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SOME MATRIX FACTORIZATION THEOREMS

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The object of this paper is to make an exhaustive study of the matrix equation

$$(1) \qquad C = ABA^{-1}B^{-1}$$

when A, B, and C are normal matrices. We shall specialize these matrices in various ways by requiring that C, A, or B lie in one or more of the well-known subclasses of the class of normal matrices (Hermitian, unitary, real skew symmetric, etc.). We shall also demand from time to time that C commute with A, or B, or both.

In § 2 we present some notation. In § 3, we prove a number of simple lemmas that will be frequently used. In § 4 we discuss (1) when C is normal and A and B are Hermitian. In § 5, we discuss (1) when C is real and normal and A and B are real and symmetric. In § 6 we present one theorem that is used several times in § 7, where we discuss (1) when C is normal, A is Hermitian, and B is unitary. In § 8 we complete a discussion of (1) when A is Hermitian and B unitary Hermitian that is partly presented in §§ 4, 5, and 7. In §§ 4–7 cases are discussed in which C commutes with A or with B, but not with both. In § 9 we analyse the situation when C commutes with both A and B.

Commutators of normal matrices have been investigated by a number of authors: Fan [1], Frobenius [2], Gotô [3], Marcus and Thompson [5], Taussky [7], Tôyama [9], Zassenhaus [10]. The results obtained in this paper will partly overlap results obtained in [5] but will, in the main, complement the results of [5]. Our principal tools are two elegant tricks due to Ky Fan, both of which appear in his paper [1].

As a consequence of our study of commutators of normal matrices, we are able, through use of the polar factorization theorem, to obtain factorization theorems for nonnormal matrices. It is interesting that we can achieve sharper results for real matrices than for nonreal matrices.

All matrices appearing in this paper, except for the zero matrix, are assumed to be nonsingular.

2. Notation and terminology.

The words symmetric, positive definite symmetric, negative definite symmetric, skew symmetric, orthogonal, will imply that the matrix in question possessing the indicated property is a matrix of real numbers. We shall make use of skew symmetric matrices over the complex number field as well. These will be called complex skew symmetric matrices. The letters $N, H, S, K, U, \mathcal{O}$ (perhaps with subscripts attached) will denote a matrix which is, respectively, normal, Hermitian, symmetric, skew symmetric, unitary, orthogonal. We let I, I_1, I_α, etc., denote identity matrices with an unspecified number of rows that will follow from context. If the subscript attached to I is to indicate the number of rows in I, this will be expressly stated.

The matrices $F(\varphi)$ and $G(\varphi)$ are, by definition,

$$F(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}, \qquad G(\varphi) = \begin{bmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{bmatrix}.$$

The transpose of A is denoted by A^T, the complex conjugate by $\bar{A}$, and $A^* = \bar{A}^T$. We let

$$A_1 \dotplus \cdots \dotplus A_n = \operatorname{diag}(A_1, \cdots, A_n) = \sum_{i=1}^{n}{}^{\boldsymbol\cdot} A_i$$

denote the direct sum of matrices $A_1, \cdots, A_n$. We set

$$[A_1, \cdots, A_k]_k = \begin{bmatrix} 0 & A_1 & 0 & 0 \cdots 0 \\ 0 & 0 & A_2 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots A_{k-1} \\ A_k & 0 & 0 & 0 \cdots 0 \end{bmatrix}$$

if $k > 1$, and $[A_1]_1 = A_1$. Here $A_1, \cdots, A_k$ are square matrices and 0 denotes a matrix of zeros of an appropriate number of dimensions. The determinant of A is $\det A$. If square A has n rows, we say A is n-square or degree $A = n$.

If complex number λ has polar form $\lambda = re^{i\varphi}$, we call $e^{i\varphi}$ the angular part of λ.

3. Some lemmas.

The results contained in some of the lemmas below are special cases of known results.

LEMMA 3.1. (i) *Let $A = H_1H_2$ be a product of two Hermitian matrices. Then, whenever λ is a nonreal eigenvalue of A, with multiplicity m, it follows that $\bar{\lambda}$ is also an eigenvalue of A, with multiplicity m.*

(ii) *If in* (i) *H_1 is positive definite then all eigenvalues of A are real.*

(iii) *If in* (i) *both H_1 and H_2 are positive definite then all eigenvalues of A are positive.*

Proof. (i) From $A = H_1H_2$ follows $A^* = H_2H_1$. Since H_2H_1 has the same eigenvalues (including multiplicities) as H_1H_2, the result follows.

(ii) Let $H_1 = XX^*$. Then $X^{-1}AX = X^*H_2X$ is Hermitian, hence all eigenvalues are real.

(iii) If H_2 is positive definite so also is X^*H_2X. The proof now follows as in (ii).

LEMMA 3.2. *Let A be real and nonsingular. Then, if $A = SK$ with S real symmetric and K real skew symmetric, it follows that the eigenvalues of A partition into sets of the following types:*

$$\alpha, -\alpha \text{ with } \alpha \text{ real }, \tag{2}$$

$$\alpha, -\alpha \text{ with } \alpha \text{ pure imaginary }, \tag{2'}$$

$$\alpha, -\alpha, \bar{\alpha}, -\bar{\alpha}, \text{ with } \alpha \text{ neither real nor pure imaginary }. \tag{3}$$

Proof. If $A = SK$ then $A^T = -KS$. Thus A and $-A$ have the same eigenvalues. Hence, if α is a real eigenvalue of A with multiplicity m then $-\alpha$ is also an eigenvalue with multiplicity m. This also holds if α is a pure imaginary eigenvalue. If α is neither real nor pure imaginary then $-\alpha \neq \bar{\alpha}$, $\alpha \neq \bar{\alpha}$, hence α, $-\alpha$, $\bar{\alpha}$ all appear with multiplicity m, and thus $-\bar{\alpha}$ also appears with multiplicity m.

LEMMA 3.3. (i) $F(\theta)F(\varphi) = F(\theta + \varphi)$;
(ii) $F(\varphi)G(\theta) = G(\varphi + \theta)$;
(iii) $G(\varphi)G(\theta) = F(\varphi - \theta)$;
(iv) $G(\theta)F(\varphi) = G(\theta - \varphi)$.

Proof. Direct computation.

LEMMA 3.4. *Let X and Y be real nonsingular matrices, both square and of the same size. Let*

$$M = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}.$$

Let the eigenvalues of XY be classified as follows: $r_1^2, r_2^2, \cdots, r_\alpha^2$ (positive reals); $-s_1^2, -s_2^2, \cdots, -s_\beta^2$ (negative reals);

$$t_1^2, \bar{t}_1^2, t_2^2, \bar{t}_2^2, \cdots, t_\gamma^2, \bar{t}_\gamma^2$$

(all nonreal). Then the eigenvalues of M are:

$$r_1, -r_1, \cdots, r_\alpha, -r_\alpha, is_1, -is_1, \cdots, is_\beta, -is_\beta, \tag{4}$$
$$t_1, \bar{t}_1, -t_1, -\bar{t}_1, \cdots, t_\gamma, \bar{t}_\gamma, -t_\gamma, -\bar{t}_\gamma .$$

Proof. Note that

$$\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}^{-1}\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ SXYS^{-1} & 0 \end{bmatrix}.$$

For a suitable S we may assume $SXYS^{-1}$ is triangular with diagonal elements $r_1^2, \cdots, r_\alpha^2, -s_1^2, \cdots, -s_\beta^2, t_1^2, \bar{t}_1^2, \cdots, t_\gamma^2, \bar{t}_\gamma^2$. Suppose X and Y are n-square. We make the same permutation of the rows and of the columns of

$$M_1 = \begin{bmatrix} 0 & I \\ SXYS^{-1} & 0 \end{bmatrix}.$$

This is a similarity transformation. We take the rows (and columns) in the order $1, n+1, 2, n+2, 3, n+3, \cdots, n, 2n$. The effect of this is to convert M_1 into a block triangular form in which the main block diagonal is

$$\begin{bmatrix} 0 & 1 \\ r_1^2 & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & 1 \\ r_\alpha^2 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & 1 \\ -s_1^2 & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & 1 \\ -s_\beta^2 & 0 \end{bmatrix}$$
$$\dotplus \begin{bmatrix} 0 & 1 \\ t_1^2 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & 1 \\ \bar{t}_1^2 & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & 1 \\ t_\gamma^2 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & 1 \\ \bar{t}_\gamma^2 & 0 \end{bmatrix}.$$

The eigenvalues of these 2-square matrices are easy to compute, completing the proof.

LEMMA 3.5. *Let A be a nonsingular real or complex n-square matrix. Let the eigenvalues of AA^* be $\lambda_1^2, \cdots, \lambda_n^2$ with*

$$\lambda_1 > 0, \cdots, \lambda_n > 0 .$$

Let γ be a nonzero number. Then the matrix

$$\begin{bmatrix} 0 & \gamma A \\ A^* & 0 \end{bmatrix}$$

is similar to a diagonal matrix and its eigenvalues are

$$\gamma^{1/2}\lambda_1, -\gamma^{1/2}\lambda_1, \cdots, \gamma^{1/2}\lambda_n, -\gamma^{1/2}\lambda_n . \tag{5}$$

Proof. This proof is similar to the proof of Lemma 3.4.

