PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 2, 1970

PROJECTIVE DISTRIBUTIVE LATTICES

RAYMOND BALBES AND ALFRED HORN

It will be shown that a countable distributive lattice is projective if and only if the product of any two join irreducible elements is join irreducible, and every element of the lattice is both a finite sum of join irreducible elements and a finite product of meet irreducible elements. For an arbitrary distributive lattice, necessary and sufficient conditions for projectivity are obtained by adding to these conditions a further condition on the set of join irreducible elements.

1. Definitions. We use sum and product notation for least upper bounds and greatest lower bounds. If A and B are meet semilattices, then a meet homomorphism $f: A \rightarrow B$ is a function such that $f(xy) = f(x)f(y)$. An element x of a lattice is called join irreducible if $x = y + z$ implies $x = y$ or $x = z$. x is called sub join irreducible if $x \leqq y + z$ implies $x \leqq y$ or $x \leqq z$. In a distributive lattice these two notions coincide. We define meet irreducible and super meet irreducible in a dual manner. A lattice is called conditionally implicative if whenever $x \nleqq y$ there exists a largest z such that $xz \leqq y$. The smallest and largest element of a lattice are denoted by 0 and 1 respectively. The cardinal of a set S is denoted by $|S|$. For definitions of projective distributive lattice and retract, see [1]. Note that epimorphisms are understood to be homomorphisms which are onto. If the term epimorphism is used as in the general theory of categories, there are no projective distributive lattices.

2. Projective distributive lattices. Consider the following properties of a lattice.

(P1) Every element is a sum of finitely many sub join irreducible elements.

(P2) Every element is a product of finitely many super meet irreducible elements.

(P3) The product of any two sub join irreducible elements is sub join irreducible.

(P4) The sum of any two super meet irreducible elements is super meet irreducible.

(P5) The lattice is conditionally implicative.

(P6) The lattice is dually conditionally implicative.

THEOREM 1. *Suppose A and B are lattices and A is a retract of B. Then we have:*

(i) *If B satisfies* (P1), *then A satisfies* (P1).

(ii) *If B satisfies* (P1) *and* (P3), *then A satisfies* (P4).

(iii) *If B satisfies* (P5), *then A satisfies* (P5).

Proof. By hypothesis, there exist homomorphisms $f: B \to A$ and $g: A \to B$ such that $fg = I_A$. Suppose B satisfies (P1). Let x be any element of A. Then $g(x) = \sum(S)$, where S is a finite nonempty set of sub join irreducible elements of B. Let T be the set of maximal elements of the set $f(S)$. Then $x = \sum(T)$, and we claim that every element a of T is sub join irreducible. Suppose $a \leqq u + v$ but $a \nleqq u$ and $a \nleqq v$. We have $a = f(b)$ for some $b \in S$. Then

$$b \leqq g(x) \leqq g(u) + g(v) + \sum(g(T - \{a\})) .$$

Hence $b \leqq g(u)$, or $b \leqq g(v)$, or $b \leqq g(c)$ for some $c \in T - \{a\}$. Applying f, we find $a \leqq u$, or $a \leqq v$, or $a \leqq c$. This contradicts the maximality of a.

Suppose B satisfies (P1) and (P3). Let a_1 and a_2 be super meet irreducible in A. Suppose $a_1 + a_2 \geqq a_3a_4$ but $a_1 + a_2 \ngeqq a_3$ and $a_1 + a_2 \ngeqq a_4$. We have $g(a_3) = \sum(S)$ and $g(a_4) = \sum(T)$, where S and T are finite sets of sub join irreducible elements of B. Hence $a_3 = \sum(f(S))$ and $a_4 = \sum(f(T))$. There exists $x \in S$ and $y \in T$ such that $f(x) \nleqq a_1 + a_2$ and $f(y) \nleqq a_1 + a_2$. Now $xy \leqq g(a_3)g(a_4) \leqq g(a_1) + g(a_2)$. By (P3), either $xy \leqq g(a_1)$ or $xy \leqq g(a_2)$. Therefore $f(x)f(y) \leqq a_1$ or $f(x)f(y) \leqq a_2$. Since a_1 and a_2 are super meet irreducible, we have $f(x) \leqq a_1 + a_2$ or $f(y) \leqq a_1 + a_2$, which is a contradiction.

The proof of (iii) is given in [2, Th. 2.9].

THEOREM 2. *For any lattice A, we have*:

(i) *If* (P1) *and* (P3), *then* (P4).

(ii) *If* (P2) *then* (P5).

(iii) *If* (P1), (P2) *and* (P3), *then* (P4), (P5) *and* (P6).

