PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 3, 1970

SOME MATRIX FACTORIZATION THEOREMS, II

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In the first part of this paper a thorough analysis was made of the matrix equation $C = ABA^{-1}B^{-1}$ when C, A, B are normal matrices. Not included, however, was the discussion of this equation when A and B are real skew-symmetric matrices. In the present paper we complete the investigation by giving this discussion.

Throughout this paper we adopt the notation and terminology of part I. We also continue the convention that all matrices appearing in this paper, except the zero matrix, are to be nonsingular. We always let K_1, K_2 denote real skew symmetric matrices.

LEMMA 1. *Let M be a matrix with linear elementary divisors, and let $M = K_1K_2$ be a product of two real skew-symmetric matrices K_1, K_2. Then each eigenvalue of M has even multiplicity.*

Proof. This is a special case of a result of H. Freudenthal [1]. Using the idea of [1], we give a short proof of the lemma. From $M = K_1K_2$ we get $\lambda I - M = (\lambda K_2^{-1} - K_1)K_2$. For any (real or complex) eigenvalue λ of M, the matrix $\lambda K_2^{-1} - K_1$ is (real or complex) skew symmetric and therefore has even rank. Because K_2 is nonsingular, it follows that $\lambda I - M$ has even rank for each λ. Since degree M is even and M has linear elementary divisors, it follows that the multiplicity of λ as an eigenvalue of M is even.

We are now ready to state our main result.

THEOREM 1. *Let N be real and normal. Then N is a commutator*

$$N = K_1K_2K_1^{-1}K_2^{-1} \tag{1}$$

of two real skew-symmetric matrices K_1, K_2 if and only if N is orthogonally similar to a direct sum of the following five types of real normal matrices:

$$\operatorname{diag}(r_1, r_1^{-1}, r_2, r_2^{-1}), \qquad r_1 > 0, r_2 > 0; \tag{2}$$

$$\operatorname{diag}(-r_1, -r_1^{-1}, -r_2, -r_2^{-1}), \qquad r_1 > 0, r_2 > 0; \tag{3}$$

$$F(\varphi) \dotplus F(\varphi); \tag{4}$$

$$R_1F(\varphi) \dotplus R_1^{-1}F(\varphi) \dotplus R_2F(\varphi) \dotplus R_2^{-1}F(\varphi), \quad R_1 > 0, R_2 > 0; \tag{5}$$

$$\operatorname{diag}(1, 1). \tag{6}$$

We remind the reader that

$$F(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}.$$

THEOREM 2. *If real normal N is a commutator* (1) *with*

$$NK_1 = K_1N \tag{7}$$

then N is symmetric and orthogonally similar to a direct sum of the types (6), (8), (9):

$$\operatorname{diag}(r, r, r^{-1}, r^{-1}), \qquad r > 0; \tag{8}$$

$$\operatorname{diag}(-r, -r, -r^{-1}, -r^{-1}), \qquad r > 0. \tag{9}$$

Conversely, if symmetric N is orthogonally similar to a direct sum of types (6), (8), (9) *then N is a commutator* (1) *of two skew matrices such that* (7) *holds, and such that K_2 is also orthogonal. We may, in addition, choose K_1 orthogonal if N is also orthogonal.*

THEOREM 3. *If real normal N is a commutator* (1) *of two skew matrices K_1, K_2 such that*

$$NK_1 = K_1N,\ NK_2 = K_2N \tag{10}$$

then N Symmetric is orthogonal and satisfies the condition

$$(\textit{multiplicity of eigenvalue } -1) \equiv 0 \pmod 4 .$$

(That is, N is orthogonally similar to a direct sum of the types (6) *and* (11):

$$\operatorname{diag}(-1, -1, -1, -1). \tag{11}$$

Conversely, if N satisfies these conditions then N can be represented as a commutator (1) *satisfying* (10) *such that K_1 and K_2 are both skew orthogonal.*