LEMMA 3.6. *Let $R = \operatorname{diag}(R_1, R_2, \cdots, R_k)$, $T = \operatorname{diag}(T_1, \cdots, T_k)$. Suppose R_i and T_j do not have any common eigenvalue, whenever $i \neq j$. Then if $RX = XT$, it follows that $X = \operatorname{diag}(X_1, \cdots, X_k)$.*

Proof. Partition $X = (X_{ij})$. Then $R_i X_{ij} = X_{ij} T_j$. Since R_i and T_j do not have any common eigenvalue, it is known that this relation implies $X_{ij} = 0$; $i \neq j$.

The following result is due to Hua [4]. We given a short proof.

LEMMA 3.7. *Let Z be a complex skew symmetric matrix. Then a unitary matrix U exists such that*

$$U^T Z U = \sum_{i=1}^{r} \dot{}\begin{bmatrix} 0 & \rho_i \\ -\rho_i & 0 \end{bmatrix} \dot{+}\, 0 \,, \quad \rho_i > 0 \ for\ 1 \leqq i \leqq r \,.$$

Proof. Since $\bar{Z}Z = -Z^*Z$, the matrix $\bar{Z}Z$ is negative semi-definite. Let $-\rho_1^2$ (with $\rho_1 > 0$) be an eigenvalue of $\bar{Z}Z$ and let v_1 be an associated unit eigenvector:

$$\bar{Z}Zv_1 = -\rho_1^2 v_1 \,, \qquad v_1^* v_1 = 1 \,.$$

Set $v_2 = -\rho_1^{-1}\bar{Z}\bar{v}_1$. Then

$$-\rho_1 v_1^* v_2 = \bar{v}_1^T \bar{Z} v_1 = \overline{(v_1^T Z v_1)} = 0$$

because Z is skew; also

$$\begin{aligned} \rho_1^2 v_2^* v_2 &= v_1^T Z^T \bar{Z} \bar{v}_1 = -v_1^T Z \bar{Z} \bar{v}_1 = -\overline{(\bar{v}_1^T \bar{Z} Z v_1)} \\ &= \overline{-(\bar{v}_1^T(-\rho_1^2)v_1)} = \rho_1^2 \,. \end{aligned}$$

Hence v_1 and v_2 are orthonormal unit vectors. We may therefore use v_1 and v_2 as the first two columns of a unitary U_1. Let $v_3, v_4, \cdots$ be the remaining columns of U_1. Then for $i > 2$ we have $v_i^T Z v_1 = -\rho_1 v_i^T \bar{v}_2 = 0$ and $\rho_1 v_i^T Z v_2 = -v_i^T Z \bar{Z} \bar{v}_1 = -v_i^T \overline{(\bar{Z} Z v_1)} = -v_i^T(-\rho_1^2)\bar{v}_1 = 0$. Hence $U_i^T Z U_1$ is block triangular, and because $U_1^T Z U_1$ is skew, we get

$$U_1^T Z U_1 = Z_1 \dot{+} Z_2$$

where Z_1, Z_2 are skew and Z_1 is 2×2. Also $v_1^T Z v_2 = -\rho_1^{-1} v_1^T Z \bar{Z} \bar{v}_1 = -\rho_1^{-1} v_1^T \overline{(\bar{Z} Z v_1)} = -\rho_1^{-1} v_1^T(-\rho_1^2)\bar{v}_1 = \rho_1$. Hence

$$Z_1 = \begin{bmatrix} 0 & \rho_1 \\ -\rho_1 & 0 \end{bmatrix} .$$

We may now carry on by inducting on the degree of Z.

4. The Commutator of two Hermitian matrices.

THEOREM 4.1. *Let N be normal. Then*

(6) $$N = H_1 H_2 H_1^{-1} H_2^{-1}$$

with H_1 and H_2 Hermitian if and only if N is unitarily similar to a direct sum of the following five types (7), (8), (9), (10), (11) *of matrices*:

$$\text{diag}\,(r, r^{-1})\,, \qquad r > 0\,; \tag{7}$$

$$\text{diag}\,(-r, -r^{-1})\,, \qquad r > 0\,; \tag{8}$$

$$\text{diag}\,(r_1e^{i\varphi}, r_1^{-1}e^{i\varphi}, r_2e^{-i\varphi}, r_2^{-1}e^{-i\varphi})\,, \quad r_1 > 0,\ r_2 > 0,\ \varphi \text{ real}\,; \tag{9}$$

$$\text{diag}\,(e^{i\varphi}, e^{-i\varphi})\,, \qquad \varphi \text{ real}\,; \tag{10}$$

$$\text{the identity matrix}\,. \tag{11}$$

THEOREM 4.2. *Let N be normal.*

(i) *If N is a commutator* (6) *of two Hermitian matrices such that*

$$NH_1 = H_1N \tag{12}$$

then N is unitarily similar to a direct sum of types (7), (8), *and* (11).

(ii) *If N is unitarily similar to a direct sum of types* (7), (8), (11) *then N can be expressed as a commutator* (6) *of two Hermitian matrices, such that* (12) *holds, and such that H_2 is also unitary.*

Proofs of Theorems 4.1 *and* 4.2. From (6) one obtains (compare Fan [1])

$$N^{*-1} = (H_2H_1)^{-1}N(H_2H_1)\,. \tag{13}$$

Thus, if γ is an eigenvalue of N with a certain multiplicity, so also is $\bar{\gamma}^{-1}$, with the same multiplicity. Note that $\gamma = \bar{\gamma}^{-1}$ if and only if $|\gamma| = 1$. After a simultaneous unitary similarity of N, H_1, H_2, we may take N diagonal, so let

$$N = \begin{bmatrix} \gamma_1 I_1 & 0 \\ 0 & \bar{\gamma}_1^{-1} I_1 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} \gamma_k I_k & 0 \\ 0 & \bar{\gamma}^{-1} I_k \end{bmatrix} \dotplus \omega_{k+1}I_{k+1} \dotplus \cdots \dotplus \omega_s I_s\,. \tag{14}$$

Here we choose our notation so that $\gamma_1, \bar{\gamma}_1^{-1}, \cdots, \gamma_k, \bar{\gamma}_k^{-1}, \omega_{k+1}, \cdots, \omega_s$ are the distinct eigenvalues of N, with

$$|\gamma_1| \neq 1, \cdots, |\gamma_k| \neq 1\,, \qquad |\omega_{k+1}| = 1, \cdots, |\omega_s| = 1\,.$$

Then, writing (13) as

$$(H_2H_1)N^{*-1} = N(H_2H_1)\,, \tag{15}$$

we obtain

$$H_2H_1 = \begin{bmatrix} 0 & A_1 \\ B_1 & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & A_k \\ B_k & 0 \end{bmatrix} \dotplus A_{k+1} \dotplus \cdots \dotplus A_s \,. \tag{16}$$

Taking the $*$ of each side of (16), we get

$$H_1H_2 = \begin{bmatrix} 0 & B_1^* \\ A_1^* & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & B_k^* \\ A_k^* & 0 \end{bmatrix} \dotplus A_{k+1}^* \dotplus \cdots \dotplus A_s^* \,. \tag{17}$$

The equation $N(H_2H_1) = H_1H_2$ now yields $B_1^* = \gamma_1 A_1, \cdots, B_k^* = \gamma_k A_k$, $\omega_{k+1}A_{k+1} = A_{k+1}^*, \cdots, \omega_s A_s = A_s^*$. Thus $A_{k+1}, \cdots, A_s$ are each normal. If we make a simultaneous unitary similarity of N, H_1, H_2 using a U of the form $U = \operatorname{diag}(I_1, I_1, \cdots, I_k, I_k, U_{k+1}, \cdots, U_s)$, we can leave $A_1, B_1, \cdots, A_k, B_k$ unchanged and diagonalize $A_{k+1}, \cdots, A_s$. Having accomplished this, we now change notation, and let

$$\begin{aligned} N = &\begin{bmatrix} \gamma_1 I_1 & 0 \\ 0 & \bar{\gamma}_1^{-1} I_1 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} \gamma_k I_k & 0 \\ 0 & \bar{\gamma}^{-1} I_k \end{bmatrix} \\ &\dotplus \operatorname{diag}(\omega_{k+1}, \cdots, \omega_s) \,, \end{aligned} \tag{18}$$

$$\begin{aligned} H_1H_2 = &\begin{bmatrix} 0 & \gamma_1 A_1 \\ A_1^* & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & \gamma_k A_k \\ A_k^* & 0 \end{bmatrix} \\ &\dotplus \operatorname{diag}(\xi_{k+1}, \cdots, \xi_s) \,. \end{aligned} \tag{19}$$

Here $\omega_{k+1}, \cdots, \omega_s$ now denote the not necessarily distinct eigenvalues of N on the unit circle. We find

$$\omega_j = \xi_j / \bar{\xi}_j \,, \qquad k < j \leqq s \,. \tag{20}$$

Because of Lemma 3.5, the eigenvalues of (19) are positive multiples of the numbers

$$\begin{aligned} &\gamma_1^{1/2}, \cdots, \gamma_1^{1/2}, -\gamma_1^{1/2}, \cdots, -\gamma_1^{1/2}, \cdots, \gamma_k^{1/2}, \cdots, \gamma_k^{1/2}, \\ &\qquad -\gamma_k^{1/2}, \cdots, -\gamma_k^{1/2}, \xi_{k+1}, \cdots, \xi_s \,. \end{aligned} \tag{21}$$

Lemma 3.1 (i) now asserts that the angular parts of numbers (21) must be real or must appear in complex conjugate pairs.