(iv) *If A is a retract of B, and B satisfies* (P1), (P2) *and* (P3), *then A satisfies all six properties* (P1)–(P6).

Proof. (i) follows immediately from Theorem 1 (ii). Suppose A satisfies (P2). Let $x, y \in A$ and $x \nleqq y$. We have $y = \pi(S)$, where S is a finite set of super meet irreducible elements. Let

$$T = \{a \in S: x \nleqq a\} ,$$

and let $z = \pi(T)$. If $a \in S - T$, then $a \geqq x \geqq xz$. If $a \in T$, then $a \geqq z \geqq xz$. Hence $xz \leqq \pi(S) = y$. Now suppose $xw \leqq y$. Then for each $a \in T$, we have $a \geqq xw$, hence $a \geqq w$. Therefore $w \leqq \pi(T) = z$. This proves (ii). (iii) follows from (i), (ii), and the dual of (ii).

Suppose B satisfies (P1), (P2), (P3) and A is a retract of B. Then by Theorem 1 (i) and its dual and by Theorem 1 (ii), A satisfies (P1), (P2) and (P4). (iv) now follows from the dual of (iii).

THEOREM 3. *Let A be a projective distributive lattice. Then A satisfies all six properties* (P1)-(P6).

Proof. A is a retract of a free distributive lattice F. It is well known that F satisfies (P1), (P2) and (P3), the sub join irreducible elements being the products of free generators. The result now follows from Theorem 2 (iv).

THEOREM 4. *Let A be a distributive lattice satisfying* (P1), *and let J be the set of join irreducible elements of A. Then any order preserving map h: $J \to B$, where B is a distributive lattice, can be extended uniquely to a join homomorphism $\hat{h}$: $A \to B$. If in addition, A satisfies* (P3) *and h is a meet homomorphism, then $\hat{h}$ is a homomorphism.*

Proof. If $x = x_1 + \cdots + x_n$, where $x_i \in J$ for all i, let

$$\hat{h}(x) = h(x_1) + \cdots + h(x_n) .$$

$\hat{h}$ is well defined because $x_1 + \cdots + x_n \leqq y_1 + \cdots + y_m$ implies each x_i is $\leqq$ some y_j. It is obvious that $\hat{h}$ is a join homomorphism and is unique. If h is a meet homomorphism, then it is easy to see that $\hat{h}$ also preserves products.

THEOREM 5. *Let J be any meet semi-lattice. There exists a unique distributive lattice $\hat{J}$ such that $\hat{J}$ satisfies* (P1) *and* (P3), *and J is the set of join irreducible elements of $\hat{J}$. $\hat{J}$ is unique up to isomorphism over J.*

Proof. Let $\hat{J}$ be the set of all finite unions of principal ideals of J. $\hat{J}$ is a ring of sets and the map $f: J \to \hat{J}$ such that $f(x) = \{y \in J: y \leqq x\}$ is a meet-monomorphism. It is clear that $f(J)$ is the set of all join irreducible elements of $\hat{J}$. The uniqueness of $\hat{J}$ follows easily from Theorem 4.

In view of Theorems 3, 4 and 5, the study of projective distributive lattices can be reduced to the question: for which meet semi-lattices J is $\hat{J}$ projective? An obvious condition is the following.

THEOREM 6. *Let A be a distributive lattice which satisfies* (P1) (P3), *and let J be the set of join irreducible elements of A. Then*

A is projective if and only if there exists a free distributive lattice F, a homomorphism $f: F \to A$ *and a meet homomorphism* $h: J \to F$ *such that* $fh(x) = x$ *for all* $x \in J$.

Proof. If A is projective, it is a retract of a free distributive lattice F. The necessity of the condition follows immediately. Conversely, given f and h, by Theorem 4, h can be extended to a homomorphism $\hat{h}: A \to F$. It is easy to see that $f\hat{h} = I_A$, and therefore A is projective.

Consider the following weakening of the condition of Theorem 6:

(P7) There exists a free distributive lattice F, a homomorphism $f: F \to A$ and an order preserving map $h: J \to F$ such that $fh(x) = x$ for all $x \in J$.

THEOREM 7. *Let A be a distributive lattice and let J be the set of join irreducible elements of A. Then A is projective if and only if A satisfies* (P1), (P2), (P3) *and* (P7).