Proof of Theorem 1. We use the notation in the proof of Theorem 5.7 of [2]. As in that proof, we agree that subscripts attached to a matrix indicate the degree of the matrix. The only exceptions to this rule are K_1 and K_2. From (1) we get $N^{-1T} = (K_2K_1)^{-1}N(K_2K_1)$. Hence the eigenvalues of N occur in reciprocal pairs. Thus after an orthogonal similarity of (1) we may assume N is given by (61) of [2] and that the agreement about the eigenvalues of the direct summands of N explained below (61) of (2) is in force. Then we derive [2, (62)], and hence from $(K_2K_1)N^{-1T} = N(K_2K_1)$ we get

$$K_2K_1 = A_\alpha \dotplus B_\beta \dotplus \sum_{i=1}^{u}{}^{\cdot} \begin{bmatrix} 0 & C_{m_i} \\ \Gamma_{m_i} & 0 \end{bmatrix} \dotplus \sum_{i=1}^{v}{}^{\cdot} \begin{bmatrix} 0 & D_{k_i} \\ \varDelta_{k_i} & 0 \end{bmatrix} \dotplus \sum_{i=1}^{w}{}^{\cdot} E_{2p_i} \dotplus \sum_{i=1}^{t}{}^{\cdot} \begin{bmatrix} 0 & F_{2q_i} \\ \mathscr{F}_{2q_i} & 0 \end{bmatrix}, \tag{12}$$

where we also have

$$\varPhi_{2p_i}(\varphi_i)E_{2p_i} = E_{2p_i}\varPhi_{2p_i}(\varphi_i), \qquad 1 \leqq i \leqq w, \tag{13}$$

$$F_{2q_i}\varPhi_{2q_i}(\theta_i) = \varPhi_{2q_i}(\theta_i)F_{2q_i}, \qquad 1 \leqq i \leqq t, \tag{14}$$

$$\mathscr{F}_{2q_i}\varPhi_{2q_i}(\theta_i) = \varPhi_{2q_i}(\theta_i)\mathscr{F}_{2q_i}, \qquad 1 \leqq i \leqq t. \tag{15}$$

Taking the transpose of each side of (12) yields an expression for K_1K_2, which when substituted into $N(K_2K_1) = K_1K_2$ produces the following formulas:

$$\begin{gathered} A_\alpha = A_\alpha^T,\ -B_\beta = B_\beta^T,\ \Gamma_{m_i} = r_iC_{m_i}^T,\ \varDelta_{k_i} = -s_iD_{k_i}^T, \\ \varPhi_{2p_i}(\varphi_i)E_{2p_i} = E_{2p_i}^T,\ R_i\varPhi_{2q_i}(\theta_i)F_{2q_i} = \mathscr{F}_{2q_i}^T. \end{gathered} \tag{16}$$

From these formulas (16) we get by Lemmas 3.4 and 3.5 of [2] that the following direct summand of K_1K_2 is similar to a diagonal matrix and has real eigenvalues:

$$A_\alpha^T \dotplus \sum_{i=1}^{u}{}^{\cdot} \begin{bmatrix} 0 & r_iC_{m_i} \\ C_{m_i}^T & 0 \end{bmatrix}. \tag{17}$$

Similarly the following direct summand of K_1K_2 is also similar to a diagonal matrix and its eigenvalues are all pure imaginaries:

$$B_\beta^T \dotplus \sum_{i=1}^{v}{}^{\cdot} \begin{bmatrix} 0 & -s_iD_{k_i} \\ D_{k_i}^T & 0 \end{bmatrix}. \tag{18}$$

Now by (13) and the fifth of equations (16), we find as in the descussion between equations (70) and (75) of [2] that E_{2p_j} is similar to a diagonal matrix and that the eigenvalues of E_{2p_j} are of the form

$$\begin{gathered} \varepsilon'_{j1}\rho'_{j1}e^{-i\varphi_j/2}, \cdots, \varepsilon'_{jp_j}\rho'_{jp_j}e^{-i\varphi_j/2}, \\ \varepsilon''_{j1}\rho''_{j1}e^{i\varphi_j/2}, \cdots, \varepsilon''_{jp_j}\rho''_{jp_j}e^{i\varphi_j/2}, \end{gathered}$$

where each ε is ± 1 and each $\rho > 0$. Since the eigenvalues of E_{2p_j} appear in conjugate pairs and $e^{i\varphi_j/2}$ is not real, we may arrange the notation so that the eigenvalues of E_{2p_j} are

$$\begin{gathered} \varepsilon_{j1}\rho_{j1}e^{-i\varphi_j/2}, \cdots, \varepsilon_{jp_j}\rho_{jp_j}e^{-\varphi_j/2}, \\ \varepsilon_{j1}\rho_{j1}e^{i\varphi_j/2}, \cdots, \varepsilon_{jp_j}\rho_{jp_j}e^{i\varphi_j/2}, \end{gathered} \tag{19}$$

where each ε is ± 1 and each $\rho > 0$. Thus the direct summand E_{2pj}^T

of K_1K_2 contributes the eigenvalues (19) to K_1K_2. The eigenvalues (19) are not real and not pure imaginary.