We now change notation once more, and rewrite the eigenvalues of N as $\gamma_1, \bar{\gamma}_1^{-1}, \cdots, \gamma_k, \bar{\gamma}_k^{-1}, \omega_{k+1}, \cdots, \omega_s$, where $\gamma_1, \bar{\gamma}_1^{-1}, \cdots, \gamma_k, \bar{\gamma}_k^{-1}$ are the eigenvalues of N, not necessarily distinct, not on the unit circle, and $\omega_{k+1}, \cdots, \omega_s$ are the eigenvalues of N, not necessarily distinct, on the unit circle. Thus we now know that the angular parts of numbers $\gamma_1^{1/2}, -\gamma_1^{1/2}, \cdots, \gamma_k^{1/2}, -\gamma_k^{1/2}, \xi_{k+1}, \cdots, \xi_s$ are real or appear in complex conjugate pairs. Moreover, (20) holds.

Let

$$\gamma_j = r_j e^{i\varphi_j}\,, \qquad 1 \leqq j \leqq k\,,$$
$$\omega_j = e^{i\rho_j}\,, \qquad k < j \leqq s\,,$$

be the polar factorizations of the γ_j and the ω_j. Then (20) yields

$$\xi_j = p_j \varepsilon_j e^{i\rho_j/2}\,, \qquad k < j \leqq s\,,$$

where $p > 0$ and $\varepsilon = \pm 1$. Thus we get that the numbers

$$(22) \qquad e^{i\varphi_1/2},\ -e^{i\varphi_1/2},\ \cdots,\ e^{i\varphi_k/2},\ -e^{i\varphi_k/2},\ \varepsilon_{k+1}e^{i\rho_{k+1}/2},\ \cdots,\ \varepsilon_s e^{i\rho_s/2}$$

are real or appear in complex conjugate pairs. The argument now splits into several cases.

Case 1. $$\overline{e^{i\varphi_1/2}} = e^{i\varphi_1/2}\,.$$

Then $e^{i\varphi_1/2}$ is real, hence $\gamma_1 = r_1$, and $\operatorname{diag}(\gamma_1, \bar\gamma_1^{-1}) = \operatorname{diag}(r_1, r_1^{-1})$. This yields type (7). Moreover, $-e^{i\varphi_1/2}$ is its own conjugate; hence the numbers remaining in (22) after deleting $\pm e^{i\varphi_1/2}$ are real or come in conjugate pairs.

Case 2. $$\overline{e^{i\varphi_1/2}} = -e^{i\varphi_1/2}\,.$$

Then $e^{i\varphi_1} = -1$, hence $\gamma_1 = -r_1$, and $\operatorname{diag}(\gamma_1, \bar\gamma_1^{-1}) = \operatorname{diag}(-r_1, -r_1^{-1})$. This yields type (8). Moreover, the conjugate of $-e^{i\varphi_1/2}$ is $e^{i\varphi_1/2}$; hence the remaining numbers (22) are real or appear in conjugate pairs.

Case 3. $$\overline{e^{i\varphi_1/2}} = e^{i\varphi_2/2}\,.$$

Hence, $e^{i\varphi_2} = e^{-i\varphi_1}$. Thus

$$\operatorname{diag}(\gamma_1, \bar\gamma_1^{-1}, \gamma_2, \bar\gamma_2^{-1}) = \operatorname{diag}(r_1 e^{i\varphi_1}, r_1^{-1}e^{i\varphi_1}, r_2 e^{-i\varphi_1}, r_2^{-1}e^{-i\varphi_1})\,.$$

This yields type (9). And here the conjugate of $-e^{i\varphi_1/2}$ is $-e^{i\varphi_2/2}$, hence the numbers remaining in (22) after deleting $\pm e^{i\varphi_1/2}$, $\pm e^{i\varphi_2/2}$ are real or come in conjugate pairs.

Case 4. $$\overline{e^{i\varphi_1/2}} = -e^{i\varphi_2/2}\,.$$

Hence again $e^{i\varphi_2} = e^{-i\varphi_1}$. This case again yields type (9). The conjugate of $-e^{i\varphi_1/2}$ is $e^{i\varphi_2/2}$, so that the remaining numbers (22) are real or come in conjugate pairs.

Case 5. The conjugate of $e^{i\varphi_1/2}$ is not any of the numbers

$$\pm e^{i\varphi_1/2},\ \cdots,\ \pm e^{i\varphi_k/2}\,.$$

Then, with suitable notation,

$$e^{-i\varphi_1/2} = \varepsilon_{k+1}e^{i\rho_{k+1}/2}, \ -e^{-i\varphi_1/2} = \varepsilon_{k+2}e^{i\rho_{k+2}/2} .$$

It then easily follows that

$$\operatorname{diag}(\gamma_1, \bar{\gamma}_1^{-1}, \omega_{k+1}, \omega_{k+2}) = \operatorname{diag}(r_1e^{i\varphi_1}, r_1^{-1}e^{i\varphi_1}, e^{-i\varphi_1}, e^{-i\varphi_1}) ,$$

which falls into the form (9) with $r_2 = 1$. Once again, after deleting $\pm e^{i\varphi_1/2}$, $\varepsilon_{k+1}e^{i\rho_{k+1}/2}$, $\varepsilon_{k+2}e^{i\rho_{k+2}/2}$ from (22), the remaining numbers in (22) must be real or come in conjugate pairs.

Case 6. $\overline{\varepsilon_{k+1}e^{i\rho_{k+1}/2}} = \varepsilon_{k+1}e^{i\rho_{k+1}/2}$.

Then $e^{i\rho_{k+1}} = 1$, hence $\omega_{k+1} = 1$. This yields (11).

Case 7. $\varepsilon_{k+1}e^{-i\rho_{k+1}/2} = \varepsilon_{k+2}e^{i\rho_{k+2}/2}$.

Then $e^{-i\rho_{k+1}} = e^{i\rho_{k+2}}$. Then $\omega_{k+2} = \overline{\omega_{k+1}}$. Thus we obtain type (10).

This completes the proof of half of Theorem 4.1. Before completing the proof of Theorem 4.1, we start the proof of Theorem 4.2. We prove that if (12) holds then N is Hermitian. Following the part of the proof of Theorem 4.1 just given, we obtain (14) and (17), where in (17) we have $B_1^* = \gamma_1A_1, \cdots, B_k^* = \gamma_kA_k$, and we can take $A_{k+1}, \cdots, A_s$ diagonal. Condition (12) now implies that H_1 partitions in the form

$$(23) \qquad H_1 = \operatorname{diag}(T_1, W_1, T_2, W_2, \cdots, T_k, W_k, T_{k+1}, T_{k+2}, \cdots, T_s) .$$

Since H_1 is nonsingular and Hermitian, each diagonal block in (23) is nonsingular and Hermitian. Then for H_1H_2 to have the form (17), we must have

$$(24) \qquad H_2 = \begin{bmatrix} 0 & P_1 \\ P_1^* & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & P_k \\ P_k^* & 0 \end{bmatrix} \dotplus P_{k+1} \dotplus \cdots \dotplus P_s ,$$

with

$$(25) \qquad T_iP_i = \gamma_iA_i, \ W_iP_i^* = A_i^* , \qquad 1 \leqq i \leqq k ,$$

and

$$(26) \qquad T_iP_i = A_i^* , \qquad k < i \leqq s .$$

Thus $A_i = P_iT_i$, for $k < i \leqq s$, is a product of two Hermitian matrices P_i and T_i. To relieve the notation let us fix our attention on $A_{k+1} = P_{k+1}T_{k+1}$. We took A_{k+1} diagonal, so let $A_{k+1} = \operatorname{diag}(\xi_1, \xi_2, \cdots)$. Then

$$(27) \qquad \omega_{k+1} = \xi_1/\bar{\xi}_1 = \xi_2/\bar{\xi}_2 = \cdots .$$

Since A_{k+1} is a product of two Hermitian matrices, its eigenvalues are real or occur in conjugate pairs. If $\bar{\xi}_1$ is real then (27) gives $\omega_1 = 1$. If $\bar{\xi}_1$ is not real, let ξ_2 be the conjugate of ξ_1; $\xi_2 = \bar{\xi}_1$. Then (27) yields $\omega_{k+1} = \bar{\omega}_{k+1}$, hence ω_{k+1} is real.

