(s)) \\ &= f(s)\ . \end{aligned}$$

COROLLARY 10. *If H is a topological group, and if G is a dense subgroup of H, then every almost periodic function on G extends to H as an almost periodic function.*

Proof. By Lemma 7 every coefficient of a continuous finite dimensional unitary representation of G extends as such to H. It is a result of Ryll-Nardzewski that the C^*-algebra of weakly almost periodic functions on a group has an invariant mean (see [1], p. 140). The result then follows immediately from Theorem 9 and the fact that almost periodic functions and strongly almost periodic functions are the same for groups.

COROLLARY 11. *Let H be a locally compact abelian topological group. Suppose G is a (topological, but not necessarily closed) subgroup of H. Then every almost periodic function on G extends to H as an almost periodic function.*

Proof. By Corollary, 10, we may assume that G is closed. The result then follows as in the proof of Corollary 10 with the observation that characters on closed subgroups of locally compact abelian groups extend to the whole group.

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