Now we examine the eigenvalues and elementary divisors of the direct summand

$$\begin{bmatrix} 0 & R_j\Phi_{2q_j}(\theta_j)F_{2q_j} \\ F^T_{2q_j} & 0 \end{bmatrix}, \tag{20}$$

in K_1K_2. The matrix (20) is similar to

$$\begin{bmatrix} 0 & I \\ R_jF^T_{2q}F_{2q_j}\Phi_{2q_j}(\theta_j) & 0 \end{bmatrix}. \tag{21}$$

Because of (14), when we make the unitary similarity that converts $\Phi_{2q_j}(\theta_j)$ to $e^{i\theta_j}I_{q_j} \dotplus e^{-i\theta_j}I_{q_j}$, we convert F_{2q_j} to $F'_{q_j} \dotplus F''_{q_j}$. Thus (21) is similar to

$$\begin{bmatrix} 0 & I \\ R_j\begin{bmatrix} e^{i\theta_j}F'^*_{q_j}F'_{q_j} & 0 \\ 0 & e^{-i\theta_j}F''^*_{q_j}F''_{q_j} \end{bmatrix} & 0 \end{bmatrix},$$

which in turn is similar to

$$\begin{bmatrix} 0 & I \\ R_je^{i\theta_j}F'^*_{q_j}F'_{q_j} & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & I \\ R_je^{-i\theta_j}F''^*_{q_j}F''_{q_j} & 0 \end{bmatrix}. \tag{22}$$

As in Lemmas 3.4 and 3.5 of [2], we find that the direct summands in (22) are each similar to diagonal matrices and that the eigenvalues of (20) have the form

$$\begin{aligned} &\pm g'_{j1}e^{i\theta_j/2}, \cdots, \pm g'_{jq_j}e^{i\theta_j/2}, \\ &\pm g''_{j1}e^{-i\theta_j/2}, \cdots, \pm g''_{jq_j}e^{-i\theta_j/2}, \end{aligned} \tag{23}$$

where each $g > 0$. Since $e^{i\theta_j/2}$ is not real or pure imaginary, and since the eigenvalues of (20) appear in conjugate pairs, we can arrange the notation in (23) so that the eigenvalues of (20) are

$$\begin{aligned} &\pm g_{j1}e^{i\theta_j/2}, \cdots, \pm g_{jq_j}e^{i\theta_j/2}, \\ &\pm g_{j1}e^{-i\theta_j/2}, \cdots, \pm g_{jq_j}e^{-i\theta_j/2}, \end{aligned} \tag{24}$$

where each $g > 0$.

We can now classify the eigenvalues of K_1K_2 into three types: (i) the real eigenvalues, arising from the direct summand (17); (ii) the pure imaginary eigenvalues, arising from the direct summand (18); (iii) the not real, not pure imaginary eigenvalues (19) and (24), which arise, respectively from the direct summands $E^T_{2p_j}$ and

$$\begin{bmatrix} 0 & \mathscr{F}_{2q_j}^T \\ F_{2q_j}^T & 0 \end{bmatrix} .$$

Since each direct summand of K_1K_2 is similar to a diagonal matrix, so is K_1K_2. By Lemma 1, we see that each distinct eigenvalue of K_1K_2 must have even multiplicity.