Thus $\omega_{k+1}, \omega_{k+2}, \cdots$ are all real (and in fact ± 1). Next, from (25) we obtain (using the fact that the W_i are Hermitian),

$$P_i^{-1} T_i P_i = \gamma_i W_i \,, \qquad 1 \leqq i \leqq k \,. \tag{28}$$

Equation (28) yields

$$\gamma_i \text{ (an eigenvalue of } W_i) = \text{an eigenvalue of } T_i \,. \tag{29}$$

Since W_i and T_i are nonsingular and Hermitian, we get from (29) that γ_i is a quotient of two reals, hence real.

Thus, we now know that all eigenvalues of N are real. Therefore N is Hermitian. We already know from the established part of Theorem 4.1 that N is unitarily similar to a direct sum of the five types (7), (8), (9), (10), (11). In type (9), $e^{i\varphi} = \pm 1$, thus type (9) can be reclassified into type (7) or type (8). Similarly type (10) can be reclassified into types (7) or (8). This completes the proof of half of Theorem 4.2.

To establish the converse parts of Theorems 4.1 and 4.2, we let N be, in turn, each of the types (7), (8), (9), (10), (11).

In type (7) we have $N = \operatorname{diag}(r, r^{-1})$. Set $H_1 = \operatorname{diag}(r, 1)$ and

$$H_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{30}$$

Then (6) and (12) hold, and moreover H_2 is also unitary.

In type (8) we have $N = \operatorname{diag}(-r, -r^{-1})$. Set $H_1 = \operatorname{diag}(-r, 1)$ and define H by (30). Then again (6) and (12) hold, and again H_2 is also unitary.

In type (11) $N = I$. Set $H_1 = H_2 = I$. Then (6) and (12) hold, and, once more, H_2 is also unitary.

The proof of Theorem 4.2 is now complete.

In type (9) we have $N = \operatorname{diag}(r_1 e^{i\varphi}, r_1^{-1} e^{i\varphi}, r_2 e^{-i\varphi}, r_2^{-1} e^{-i\varphi})$. Set

$$\begin{aligned} H_1 &= \begin{bmatrix} 0 & 0 & r_1^{1/2} e^{i\varphi} & 0 \\ 0 & 0 & 0 & r_2^{-1/2} \\ r_1^{1/2} e^{-i\varphi} & 0 & 0 & 0 \\ 0 & r_2^{-1/2} & 0 & 0 \end{bmatrix} , \\ H_2 &= \begin{bmatrix} 0 & 0 & 0 & r_2^{1/2} r_1^{-1/2} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ r_2^{1/2} r_1^{-1/2} & 0 & 0 & 0 \end{bmatrix} . \end{aligned} \tag{31}$$

Then

$$H_1H_2 = \begin{bmatrix} 0 & r_1^{1/2}e^{i\varphi} \\ r_1^{-1/2} & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & r_2^{1/2}e^{-i\varphi} \\ r_2^{-1/2} & 0 \end{bmatrix}.$$

Taking the * of each side of this expression for H_1H_2, we compute H_2H_1. Then it is easy to verify that $N(H_2H_1) = H_1H_2$.

In type (10) with $N = \operatorname{diag}(e^{i\varphi}, e^{-i\varphi})$, note that N is unitarily similar to $F(\varphi)$. Set $S_1 = G(\theta_1)$, $S_2 = G(\theta_2)$. Then S_1 and S_2 are both orthogonal symmetric. Moreover, using Lemma 3.3, we find that $F(\varphi)S_2S_1 = S_1S_2$ if $\theta_1 - \theta_2 = \varphi/2$.

The proof of Theorem 4.1 is now complete.

THEOREM 4.3. *Let N be normal.*

(i) *If N is a commutator* (6) *of two Hermitian matrices with H_1 positive definite then N is positive definite Hermitian with the eigenvalues γ of N for which $\gamma \neq 1$ occurring in reciprocal pairs γ, γ^{-1}. (That is, N is unitarily similar to a direct sum of types* (7) *and* (11).)

(ii) *If positive definite Hermitian N has its eigenvalues γ for which $\gamma \neq 1$ occurring in reciprocal pairs γ, γ^{-1} then N is a commutator* (6) *of two Hermitian matrices with H_1 positive definite and commutative with N and H_2 unitary Hermitian.*

Proof. Suppose that (6) holds with H_1 positive definite. We follow the proof of Theorem 4.1 until we reach the point where

$$H_1H_2 = \begin{bmatrix} 0 & \gamma_1A_1 \\ A_1^* & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & \gamma_kA_k \\ A_k^* & 0 \end{bmatrix} \dotplus A_{k+1}^* \dotplus \cdots \dotplus A_s^* \,, \tag{32}$$

with $A_{k+1}, \cdots, A_s$ diagonal. By Lemma 3.1, the eigenvalues of H_1H_2 are real. Thus $A_{k+1}, \cdots, A_s$ are each real and diagonal. Then using (20), we get that each $\omega_j = 1$. By Lemma 3.5, the eigenvalues of

$$\begin{bmatrix} 0 & \gamma_1A_1 \\ A_1^* & 0 \end{bmatrix}$$

are positive multiples of $\pm\gamma_1^{1/2}$. Since the eigenvalues of H_1H_2 are to be real, we must have $\gamma_1 > 0, \cdots, \gamma_k > 0$. Hence each eigenvalue of N is positive, therefore N is positive definite. In this case type (8) must be absent in N, type (9) can be reclassified under type (7), and type (10) under type (11). Thus the condition of Theorem 4.3 is necessary. For the converse one need only note that if $N = \operatorname{diag}(r, r^{-1})$ with $r > 0$, then with $H_1 = \operatorname{diag}(r, 1)$ and H_2 given by (30), we have

(6) and (12) and here H_1 is positive definite and H_2 is unitary Hermitian, as required.

The following Theorem 4.4 is a special case of Theorem 1 of [5].

THEOREM 4.4. *Let N be normal, let H_1 be positive definite Hermitian, let H_2 be Hermitian such that*

$$NH_2 = H_2N \,, \tag{33}$$

and suppose that (6) *holds. Then $N = I$.*

Proof. We follow the proof of Theorem 4.1 until (14) and (32) are obtained, with $A_{k+1}, \cdots, A_s$ diagonal. Then (33) yields

$$H_2 = \begin{bmatrix} C_1 & 0 \\ 0 & D_1 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} C_k & 0 \\ 0 & D_k \end{bmatrix} \dotplus C_{k+1} \dotplus \cdots \dotplus C_s \,. \tag{34}$$

Then for H_1H_2 to be given by (32), we must have

$$H_1 = \begin{bmatrix} 0 & M_1 \\ M_1^* & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & M_k \\ M_k^* & 0 \end{bmatrix} \dotplus M_{k+1} \dotplus \cdots \dotplus M_s \,.$$

But

$$\begin{bmatrix} 0 & M_1 \\ M_1^* & 0 \end{bmatrix} \tag{35}$$

is a direct summand of the positive definite matrix H_1, hence is positive definite, a contradiction since (35) has zero trace. Thus in N no γ_i can appear and so the eigenvalues of N must lie on the unit circle. Since $A_{k+1}^* = M_{k+1}C_{k+1}$, A_{k+1} is a product of two Hermitian matrices with one factor definite. Thus the eigenvalues of A_{k+1} are real. Owing to (20), this implies that each $\omega_i = 1$. Hence $N = I$.

THEOREM 4.5. *Let H_1 and H_2 be positive definite. If N, given by* (6) *is normal, then $N = I$.*

Proof. We obtain as in the proof of Theorem 4.1 that (32) holds. By Lemma 3.1, H_1H_2 has positive eigenvalues. Since

$$\begin{bmatrix} 0 & \gamma_1 A_1 \\ A_1^* & 0 \end{bmatrix}$$

is a direct summand of H_1H_2 (hence has positive eigenvalues) and has trace zero, it follows that all eigenvalues of N are on the unit circle. Then each ξ_i is positive and so by (20) each ω_i is 1.

In the next few theorems, we give some more special results that follow from Theorems 4.1 to 4.5 or from the proofs of these theorems.

THEOREM 4.6. *Let U be unitary.*

(i) *If U is a commutator of two Hermitian matrices,*

$$U = H_1 H_2 H_1^{-1} H_2^{-1} , \tag{36}$$

then U has real characteristic polynomial and $\det U = 1$.

(ii) *If U has real characteristic polynomial and* $\det U = 1$, *then U is a commutator* (36) *with both H_1 and H_2 unitary Hermitian.*

(iii) *If U is a commutator* (36) *of two Hermitian matrices such that $UH_1 = H_1 U$ then U is Hermitian and* $\det U = 1$. *Conversely, if U is Hermitian and* $\det U = 1$, *then U is a commutator* (36) *of two unitary Hermitian matrices H_1, H_2 with $H_1 U = UH_1$ and $H_2 U = UH_2$.*

(iv) *If* (36) *holds with H_1 definite then $U = I$.*

Proof. (i) Suppose (36) holds. Then by Theorem 4.1 U is unitarily similar to a direct sum of types (7)-(11). Because U is unitary, in types (7), (8), (9) we have $r = r_1 = r_2 = 1$. Thus the nonreal eigenvalues of N occur in conjugate pairs and -1 occurs an even number of times. This proves (i).