Proof. The necessity of the conditions follows from Theorems 3 and 6. Suppose A satisfies the conditions. Let $f: F \to A$ and $h: J \to F$ be as in (P7). By Theorem 4, h can be extended to a join homomorphism $\hat{h}: A \to F$. It is clear that $f\hat{h} = I_A$. For each $x \in J$, there exists a finite set $S(x)$ of meet irreducible elements of A such that $x = \pi(S(x))$. Define $g: J \to F$ as follows: $g(x) = \pi(\hat{h}(S(x)))$. For any $x \in J$, $fg(x) = \pi(f\hat{h}(S(x))) = \pi(S(x)) = x$. If $x, y \in J$, $x \leqq y$ and $z \in S(y)$, then $\pi(S(x)) \leqq z$. Since z is super meet irreducible, z must be $\geqq$ some member of $S(x)$. Hence every member of $\hat{h}(S(y))$ is $\geqq$ some member of $\hat{h}(S(x))$. Thus $g(y) \geqq g(x)$ and we have shown that g preserves order. The proof will be complete by Theorem 6 if we show that g preserves products. Suppose $x, y, z \in J$ and $xy = z$. Then $\pi(S(z)) = \pi(S(x) \cup S(y))$, and therefore each member of $S(z)$ is $\geqq$ some member of $S(x) \cup S(y)$. Therefore

$$g(z) = \pi(\hat{h}(S(z))) \geqq \pi(\hat{h}(S(x) \cup S(y))) = g(x)g(y) .$$

Since g preserves order, we have $g(z) \leqq g(x)g(y)$, and the proof is complete.

THEOREM 8. *Let A be a countable distributive lattice and let J be the set of join irreducible elements of A. If A satisfies* (P1) *and* (P3) *then A satisfies* (P7).

Proof. There exists an epimorphism $f: F \to A$, where F is a free distributive lattice. For each $x \in J$, select an element $g(x)$ such that $fg(x) = x$. Arrange the members of J in sequence $x_0, x_1, \cdots$. Define $h: J \to F$ inductively as follows: $h(x_0) = g(x_0)$, and

$$h(x_n) = g(x_n) \prod \{h(x_i): i < n, x_i > x_n\} + \sum \{h(x_i): i < n, x_i < x_n\} .$$

By induction, it is easy to see that $fh(x_n) = x_n$ and h preserves order on the set $\{x_0, \cdots, x_n\}$. This proves (P7).

THEOREM 9. *If A is a countable distributive lattice, then A is projective if and only if A satisfies* (P1), (P2) *and* (P3).

Proof. This follows from Theorems 7 and 8.

COROLLARY ([1, Th. 7.1]). *If A is finite, then A is projective if and only if A satisfies* (P3).

Proof. Every finite distributive lattice satisfies (P1) and (P2).

Theorem 7 suggests the following question: for which semi-lattices J does $S = \hat{J}$ satisfy (P7)? Theorem 8 states that countability is a sufficient condition. Another sufficient condition is that J be projective in the category of semi-lattices. Condition (P7) may be replaced by a condition which refers more explicitly to J itself. First, if A is projective, every epimorphism $f: F \to A$ has a right inverse. Therefore in Theorem 7, we may replace (P7) by

(P8) There exists a free distributive lattice F whose set of free generators is T, a homomorphism $f: F \to A$ such that $f(T) = J$, and an order preserving map $h: J \to F$ such that $fh(x) = x$ for all $x \in J$.

THEOREM 10. *Let J be a meet semi-lattice. Then $A = \hat{J}$ satisfies* (P8) *if and only if for each $x \in J$ there exists a finite sequence $S_{x,0}, \cdots, S_{x,p(x)}$ of nonempty finite subsets of J such that*

(i) $\pi(S_{x,0}) = x$.

(ii) $\pi(S_{x,j}) \leqq x$ *for all j.*

(iii) *if $x \leqq y$, there for every j there is a k such that* $S_{x,j} \supseteq S_{y,k}$.

Proof. Assume (P8) holds. Let $x \in J$. Then

$$h(x) = \pi(T_{x,0}) + \cdots + \pi(T_{x,p(x)}) ,$$

where each $T_{x,j}$ is a finite subset of T. Let $S_{x,j} = f(T_{x,j})$. Then $x = \pi(S_{x,0}) + \cdots + \pi(S_{x,p(x)})$. Since x is join irreducible, we have (i) and (ii) after renumbering indices. If $x \leqq y$, then $h(x) \leqq h(y)$. From this it follows that every $T_{x,j}$ contains some $T_{y,k}$ (see [1, Lemma 4.5]). This proves (iii).