Let us first consider the real eigenvalues of K_1K_2. We study (17). Let $\#^+$ be the number of positive eigenvalues of the symmetric matrix A_α and $\#^-$ be the number of negative eigenvalues of A_α. Then by Lemma 3.5 of [2], the number of positive eigenvalues of (17) is

$$\#^+ + \sum_{i=1}^{u} m_i , \tag{25}$$

and the number of negative eigenvalues is

$$\#^- + \sum_{i=1}^{u} m_i . \tag{26}$$

Each of (25), (26) has to be an even integer. If $\sum_{i=1}^{u} m_i$ is even, then both $\#^+$ and $\#^-$ are even and hence $\alpha = \#^+ + \#^-$ is even. In this event the direct summands of all the $\Omega_{2m_i}(r_i)$, $1 \leqq i \leqq u$, of N can be brought together in pairs and so classified into $(\sum_{i=1}^{u} m_i)/2$ replicas of type (2), and as α is even, the direct summand I_α classifies into $\alpha/2$ copies of type (6). If $\sum_{i=1}^{u} m_i$ is odd, then both $\#^+$ and $\#^-$ are odd, hence α is even again. By classifying the direct summand I_α into $(\alpha - 2)/2$ copies of type (6), and reclassifying one copy of I_2 as $\Omega_2(1)$, we can now group together the direct summands of the $\Omega_{2m_i}(r_i)$ in pairs and so obtain $(1 + \sum_{i=1}^{u} m_i)/2$ sets of type (2). Thus the real eigenvalues of K_1K_2 give rise to types (2), (6).

Now let us consider the pure imaginary eigenvalues of K_1K_2. We study (18). The eigenvalues of (18) are pure imaginaries of total number

$$\beta + \sum_{i=1}^{v} 2k_i .$$

Since the eigenvalues must appear in conjugate pairs, we may count only the eigenvalue of each pair in the upper half plane, and hence conclude that (18) has

$$\beta/2 + \sum_{i=1}^{v} k_i \tag{27}$$

eigenvalues in the upper half plane, each of which must therefore have even multiplicity. (Note that β is even since B_β is a nonsingular skew matrix.) Let us reclassify the direct summand $-I_\beta$ of N as the direct sum of $\beta/2$ copies of $\Omega_2(-1)$. Then N has an even number of blocks of the type $\Omega_2(-r)$, $r > 0$; hence we may group these blocks into pairs of type (3). Thus the type (3) blocks in N arise from the pure imaginary eigenvalues of K_1K_2.

We now study the eigenvalues of K_1K_2 not on the real or imaginary axes. These are given by (19), where $1 \leqq j \leqq w$, and (24), where $1 \leqq j \leqq t$. Each eigenvalue in the union of these sets must appear with even multiplicity. To simplify the discussion, we now change notation somewhat. We now assume the not real, not pure imaginary, eigenvalues of N on the unit circle arise from blocks $\Phi_2(\varphi_i) = F(\varphi_i)$, $1 \leqq i \leqq w$, and that the eigenvalues of N not on the real or imaginary axes nor the unit circle arise from blocks $\Psi_4(R_i, \theta_i)$, $1 \leqq i \leqq t$. Now, of course $\Phi_2(\varphi_i)$ and $\Phi_2(\varphi_j)$ may have a common eigenvalue if $i \neq j$, but if this happens we arrange matters such that $\varphi_i = \varphi_j$. Also $\Psi_4(R_i, \theta_i)$ and $\Psi_4(R_j, \theta_j)$ may have a common eigenvalue if $i \neq j$, but if this happens then the four eigenvalues of $\Psi_4(R_i, \theta_i)$ coincide in some order with the four eigenvalues of $\Psi_4(R_j, \theta_j)$. Then in place of (19) we get the pair of eigenvalues

$$\varepsilon_j \rho_j e^{-i\varphi_j/2},\ \varepsilon_j \rho_j e^{i\varphi_j/2}, \qquad \varepsilon_j = \pm 1,\ \rho_j > 0\ , \tag{28}$$

as the eigenvalues of K_1K_2 associated with the direct summand $\Phi_2(\varphi_j)$ of N, $1 \leqq j \leqq w$, and we get the set of four eigenvalues,

$$\pm g_j e^{i\vartheta_j/2},\ \pm g_j e^{-i\vartheta_j/2}\ , \qquad g_j > 0\ , \tag{29}$$

as the set of eigenvalues of K_1K_2 associated with the direct summand $\Psi_4(R_j, \theta_j)$ of N, $1 \leqq j \leqq t$. Then in the union of the sets (28), (29), each eigenvalue appears with even multiplicity.