(ii) The conditions imply that the nonreal eigenvalues occur in conjugate pairs and -1 occurs an even number of times. Thus N is unitarily similar to a direct sum of types (10) and (11). The proof of Theorem 4.1 showed how to express each type (10), (11) as a commutator of two unitary Hermitian matrices.

(iii) If (36) holds with $UH_1 = H_1 U$, then Theorem 4.2 shows U is Hermitian and $\det U = 1$. Conversely, it suffices to consider $U = \operatorname{diag}(-1, -1)$. This $U = F(\pi)$ is known from the proof of Theorem 4.1 to be a commutator of two unitary Hermitian matrices, both of which must commute with $\operatorname{diag}(-1, -1)$.

(iv) By Theorem 4.3, U is positive definite. Hence $U = I$.

THEOREM 4.7. *Let H be Hermitian. Then H is a commutator,*

$$H = H_1 H_2 H_1^{-1} H_2^{-1} \tag{37}$$

of two Hermitian matrices if and only if the eigenvalues γ of H other than one come in reciprocal pairs γ, γ^{-1}. (That is, H is unitarily similar to a direct sum of types (7) *and* (11).) *If this condition is satisfied then H_1 may always be chosen to commute with H and H_2 to be both Hermitian and unitary.*

REMARK. Theorem 4.7 is contained in **[1]**.

Proof. If (37) holds, then H is unitarily similar to a direct sum of types (7), (8), (9), (10), (11). As H is Hermitian, in types (9) and (10) we must have $e^{i\varphi} = \pm 1$. Thus, in fact, H is unitarily similar to a direct sum of types (7), (8), (11). For the converse observe that in the proof of Theorems 4.1 and 4.2, types (7), (8) and (11) were each expressed as a commutator $H_1H_2H_1^{-1}H_2^{-1}$ commuting with H_1 and with H_2 unitary and Hermitian.

THEOREM 4.8. *Let θ be a nonreal number on the unit circle: $|\theta| = 1$. Let H be Hermitian. If*

$$\theta H = H_1H_2H_1^{-1}H_2^{-1} \tag{38}$$

is a commutator of two Hermitian matrices then $\theta = \pm i$ and H is unitarily similar to a direct sum of copies of the following two types:

$$\operatorname{diag}(1, -1), \tag{39}$$

$$\operatorname{diag}(r_1, r_1^{-1}, -r_2, -r_2^{-1}) \qquad r_1 > 0, r_2 > 0 . \tag{40}$$

Conversely, if H is unitarily similar to a direct sum of copies of (39) *or* (40), *then*

$$iH = H_1H_2H_1^{-1}H_2^{-1} \tag{41}$$

is a commutator of two Hermitian matrices. In (41) *H and H_1 never commute and H_1 is never definite. Similar results hold for $-iH$.*

Proof. If θH is a commutator of two Hermitian matrices then, as θH has no real eigenvalues, θH must be unitarily similar to a direct sum of types (9) and (10). If either type appears then for two eigenvalues γ_1 and γ_2 of H we have $\theta\gamma_1 = \mu e^{i\varphi}$, $\theta\gamma_2 = \nu e^{-i\varphi}$, with μ, ν real. Thus $\theta/\bar{\theta}$ is real, hence $\theta = \pm i$. Thus in either event $\theta = \pm i$. Thus type (10) takes the form $\operatorname{diag}(i, -i)$, and type (9) the form $i \operatorname{diag}(r_1, r_1^{-1}, -r_2, -r_2^{-1})$ with $r_1 > 0$ and $r_2 > 0$. The converse follows from Theorem 4.1. The additional assertions follow from Theorems 4.2 and 4.3.

THEOREM 4.9. *Let H be positive definite.*

(i) *Let θ be real or nonreal, with $|\theta| = 1$. If θH is a commutator* (38) *of two Hermitian matrices then $\theta = \pm 1$.*

(ii) *$-H$ is a commutator of two Hermitian matrices,*

$$-H = H_1H_2H_1^{-1}H_2^{-1}\ , \tag{42}$$

if and only if all eigenvalues γ of H (including 1) appear in reciprocal pairs γ, γ^{-1} (that is, H is unitarily similar to a direct sum of type (7)). If this condition is satisfied then in (42) H_1 may always be chosen to commute with H and H_2 may always be chosen to be unitary and Hermitian. It is never possible to choose H_1 definite.

Proof. (i) Suppose (38) holds. By Theorem 4.8, $\theta = \pm i$ or $\theta = \pm 1$. If $\theta = \pm i$ then Theorem 4.8 shows that H is not definite. Hence $\theta = \pm 1$.

(ii) If (42) holds, Theorem 4.7 shows that all eigenvalues γ of H appear in reciprocal pairs. Conversely, in the proof of Theorem 4.1 it was shown how to express $\operatorname{diag}(-r, -r^{-1}) = H_1H_2H_1^{-1}H_2^{-1}$ such that the commutator commutes with H_1 and H_2 is unitary Hermitian.

For 2-square matrices, the conclusion of Theorem 4.3 is valid under a weaker hypothesis.

THEOREM 4.10. *Let N be a normal 2-square matrix such that*

$$N = HLH^{-1}L^{-1} \tag{43}$$

where H is positive definite. (No assumptions are made about L other than that it is nonsingular.) Then N is positive definite.

Proof. We may assume $N = \operatorname{diag}(\lambda, \lambda^{-1})$. Let

$$H^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ \bar{h}_{12} & h_{22} \end{bmatrix} .$$

From $H^{-1}N = LH^{-1}L^{-1}$ we get by taking traces,

$$\lambda h_{11} + \lambda^{-1}h_{22} = h_{11} + h_{22}\ . \tag{44}$$

Let $\alpha = h_{11}(h_{11} + h_{22})^{-1}$, $1 - \alpha = h_{22}(h_{11} + h_{22})^{-1}$, and let $\lambda = re^{i\varphi}$ be the polar factorization of λ. Then $0 \leqq \alpha \leqq 1$, and (44) yields

$$r\alpha \cos\varphi + r^{-1}(1 - \alpha)\cos\varphi = 1\ , \tag{45}$$

$$r\alpha \sin\varphi - r^{-1}(1 - \alpha)\sin\varphi = 0\ . \tag{46}$$

From (45) follows $\cos\varphi > 0$. If $\sin\varphi \neq 0$, (46) gives $\alpha = r^{-1}(r + r^{-1})^{-1}$. Then from (45) we get

$$\cos\varphi = (r + r^{-1})/2\ .$$

Since $r + r^{-1} \geqq 2$, and $\cos\varphi \leqq 1$, we get $\cos\varphi = 1$. Thus $\varphi = 0$, contradicting $\sin\varphi \neq 0$. Hence $\sin\varphi = 0$, and therefore $\lambda > 0$.

We now introduce a trick of Professor Fan. Let

$$N = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

with $\det N = 1$. Set $N_1 = \operatorname{diag}(\mu_1, \mu_2, \cdots, \mu_n)$ and

$$N_2 = \operatorname{diag}(\nu_1, \nu_2, \cdots, \nu_n)\ ,$$

where

$$\begin{aligned} \mu_j &= \prod_{i=1}^{j} \lambda_i && \text{if } j \text{ is odd}\ , \\ \mu_j &= \left(\prod_{i=1}^{j-1} \lambda_i\right)^{-1} && \text{if } j \text{ is even}\ ; \\ \nu_j &= \prod_{i=1}^{j} \lambda_i && \text{if } j \text{ is even}\ , \\ \nu_j &= \left(\prod_{i=1}^{j-1} \lambda_i\right)^{-1} && \text{if } j \text{ is odd}\ ;\ 1 \leqq j \leqq n\ . \end{aligned} \tag{47}$$

Then $N = N_1N_2$. We have $\mu_{2j} = \mu_{2j-1}^{-1}$ for all $j \leqq n/2$ and $\mu_n = 1$ for odd n. We have $\nu_{2j+1} = \nu_{2j}^{-1}$ for all $j \leqq (n-1)/2$, $\nu_1 = 1$, and $\nu_n = 1$ if n is even. Thus N has its eigenvalues μ in reciprocal pairs μ, μ^{-1} together with possibly $\mu = 1$ as an eigenvalue. Furthermore N_2 also has its eigenvalues ν in reciprocal pairs ν, ν^{-1} together with $\nu = 1$ as an eigenvalue. We shall refer to this factorization of N as Fan's factorization.

THEOREM 4.11. *Let U be unitary with* $\det U = 1$. *Then*

$$U = (H_1H_2H_1^{-1}H_2^{-1})(H_3H_4H_3^{-1}H_4^{-1}) \tag{48}$$

is a product of two commutators of Hermitian matrices. In fact we may have H_1, H_2, H_3, H_4 all unitary Hermitian.

Proof. By Fan's factorization, $U = U_1U_2$ where U_i is unitary with its eigenvalues in reciprocal pairs; $i = 1, 2$. By Theorem 4.6, U_1 and U_2 each may be written as a commutator of Hermitian unitary matrices.