Assume (i), (ii) and (iii). Let F be a free distributive lattice with a free generating set T with the same cardinal as J. There exists a homomorphism $f: F \rightarrow A$ which maps T onto J in a one-to-one way. For each $S_{x,j}$ let $T_{x,j}$ be the subset of T such that $f(T_{x,j}) = S_{x,j}$. Define $h: J \rightarrow F$ as follows:

$$h(x) = \pi(T_{x,0}) + \cdots + \pi(T_{x,p(x)}) .$$

By (i) and (ii), $fh(x) = x$ for all $x \in J$, and by (iii) h is order preserving. This completes the proof.

If P is a partially ordered set, there exists a distributive lattice P^* containing P such that P generates P^* and every order preserving map from P to a distributive lattice B can be extended to a homomorphism from P^* to B. (See for example [2, Definition 1.10].) In Lemma 3.8 of [2] it was shown that P^* is projective if and only if for each $x \in P$ there exists a finite sequence $S_{x,0}, \cdots, X_{x,p(x)}$ of non-empty finite subsets of P such that

(i) $x \in S_{x,0}$ and every member of $S_{x,0}$ is $\geqq x$.

(ii) for each j, $S_{x,j}$ has a member $\leqq x$

(iii) if $x \leqq y$, then for every j there is a k such that $S_{x,j} \supseteqq S_{y,k}$.

Comparing with the conditions of Theorem 9, we find the following.

THEOREM 11. *If J is meet semi-lattice and J^* is projective, then $\hat{J}$ satisfies* (P8).

A sufficient condition for the projectivity of J^* is given in [2, Th. 3.12].

3. Direct products.

LEMMA 1. *Let A_1 and A_2 be projective distributive lattices. If A_1 and A_2 have a 0 and 1, then $A_1 \times A_2$ is projective.*

Proof. We may assume $|A_i| > 1$ for $i = 1, 2$. Let F be a free distributive lattice with the free generating set $T_1 \cup T_2$, where T_1 and T_2 are disjoint and $|T_i| = |A_i|$, $i = 1, 2$. There exists an epimorphism $f: F \rightarrow A_1 \times A_2$ such that $f(T_1) = A_1 \times \{0\}$ and $f(T_2) = \{0\} \times A_2$. Let F_i be the sublattice of F generated by T_i. Then $f(F_1) = A_1 \times \{0\}$ and $f(F_2) = \{0\} \times A_2$. Define $f_i: F_i \rightarrow A_i$ by $f_i = \pi_i \cdot (f \mid F_i)$, where $\pi_i: A_1 \times A_2 \rightarrow A_i$ is the natural projection. Since A_i is projective, there exists a homomorphism $g_i: A_i \rightarrow F_i$ such that $f_i g_i = I_{A_i}$. Define $g: A_1 \times A_2 \rightarrow F$ by

$$g(x, y) = g_1(x) + g_2(y) + g_1(1)g_2(1) .$$

Then $fg(x, y) = (x, 0) + (0, y) + (1, 0)\cdot(0, 1) = (x, y)$, and g is a homomorphism. Therefore $A_1 \times A_2$ is a retract of F, and the proof is complete.

LEMMA 2. *An element x of a direct product $\prod_{i \in I} A_i$ of distributive lattices is join irreducible if and only if for some $i \in I$, we have:*

(i) $x(j) = 0$ *for all* $j \neq i$.

(ii) $x(i)$ *is join irreducible.*

Proof. The proof is easy and will be omitted.

THEOREM 12. *Let $\langle A_i : i \in I\rangle$ be a family of distributive lattices. Suppose $|I| > 1$ and $|A_i| > 1$ for all i. Then $\prod_{i \in I} A_i$ is projective if and only if*

(i) A_i *is projective for each* i

(ii) I *is finite, and*

(iii) *each* A_i *has a* 0 *and* 1.

Proof. The sufficiency follows from Lemma 1. Suppose A is projective. By hypothesis there exists $x \in A$ and $i_1, i_2 \in I$ such that $i_1 \neq i_2$, $x(i_1) \neq 0$ and $x(i_2) \neq 0$. Since x is a sum of join irreducible elements, it follows from Lemma 2 that A_i has a 0 for all $i \in I$. By duality, each A_i has a 1, which proves (iii). Finally if I is infinite, then the 1 element of A cannot be a finite sum of join irreducible elements by Lemma 2, since $|A_i| > 1$ for all i.

REFERENCES

1. R. Balbes, *Projective and injective distributive lattices*, Pacific J. Math. **21** (1967), 405-420.

2. R. Balbes and A. Horn, *Order sums of distributive lattices*, Pacific J. Math. **21** (1967), 421-435.

Received October 1, 1969. This research was supported in part by National Science Foundation grants GP-11893 and GP-9044.

UNIVERSITY OF MISSOURI, ST. LOUIS
UNIVERSITY OF CALIFORNIA, LOS ANGELES