Note that if the two sets

$$\pm g_1 e^{i\theta_1/2},\ \pm g_1 e^{-i\theta_1/2}\ ;$$
$$\pm g_2 e^{i\theta_2/2},\ \pm g_2 e^{-i\theta_2/2}\ ;$$

have a common eigenvalue, then all four of the eigenvalues in one of these sets appear in the other set. This situation gives rise in N to the pairing of the blocks $\Psi_4(R_1, \theta_1)$, $\Psi_4(R_2, \theta_2)$ and so leads to the block

$$R_1F(\theta_1) \dotplus R_1^{-1}F(\theta_1) \dotplus R_2F(\theta_2) \dotplus R_2^{-1}F(\theta_2)\ ,$$

of type (5) as a direct summand of N. (A change of notation brings θ_2 to equal θ_1.) Deleting such pairings from the sets (29), we obtain a new smaller collection of sets (26), (29) of eigenvalues such that each eigenvalue appears with even multiplicity in the union of these sets and such that no common eigenvalue appears in two of the sets (29).

Now the eigenvalue equal to

$$\varepsilon_1\rho_1 e^{i\varphi_1/2}$$

may appear in some other set (28). (We don't have $\varepsilon_1\rho_1 e^{i\varphi_1/2} = \varepsilon_1\rho_1 e^{-\varphi_1/2}$.) So assume that $\varepsilon_1\rho_1 e^{i\varphi_1/2}$ is one of

$$\varepsilon_2\rho_2 e^{-i\varphi_2/2},\ \varepsilon_2\rho_2 e^{i\varphi_2/2}\ .$$

Then $\rho_1 = \rho_2$. We can't have $\varepsilon_1 e^{i\varphi_1/2} = \varepsilon_2 e^{-i\varphi_2/2}$ since then $F(\varphi_1)$, $F(\varphi_2)$ have a common eigenvalue and $\varphi_1 \neq \varphi_2$. So $\varepsilon_1\rho_1 e^{i\varphi_1/2} = \varepsilon_2\rho_2 e^{i\varphi_2/2}$, hence $e^{i\varphi_1} = e^{i\varphi_2}$, so that $\varphi_1 = \varphi_2$. Thus we get a direct summand $F(\varphi_1) \dotplus F(\varphi_1)$ in N, and moreover after deleting

$$\varepsilon_1\rho_1 e^{-i\varphi_1/2},\ \varepsilon_1\rho_1 e^{i\varphi_1/2}\ ,$$

$$\varepsilon_2\rho_2 e^{-i\varphi_2/2},\ \varepsilon_2\rho_2 e^{i\varphi_2/2}\ ,$$

from the union of sets (28), the eigenvalues remaining in the sets (28), (29) each appear with even multiplicity.

Thus we may reduce ourselves to the situation where different sets (28) do not have a common eigenvalue, and different sets (29) do not have a common eigenvalue. In this circumstance we must have for a certain choice of the $\pm$ sign and perhaps after a notational change (including possibly the change of θ_1 to $-\theta_1$),

$$\varepsilon_1\rho_1 e^{i\varphi_1/2} = \pm g_1 e^{i\theta_1/2}\ . \tag{30}$$

Then

$$\varepsilon_1\rho_1 e^{-i\varphi_1/2} = \pm g_1 e^{-i\theta_1/2} \tag{31}$$

and so $\mp g_1 e^{i\theta_1/2}$ must also appear in one of the sets (28), say

$$\mp g_1 e^{i\theta_1/2} = \varepsilon_2\rho_2 e^{i\varphi_2/2}\ . \tag{32}$$

(It may be necessary to replace φ_2 with $-\varphi_2$ to achieve (32).) Then

$$\mp g_1 e^{-i\theta_1/2} = \varepsilon_2\rho_2 e^{-i\varphi_2/2}\ . \tag{33}$$

In this case the four eigenvalues of the set (29) with $j = 1$ find their partners in the sets $j = 1, j = 2$ of (28). After deleting from (28) the pairs with $j = 1, 2$ and deleting from (29) the set with $j = 1$, the eigenvalues in the remaining sets (28), (29) must still have even multiplicity.