THEOREM 4.12 (*Fan*). *Let H be Hermitian with* $\det H = 1$. *Then*

$$H = (H_1H_2H_1^{-1}H_2^{-1})(H_3H_4H_3^{-1}H_4^{-1}) \tag{49}$$

is a product of two commutators of Hermitian matrices, with H_1 and H_3 commutative with H and H_2 and H_4 unitary Hermitian. If H is positive definite we may, in addition, choose H_1 and H_3 to be definite.

Proof. The proof is the same as the proof of Theorem 4.11, except that one appeals to Theorem 4.7.

THEOREM 4.12. *Let A be any matrix with* $\det A = 1$. *Then*

$$A = (H_1H_2H_1^{-1}H_2^{-1})(H_3H_4H_3^{-1}H_4^{-1})(H_5H_6H_5^{-1}H_6^{-1})(H_7H_8H_7^{-1}H_8^{-1}) \tag{50}$$

is a product of four commutators of Hermitian matrices. In (50), *H_5 and H_7 may be taken positive definite, and H_1, H_2, H_3, H_4, H_6, H_8 may all be taken to be unitary Hermitian.*

Proof. Let $A = UH$ be the polar factorization of A. Since $H^2 = A^*A$, we get $\det H = 1$. Then $\det U = 1$ also. Now use Theorems 4.11 and 4.12.

For 2-square matrices, the number of commutators required in (50) may be reduced from four to two; in (48) and (49) from two to one.

THEOREM 4.14. (i) *Any unitary 2-square U with* $\det U = 1$ *is a commutator* (36) *of Hermitian unitary matrices.*

(ii) *Any Hermitian* 2-*square H with* $\det H = 1$ *is a commutator* (37) *of Hermitian matrices. In* (37), *H_2 may be chosen Hermitian unitary, and H_1 may be chosen to commute with H and also may be chosen to be definite if H is positive definite.*

(iii) *An* 2-*square A with* $\det A = 1$ *is a product*

$$A = (H_1H_2H_1^{-1}H_2^{-1})(H_3H_4H_3^{-1}H_4^{-1}) \tag{51}$$

of two commutators of Hermitian matrices, with H_3 definite and H_1, H_2, H_4 unitary Hermitian.

Proof. (i), (ii). If U or H is 2-square and $\det U = 1$ or $\det H = 1$, then the eigenvalues of U or H must be reciprocal pairs.

(iii) As in Theorem 4.13, write $A = UH$ and use (i) and (ii) of this theorem.

5. The real analogues of the theorems of § 4.

For certain of the theorems of § 4, the analogues over the real number field are essentially the same. For others, however, this is not so. Moreover, factorization theorems involving real skew symmetric matrices do not always immediately follow from the real symmetric or Hermitian cases by inserting a factor i. In § 5 we therefore will also discuss commutators involving real symmetric or skew symmetric matrices.

THEOREM 5.1. *Let N be a real normal matrix. If N is a* *commutator of two real symmetric matrices,*

$$N = S_1 S_2 S_1^{-1} S_2^{-1} , \tag{52}$$

then the eigenvalues γ of N, excluding $\gamma = 1$, occur in reciprocal pairs γ, γ^{-1}. Conversely, if this condition is satisfied, N can be expressed as a commutator (52) *of two real symmetric matrices, with S_2 both symmetric and orthogonal.*

Proof. Suppose (52) holds. Then

$$N^{T-1} = (S_2 S_1)^{-1} N (S_2 S_1) .$$

Thus, if γ is an eigenvalue of N with a certain multiplicity, so also is γ^{-1} with the same multiplicity. Now $\gamma = \gamma^{-1}$ if and only if $\gamma = \pm 1$. Thus the eigenvalues γ of N for which $\gamma \neq \pm 1$ appear in reciprocal pairs. Since $\det N = 1$, the eigenvalue $\gamma = -1$ must appear an even number of times, hence also appears in reciprocal pairs. Thus the condition of the theorem is necessary.

Suppose now that the condition of Theorem 5.1 is satisfied. Then N is orthogonally similar to a direct sum of blocks of type (7), (8), (11), (53), or (54), where (53) and (54) are given by

$$rF(\varphi) \dotplus r^{-1}F(\varphi) , \qquad r > 0, \ \varphi \text{ real} , \tag{53}$$

$$F(\varphi) , \qquad \varphi \text{ real} . \tag{54}$$

In the proof of Theorem 4.1 and 4.2, it is demonstrated that if N is given by (7), (8), or (11), then N is a commutator (52) of two real symmetric matrices with S_2 symmetric and orthogonal, and S_1 commutative with N. It was also shown that if $N = F(\varphi)$, then N is a commutator of two real symmetric orthogonal matrices. So let N be given by (53).

Let θ and Ψ be any angles. Set $S_1 = \operatorname{diag}(rG(2\theta + \varphi - \Psi), G(\Psi))$ and

$$S_2 = \begin{bmatrix} 0 & G(\theta) \\ G(\theta) & 0 \end{bmatrix} .$$

Then using Lemma 3.3 one easily checks that $NS_2S_1 = S_1S_2$. Moreover S_1 is symmetric and S_2 symmetric orthogonal as required. This completes the proof.

THEOREM 5.2. *The conclusions of Theorem* 4.2 *remain valid if all matrices in Theorem* 4.2 *are required to have real entries.*

THEOREM 5.3. *The conclusions of Theorem* 4.3 *remain valid if* *all matrices in Theorem 4.3 are required to have real entries.*

The real analogues of Theorem 4.4 and 4.5 are special cases of these theorems. We next consider the real counterpart of Theorem 4.6.

THEOREM 5.4. *The conclusions of Theorem 4.6 remain valid if all matrices in Theorem 4.6 are required to have real entries. In particular, a proper orthogonal $\mathcal{O}$ may always be expressed as*

$$\mathcal{O} = S_1 S_2 S_1^{-1} S_2^{-1} \tag{55}$$

where S_1 and S_2 are symmetric orthogonal.

Proof. Let $\mathcal{O}$ be proper orthogonal. Then $\mathcal{O}$ is orthogonally similar to a direct sum of 2-square blocks of the type $F(\varphi)$ and (perhaps) an identity matrix. In the proof of Theorem 4.1, $F(\varphi)$ was expressed as a commutator of two symmetric orthogonal matrices.

If we take N to be symmetric in Theorems 5.1, 5.2, and 5.3 we obtain necessary and sufficient conditions for a symmetric matrix to be a commutator of symmetric matrices, subject to various side condition. In Theorem 5.5 we establish the real analogue of Theorem 4.12.

THEOREM 5.5. *Let A be symmetric with* $\det S = 1$. *Then*

$$S = (S_1 S_2 S_1^{-1} S_2^{-1})(S_3 S_4 S_3^{-1} S_4^{-1})$$

is a product of two commutators of symmetric matrices, with S_2 and S_4 symmetric orthogonal, and S_1, S_3 commutative with S. If S is positive definite, we may in addition require that S_1 and S_3 be definite.

Proof. Use Fan's factorization to express S as a product of two symmetric matrices, each of which has its eigenvalues γ (other than $\gamma = 1$) in inverse pairs γ, γ^{-1}. Then use the proofs of Theorem 4.1 or 5.1.

THEOREM 5.6. *Let A be real with* $\det A = 1$. *Then*

$$A = (S_1 S_2 S_1^{-1} S_2^{-1})(S_3 S_4 S_3^{-1} S_4^{-1})(S_5 S_6 S_5^{-1} S_5^{-1})$$

is a product of three commutators of real symmetric matrices, with S_1, S_2, S_4, S_6 symmetric orthogonal, and S_3, S_5 definite. If A is 2-square, two commutators suffice,

$$A = (S_1 S_2 S_1^{-1} S_2^{-1})(S_3 S_4 S_3^{-1} S_4^{-1})$$

with S_1, S_2, S_4 symmetric orthogonal and S_3 definite.

Proof. Write $A = \mathscr{O}S$, by the polar factorization theorem. Then apply Theorem 5.4 to $\mathscr{O}$ and Theorem 5.5 to S.

In the next theorems we investigate commutators of the form $SKS^{-1}K^{-1}$.

THEOREM 5.7. *Let N be real normal. If N is a commutator of a real symmetric S and a real skew symmetric K,*

$$N = SKS^{-1}K^{-1}\ , \tag{56}$$

then N is orthogonally similar to a direct sum of blocks of types (7), (8), *and* (53). *Conversely, if N is orthogonally similar to a direct sum of blocks* (7), (8), *and* (53), *then N can be expressed as a commutator* (56) *with K both skew symmetric and orthogonal.*

Proof. In this proof, a subscript on a matrix will always indicate the degree of the matrix. We introduce some additional notation:

$$\Omega_{2m}(r) = rI_m \dotplus r^{-1}I_m\ , \tag{57}$$

$$\Phi_{2m}(\varphi) = F(\varphi) \dotplus \cdots \dotplus F(\varphi)\ , \tag{58}$$

$$\Psi_{4m}(r, \varphi) = r\Phi_{2m}(\varphi) \dotplus r^{-1}\Phi_{2m}(\varphi)\ . \tag{59}$$

In (58) there are m direct summands $F(\varphi)$.

Suppose that (56) holds. From Theorem 4.1 we can conclude a good deal about the structure of N. The major hurdle to be overcome is to show that if a not diagonal block of type $F(\varphi)$ occurs in N, it does so with even multiplicity. We have

$$N^{-1T} = (KS)^{-1}N(KS)\ . \tag{60}$$

Thus the eigenvalues of N appear in reciprocal pairs. Thus, after a simultaneous orthogonal similarity of N, K, S, we may assume that

$$\begin{aligned} N = I_\alpha \dotplus -I_\beta \dotplus \sum_{i=1}^{u}{}^{\cdot}\, \Omega_{2m_i}(r_i) \dotplus \sum_{i=1}^{v}{}^{\cdot}\, \Omega_{2k_i}(-s_i) \\ \dotplus \sum_{i=1}^{w}{}^{\cdot}\, \Phi_{2p_i}(\varphi_i) \dotplus \sum_{i=1}^{t}{}^{\cdot}\, \Psi_{4q_i}(R_i, \theta_i)\ . \end{aligned} \tag{61}$$

In (61) we have separated the various types of blocks according to the character of their eigenvalues, as follows: I_α has eigenvalue $+1$; $-I_\beta$ has eigenvalue -1; each $r_i > 1$ and $r_i \neq r_j$ if $i \neq j$, $1 \leqq i$, $j \leqq u$; each $s_i > 1$ and $s_i \neq s_j$ if $i \neq j$, $1 \leqq i$, $j \leqq v$; each $\Phi_{2p_i}(\varphi_i)$ has nonreal eigenvalues on the unit circle and $\Phi_{2p_i}(\varphi_i)$, $\Phi_{2p_j}(\varphi_j)$ do not have a common eigenvalue for $i \neq j$, $1 \leqq i$, $j \leqq w$; each $\Psi_{4q_i}(R_i, \theta_j)$ has nonreal eigenvalues not on the unit circle and $\Psi_{4q_i}(R_i, \theta_i)$, $\Psi_{4q_j}(R_j, \theta_j)$ do not have a common eigenvalue for $i \neq j$, $1 \leqq i$, $j \leqq t$. Thus in (61) distinct direct summands do not have a common eigenvalue. From (61) follows

$$(62)\qquad \begin{aligned} N^{-1T} = I_\alpha \dotplus -I_\beta \dotplus \sum_{i=1}^{u}{}^{\boldsymbol\cdot}\, \Omega_{2m_i}(r_i^{-1}) \dotplus \sum_{i=1}^{v}{}^{\boldsymbol\cdot}\, \Omega_{2k_i}(-s_i^{-1}) \\ \dotplus \sum_{i=1}^{w}{}^{\boldsymbol\cdot}\, \Phi_{2p_i}(\varphi_i) \dotplus \sum_{i=1}^{t}{}^{\boldsymbol\cdot}\, \Psi_{4q_i}(R_i^{-1}, \theta_i) \,. \end{aligned}$$

From (60), we get $(KS)N^{-1T} = N(KS)$, and then (61) and (62) force a partitioning on KS, as follows:

$$(63)\qquad \begin{aligned} KS = A_\alpha \dotplus B_\beta \dotplus \sum_{i=1}^{u}{}^{\boldsymbol\cdot} \begin{bmatrix} 0 & C_{m_i} \\ \Gamma_{m_i} & 0 \end{bmatrix} \dotplus \sum_{i=1}^{v}{}^{\boldsymbol\cdot} \begin{bmatrix} 0 & D_{k_i} \\ \Delta_{k_i} & 0 \end{bmatrix} \dotplus \sum_{i=1}^{w}{}^{\boldsymbol\cdot}\, E_{2p_i} \\ \dotplus \sum_{i=1}^{t}{}^{\boldsymbol\cdot} \begin{bmatrix} 0 & F_{2q_i} \\ \mathscr{F}_{2q_i} & 0 \end{bmatrix}, \end{aligned}$$

where we also have

$$(64)\qquad E_{2p_i}\Phi_{2p_i}(\varphi_i) = \Phi_{2p_i}(\varphi_i)E_{2p_i}\,, \qquad 1 \leqq i \leqq w\,.$$

Taking the transpose of each side of (63) yields

$$(65)\qquad \begin{aligned} SK = -A_\alpha^T \dotplus -B_\beta^T \dotplus \sum_{i=1}^{u}{}^{\boldsymbol\cdot} \begin{bmatrix} 0 & -\Gamma_{m_i}^T \\ -C_{m_i}^T & 0 \end{bmatrix} \\ \dotplus \sum_{i=1}^{v}{}^{\boldsymbol\cdot} \begin{bmatrix} 0 & -\Delta_{k_i}^T \\ -D_{k_i}^T & 0 \end{bmatrix} \dotplus \sum_{i=1}^{w}{}^{\boldsymbol\cdot} -E_{2p_i}^T \\ \dotplus \sum_{i=1}^{t}{}^{\boldsymbol\cdot} \begin{bmatrix} 0 & -\mathscr{F}_{2q_i}^T \\ -F_{2q_i}^T & 0 \end{bmatrix}. \end{aligned}$$

The equation $NKS = SK$ now yields a number of equations, of which we single out the following:

$$(66)\qquad A_\alpha = -A_\alpha^T\,,$$

$$(67)\qquad B_\beta = B_\beta^T\,,$$

$$(68)\qquad \Phi_{2p_i}(\varphi_i)E_{2p_i} = -E_{2p_i}^T\,, \qquad 1 \leqq i \leqq w\,.$$

Because of (66), the eigenvalues of A_α occur in pure imaginary pairs $\pm ri$, r real. By Lemma 3.4, each of the blocks

$$\begin{bmatrix} 0 & -\Gamma_{m_i}^T \\ -C_{m_i}^T & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\Delta_{k_i}^T \\ -D_{k_i}^T & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\mathscr{F}_{2q_i}^T \\ -F_{2q_i}^T & 0 \end{bmatrix}$$

has its eigenvalues in sets of the types: $\pm r$ (r real); $\pm ri$ (r real);

λ, $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$, (λ neither real nor pure imaginary). By Lemma 3.2 the eigenvalues of SK partition into sets of these three types. Hence the eigenvalues of

$$-B_\beta^T \dotplus \sum_{i=1}^{w}{}^{\cdot} -E_{2p_i}^T \tag{69}$$

must also partition into sets of these three types.

Because of (67), the eigenvalues of $-B_\beta^T$ are real.

We wish now to discuss the eigenvalues of E_{2p_i}. To relieve the notation let

$$E_{2p} = E_{2p_i}\ , \qquad \Phi_{2p}(\varphi) = \Phi_{2p_i}(\varphi_i)\ , \qquad i \text{ fixed}\ .$$

Because of (64) and (68), we have

$$E_{2p}\Phi_{2p}(\varphi) = \Phi_{2p}(\varphi)E_{2p}\ , \tag{70}$$

$$\Phi_{2p}(\varphi)E_{2p} = -E_{2p}^T\ . \tag{71}$$

We may make a simultaneous unitary similarity of E_{2p} and $\Phi_{2p}(\varphi)$ so that $\Phi_{2p}(\varphi)$ is converted to $e^{i\varphi}I_p \dotplus e^{-i\varphi}I_p$. Because of (70), E_{2p} becomes $E'_p \dotplus E''_p$. Owing to (71), we have

$$e^{i\varphi}E'_p = -E'^*_p\ , \tag{72}$$

$$e^{-i\varphi}E''_p = -E''^*_p\ . \tag{73}$$

Because of (72) and (73), E'_p and E''_p are normal. Unitary similarities of (72) and (73) render E'_p and E''_p diagonal. Using (72) and (73) again, we find

$$E'_p = \operatorname{diag}(\varepsilon'_1\rho'_1 ie^{-i\varphi/2}, \cdots, \varepsilon'_p\rho'_p ie^{-i\varphi/2})\ ,$$
$$E''_p = \operatorname{diag}(\varepsilon''_1\rho''_1 ie^{i\varphi/2}, \cdots, \varepsilon''_p\rho''_p ie^{i\varphi/2})\ ,$$

where each ε is ± 1 and each $\rho > 0$. Restoring subscripts, we have that E_{2p_j} is unitarily similar to $E'_{p_j} \dotplus E''_{p_j}$, where

$$E'_{p_j} = \operatorname{diag}(\varepsilon'_{j1}\rho'_{j1} ie^{-i\varphi_j/2}, \cdots, \varepsilon'_{jp_j}\rho'_{jp_j} ie^{-i\varphi_j/2})\ , \tag{74}$$

$$E''_{p_j} = \operatorname{diag}(\varepsilon''_{j1}\rho''_{j1} ie^{i\varphi_j/2}, \cdots, \varepsilon''_{jp_j}\rho''_{jp_j} ie^{i\varphi_j/2})\ . \tag{75}$$

We ask: can it happen that E_{2p_j} has a real eigenvalue? If so, for some choice of the $\pm$ signs,

$$\pm ie^{\pm i\varphi_j/2} = \pm 1\ ,$$

hence

$$e^{i\varphi_j} = -1\ .$$

This is not so owing to the classification of eigenvalues made in (61). We ask: can it happen that E_{2p_j} has a pure imaginary eigenvalue? If so

$$\pm ie^{\pm i\varphi_j/2} = \pm i ,$$

hence

$$e^{i\varphi_j} = 1 .$$

Again, this is not so because of the choices made in (61). We ask: if λ is an eigenvalue of E_{2p_j}, can any of λ, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ be an eigenvalue of E_{2p_s}, for $s \neq j$? If so

$$\pm ie^{\pm i\varphi_j/2} = \pm ie^{\pm i\omega_s/2} ,$$

hence

$$e^{i\omega_j} = e^{\pm i\omega_s} .$$

This means that $\Phi_{2p_j}(\varphi_j)$ and $\Phi_{2p_s}(\varphi_s)$ have a common eigenvalue, which is not so.