The equations (30), (32) imply $g_1 = \rho_1 = \rho_2$, and $e^{i\theta_1} = e_i^{\varphi_1} = e^{i\varphi_2}$ and so $\theta_1 = \varphi_1 = \varphi_2$. Thus, before we changed the signs of θ_1, θ_2, we had $\theta_1 = \pm\varphi_1 = \pm\varphi_2$. Without loss of generality we may make a diagonal similarity of N to achieve $\theta_1 = \varphi_1 = \varphi_2$. We now group together the following direct summands of N:

$$R_1F(\theta_1) \dotplus R_1^{-1}F(\theta_1) \dotplus F(\varphi_1) \dotplus F(\varphi_2)\ . \tag{34}$$

This block (34) can be classified under the type (5) with $R_2 = 1$.

Thus we have demonstrated that N is orthogonally similar to a direct sum of types (2)–(6).

For the converse we express each of the types (2)–(6) in turn as a commutator of two skew symmetric matrices.

Let $N = \operatorname{diag}(r_1, r_1^{-1}, r_2, r_2^{-1})$. Put

$$K_1 = \begin{bmatrix} 0 & 0 & -r_2^{1/2}r_1^{1/2} & 0 \\ 0 & 0 & 0 & -1 \\ r_2^{1/2}r_1^{1/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{35}$$

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & r_1^{1/2}r_2^{-1/2} & 0 \\ 0 & -r_1^{1/2}r_2^{-1/2} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{36}$$

Then

$$K_1K_2 = \begin{bmatrix} 0 & r_1 \\ 1 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & r_1^{1/2}r_2^{1/2} \\ r_2^{-1/2}r_1^{1/2} & 0 \end{bmatrix}.$$

Taking the transpose we obtain K_2K_1 and then we easily see that $NK_2K_1 = K_1K_2$.

Now let $N = \operatorname{diag}(-r_1, -r_1^{-1}, -r_2, -r_2^{-1})$. Let

$$K_1 = \begin{bmatrix} 0 & 0 & r_1^{1/2}r_2^{1/2} & 0 \\ 0 & 0 & 0 & -1 \\ -r_1^{1/2}r_2^{1/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{37}$$

and let K_2 be given by (36). Then $N = K_1K_2K_1^{-1}K_2^{-1}$.

Now let $N = \operatorname{diag}(1, 1)$. Put

$$K_1 = K_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $N = K_1K_2K_1^{-1}K_2^{-1}$.

Next let $N = F(\varphi) \dotplus F(\varphi)$. Let θ_1, θ_2 be any two angles with $\theta_1 - \theta_2 = \varphi/2$. Put

$$K_1 = \begin{bmatrix} 0 & G(\theta_1) \\ -G(\theta_1) & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & G(\theta_2) \\ -G(\theta_2) & 0 \end{bmatrix}.$$

(The matrix $G(\theta)$ is described in [2].) Using Lemma 3.3 of [2], we see that $N = K_1K_2K_1^{-1}K_2^{-1}$. Clearly K_1, K_2 are skew orthogonal.

Finally let $N = R_1F(\varphi) \dotplus R_1^{-1}F(\varphi) \dotplus R_2F(\varphi) \dotplus R_2^{-1}F(\varphi)$. Let $\theta_1, \theta_2, \alpha_1, \alpha_2$ be any four angles such that $\varphi = \theta_1 + \theta_2 - \alpha_1 - \alpha_2$. Put

$$K_1 = \begin{bmatrix} 0 & 0 & (R_1R_2)^{1/2}G(\theta_1) & 0 \\ 0 & 0 & 0 & G(\theta_2) \\ -(R_1R_2)^{1/2}G(\theta_1) & 0 & 0 & 0 \\ 0 & -G(\theta_2) & 0 & 0 \end{bmatrix}. \tag{38}$$

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & (R_2/R_1)^{1/2}G(\alpha_1) \\ 0 & 0 & G(\alpha_2) & 0 \\ 0 & -G(\alpha_2) & 0 & 0 \\ -G(\alpha_1)(R_2/R_1)^{1/2} & 0 & 0 & 0 \end{bmatrix}. \tag{39}$$

Using Lemma 3.3 of [2],

$$K_1K_2 = \begin{bmatrix} 0 & -(R_1R_2)^{1/2}F(\theta_1 - \alpha_2) \\ -(R_2/R_1)^{1/2}F(\theta_2 - \alpha_1) & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & -R_2F(\theta_1 - \alpha_1) \\ -F(\theta_2 - \alpha_2) & \end{bmatrix}.$$

By taking transposes one finds K_2K_1. It is then a simple matter to verify that $NK_2K_1 = K_1K_2$.

The proof of Theorem 1 is now complete.

Proof of Theorem 2. From (1) and (7) we see that N is a commutator of the Hermitian matrices iK_1, iK_2, commuting with iK_1. By [2, Th. 4.2] it follows that N is symmetric. The formula (61) of [2] therefore simplifies to

$$N = I_\alpha \dotplus -I_\beta \dotplus \sum_{i=1}^{u}{}^{\cdot}\, \Omega_{2m_i}(r_i) \dotplus \sum_{i=1}^{v}{}^{\cdot}\, \Omega_{2k_i}(-s_i)\ , \tag{40}$$

where $r_i > 1$, $s_i > 1$, and distinct direct summands in (40) do not have a common eigenvalue. Then, as in the proof of Theorem 1, we obtain

$$K_2K_1 = A_\alpha \dotplus B_\beta \dotplus \sum_{i=1}^{u}{}^{\cdot} \begin{bmatrix} 0 & C_{m_i} \\ \Gamma_{m_i} & 0 \end{bmatrix} \dotplus \sum_{i=1}^{v}{}^{\cdot} \begin{bmatrix} 0 & D_{k_i} \\ \Delta k_i & 0 \end{bmatrix}. \tag{41}$$

From $NK_1 = K_1N$ we see that K_1 has the the form

$$K_1 = U_\alpha \dotplus V_\beta \dotplus \sum_{i=1}^{u}{}^{\cdot} \begin{bmatrix} W_{m_i} & 0 \\ 0 & \tilde{W}_{m_i} \end{bmatrix} \dotplus \sum_{i=1}^{v}{}^{\cdot} \begin{bmatrix} X_{k_i} & 0 \\ 0 & \tilde{X}_{k_i} \end{bmatrix}. \tag{42}$$

The direct summands in (42) must each be skew. Thus α, β, m_i, k_i

all must be even. Then each $\Omega_{2m_i}(r_i)$ is the direct sum of $m_i/2$ copies of type (8) and each $\Omega_{2k_i}(-s_i)$ is the direct sum of $k_i/2$ copies of type (9). Furthermore I_α is the direct sum of $\alpha/2$ copies of (6). If we can prove that $\beta \equiv 0 \pmod 4$ then we can classify $-I_\beta$ as the direct sum of $\beta/4$ copies of type (9).

From the forms (41) of K_2K_1 and (42) of K_1, it follows that a direct summand Y_β of K_2 exists such that $B_\beta = Y_\beta V_\beta$. We also have (see (16)) $B_\beta = -B_\beta^T$; hence B_β is a real skew matrix which is the product of two other real skew matrices. By Lemma 1 we know that each eigenvalue of B_β has even multiplicity. Thus the eigenvalues of B_β come in sets of four of the form $ri, ri, -ri, -ri,$ with $r > 0$. This implies $\beta \equiv 0 \pmod 4$.

The conditions of Theorem 2 are therefore necessary. To prove sufficiency, we examine types (8), (9), (6) in turn.

Let $N = rI_2 \dotplus r^{-1}I_2$. Set

$$K_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & r^{-1} \\ -r^{-1} & 0 \end{bmatrix}, \tag{43}$$

$$K_2 = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}. \tag{44}$$

Plainly K_2 is skew orthogonal. It is easy to see that $N = K_1K_2K_1^{-1}K_2^{-1}$ and $NK_1 = K_1N$. This works whether r is positive or negative. Now let $N = I_2$. Here we may take

$$K_1 = K_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and again K_2 is skew orthogonal. The proof of Theorem 2 is complete.

Proof of Theorem 3. By [2, Th. 9.1], N is unitary; hence in types (8) and (9) we have $r = 1$; and so we obtain types (6) and (11). Conversely, if N is given by (11), then let K_1 be given by (43), with $r = -1$ in (43), and let K_2 be given by (44). Then (1) and (10) are satisfied.

THEOREM 4. *Let N be positive definite symmetric and n-square. Then N is a commutator* (1) *of two skew symmetric matrices K_1, K_2 if and only if*:

(i) *for $n \equiv 0 \pmod 4$, N is orthogonally similar to a direct sum of blocks of the type* $\operatorname{diag}(r, r^{-1})$, $r > 0$;

(ii) *for $n \equiv 2 \pmod 4$, N is orthogonally similar to*

$$\operatorname{diag}(1, 1) \dotplus N_1,$$

where N_1 satisfies the condition (i).

THEOREM 5. *Proper orthogonal $\mathcal{O}$ is a commutator*

$$\mathcal{O} = K_1K_2K_1^{-1}K_2^{-1}$$

of two skew symmetric matrices if and only if:

(i) *each eigenvalue γ of $\mathcal{O}$ for which $\gamma \neq -1$ has even multiplicity;*

(ii) *the eigenvalue $\gamma = -1$ of $\mathcal{O}$ has multiplicity $\equiv 0 \pmod 4$.* *If these conditions are satisfied, we may choose both K_1 and K_2 to be skew orthogonal.*

Proofs. These results follow by observing what happens to types (2)-(6) when N is positive definite or orthogonal. The proof of Theorem 1 showed how to choose K_1, K_2 to be skew orthogonal if N is orthogonal.

THEOREM 6. *Let $n \equiv 0 \pmod 4$. Let S be positive definite symmetric and n-square and let* $\det S = 1$. *Then*

$$S = (K_1K_2K_1^{-1}K_2^{-1})(K_3K_4K_3^{-1}K_4^{-1})$$

is a product of two commutators of skew symmetric matrices.

Proof. By Fan's factorization applied to S, we write $S = S_1S_2$ where S_1 and S_2 satisfy the conditions of Theorem 4.

THEOREM 6. *Let $n \equiv 0 \pmod 4$. Let $\mathcal{O}$ be proper orthogonal and n-square. Then*

$$\mathcal{O} = (K_1K_2K_1^{-1}K_2^{-1})(K_3K_4K_3^{-1}K_4^{-1})$$

is a product of two commutators of skew orthogonal matrices K_1, K_2, K_3, K_4.

Proof. Any proper orthogonal $\mathcal{O}$ is orthogonally similar to a direct sum of blocks of type $F(\varphi_1) \dotplus F(\varphi_2)$. But

$$F(\varphi_1) \dotplus F(\varphi_2) = (F(\alpha_1) \dotplus F(\alpha_1))(F(\alpha_2) \dotplus F(-\alpha_2))$$

where $\alpha_1 = (\varphi_1 + \varphi_2)/2$, $\alpha_2 = (\varphi_1 - \varphi_2)/2$. Each of

$$F(\alpha_1) \dotplus F(\alpha_1),\ F(\alpha_2) \dotplus F(-\alpha_2)$$

satisfies the conditions of Theorem 5.

THEOREM 7. *Let $n \equiv 0 \pmod 4$. Let A be any real n-square matrix with* $\det A = 1$. *Then*

$$A = (K_1K_2K_1^{-1}K_2^{-1})(K_3K_4K_3^{-1}K_4^{-1})(K_5K_6K_5^{-1}K_6^{-1})(K_7K_8K_7^{-1}K_8^{-1})$$

is a product of four commutators of real skew symmetric matrices, with K_5, K_6, K_7, K_8 *all skew orthogonal.*

Proof. Use the polar factorization theorem, as in [2], in combination with Theorems 5 and 6.

THEOREM 8. *Real normal* N *is a commutator* (1) *with* K_1 *skew and* K_2 *skew orthogonal, if and only if* N *is orthogonally similar to a direct sum of types*

$$\begin{aligned}
&\operatorname{diag}(r, r^{-1}, r, r^{-1}) , && r > 0 ,\\
&\operatorname{diag}(-r, -r^{-1}, -r, -r^{-1}) , && r > 0 ,\\
&\operatorname{diag}(1, 1) , && \\
&F(\varphi) \dotplus F(\varphi) , && \\
RF(\varphi) \dotplus R^{-1}F(\varphi) &\dotplus RF(\varphi) \dotplus R^{-1}F(\varphi) , && R > 0 .
\end{aligned}$$

Proof. Sufficiency follows from sufficiency part of the proof of Theorem 1. Necessity follows by using the condition (i) of Theorem 7.10 of [2] and reclassifying the types (2)–(6) of Theorem 1 above.

The author wishes to thank Mr. David Riley for his assistance in the preparation of this paper.

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Received October 18, 1967. The preparation of this paper was supported in part by the U. S. Air Force Office of Scientific Research, under Grant 698-67.

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