Now we know that the eigenvalues of (69) partition into sets of the three types: $\pm r$ (r real); $\pm ri$ (r real); λ, $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$ (λ not real or pure imaginary). But $-B_\beta^T$ can have only real eigenvalues and the $E_{2p_j}^T$ can have only eigenvalues not on the real or imaginary axes. Thus each of the direct sums in (69) must have its eigenvalues classify into sets of the three types, with only the type $\pm r$ (r real) possible for $-B_\beta^T$, and only the type λ, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ (λ not real or pure imaginary) possible for each E_{2p_j}. Thus degree B_β is even and degree $E_{2p_j} \equiv 0 \pmod 4$. Hence each p_i is even.

Thus we know in (61) that β is even and each p_i is even. Since degree N is even, it follows that α is even also.

Now, in (61), the direct summands I_α and $\Omega_{2m_i}(r_i)$, $1 \leqq i \leqq u$, can be classified under type (7) (possibly $r = 1$ in type (7)). The direct summands $-I_\beta$ and $\Omega_{2k_i}(-s_i)$, $1 \leqq i \leqq v$, can be classified under type (8) (possibly $r = 1$ in type (8)). Because p_i in even, the direct summand $\Phi_{2p_i}(\varphi_i)$ can be classified as $p_i/2$ copies of the type (53) (with $r = 1$ in (53)); $1 \leqq i \leqq w$. And the direct summand $\Psi_{4q_i}(R_i, \varphi_i)$ can also be classified as a direct sum of q_i copies of type (53); $1 \leqq i \leqq t$. We have thus established that the condition of the theorem is necessary.

To establish the converse, it suffices to assume that N is (7), or (8), or (53). If $N = \operatorname{diag}(r, r^{-1})$, set $S = \operatorname{diag}(r, 1)$ and

(76) $$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Then (56) holds, N and S are commutative, and K is orthogonal and skew symmetric. If $N = \text{diag}\,(-r, -r^{-1})$, set $S = \text{diag}\,(-r, 1)$, define K by (76). Then again (56) holds, S is symmetric and commutative with N, and K is orthogonal and skew. If N is given by (53) set $S = \text{diag}\,(G(\psi), r^{-1}G(\varphi + 2\theta - \psi))$, and put

$$K = \begin{bmatrix} 0 & G(\theta) \\ -G(\theta) & 0 \end{bmatrix}.$$

Using Lemma 3.3 one easily computes that for any choice of the angles ψ and θ, we have $NKS = SK$. This S is symmetric (and also orthogonal if $r = 1$) and this K is skew orthogonal. The proof is complete.

THEOREM 5.8. *Let N be real and normal. If N is a commutator (56) of a symmetric S and a skew symmetric K with*

$$NS = SN \tag{77}$$

then N is symmetric with all eigenvalues (including 1) occurring as reciprocal pairs γ, γ^{-1}. Conversely, this condition is satisfied then N can be expressed as a commutator (56) such that (77) holds and such that K is orthogonal and skew.

Proof. Suppose that (56) and (77) hold. If we write $N = S(iK)S^{-1}(iK)^{-1}$ then we may deduce from Theorem 4.2 that N is a direct sum of types (7) and (8). The converse was established in the proof of Theorem 5.7.

THEOREM 5.9. *Let N be real and normal. Then N is a commutator (56) of a symmetric S and a skew symmetric K such that*

$$NK = KN \tag{78}$$

if and only if N is orthogonally similar to a direct sum of the following three types (79), (80), (81):

$$\text{diag}\,(1, 1)\,, \tag{79}$$

$$\text{diag}\,(-1, -1)\,, \tag{80}$$

$$\text{diag}\,(r, r, r^{-1}, r^{-1})\,, \qquad r \neq 0, 1, -1\,. \tag{81}$$

If this condition is satisfied, S may always be taken orthogonal symmetric.

Proof. Suppose (56) and (78) hold. Then we have $N =$

$S(iK)S^{-1}(iK)^{-1}$ and $N(iK) = (iK)N$, so that by Theorem 4.2 N is symmetric. Thus by Theorem 5.7 we may assume

$$N = I_\alpha \dotplus -I_\beta \dotplus \sum_{i=1}^{u}{}^{\cdot}\, \Omega_{2m_i}(r_i) \dotplus \sum_{i=1}^{v}{}^{\cdot}\, \Omega_{2k_i}(-s_i) ,$$

where the r_i are distinct, each $r_i > 1$, the s_i are distinct, and each $s_i > 1$. Then $NK = KN$ yields

$$K = K_\alpha \dotplus K_\beta \dotplus \sum_{i=1}^{u}{}^{\cdot} \begin{bmatrix} Q_{m_i} & 0 \\ 0 & \tilde{Q}_{m_i} \end{bmatrix} \dotplus \sum_{i=1}^{v}{}^{\cdot} \begin{bmatrix} T_{k_i} & 0 \\ 0 & \tilde{T}_{k_i} \end{bmatrix} .$$

Since K is skew each Q_{m_i} and each T_{k_i} is skew and nonsingular, hence has even degree. Thus each m_i and each k_i is even. Thus the conditions of the theorem are necessary.

For the converse it suffices to consider two cases: $N = \operatorname{diag}(-1, -1)$ and $N = \operatorname{diag}(x, x, x^{-1}, x^{-1})$ with x positive or negative. Now

$$\operatorname{diag}(-1, -1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \tag{82}$$

and $\operatorname{diag}(x, x, x^{-1}, x^{-1}) = SKS^{-1}K^{-1}$ where

$$S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}.$$

This completes the proof.

THEOREM 5.10. *Let N be real and normal. If N is a commutator* (56) *of a definite S and a skew K then N is positive definite with its eigenvalues* (*including* 1) *occurring in pairs γ, γ^{-1}. Conversely, is N satisfies these conditions then N is a commutator* (56) *of a definite S commutative with N and a skew orthogonal K.*

Proof. Suppose (56) holds. Then from $N = S(iK)S^{-1}(iK)^{-1}$ one deduces from Theorem 4.3 that N is positive definite with the eigenvalues γ of N for which $\gamma \neq 1$ occurring in pairs γ, γ^{-1}. Since degree N is even, the multiplicity of the eigenvalue $\gamma = 1$ is even, hence this eigenvalue also occurs in reciprocal pairs. For the converse it suffices to observe that

$$\begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & r^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} .$$

THEOREM 5.11. *Let N be real and normal. Then if N is a commutator* (56) *of a definite S and a skew K such that N and K commute, then $N = I$.*

Proof. This follows from Theorem 4.4 or Theorem 1 of [5].

THEOREM 5.12. *Let $\mathscr{O}$ be proper orthogonal. Then if $\mathscr{O}$ is a commutator of a symmetric S and a skew K,*

$$\mathscr{O} = SKS^{-1}K^{-1}\,, \tag{83}$$

it follows that each eigenvalue of $\mathscr{O}$ has even multiplicity. If S commutes with $\mathscr{O}$ or if K commutes with $\mathscr{O}$ then $\mathscr{O}$ is also symmetric. If S is definite then $\mathscr{O} = I$. Conversely, if each eigenvalue of $\mathscr{O}$ has even multiplicity, $\mathscr{O}$ is a commutator (83) *with S symmetric orthogonal and K shew orthogonal, and if $\mathscr{O}$ is symmetric we may also make both S and K commutative with $\mathscr{O}$.*

Proof. Suppose (83) holds. Then by Theorem 5.7 $\mathscr{O}$ is orthogonally similar to a direct sum of blocks of type (7), (8), (53). Since $\mathscr{O}$ is orthogonal, in blocks (7), (8), (53) we have $r = 1$. This shows that each eigenvalue of $\mathscr{O}$ has even multiplicity. The second result in the theorem follows from Theorems 5.8 and 5.9. The third result follows from Theorem 5.10. For the converse note that if each eigenvalue of $\mathscr{O}$ has even multiplicity then $\mathscr{O}$ is orthogonally similar to a direct sum of blocks of the type $\mathrm{diag}(1, 1)$, $\mathrm{diag}(-1, -1)$, $F(\varphi) \dotplus F(\varphi)$. In the proof of Theorem 5.7 it was shown how to express each of these three matrices in the form (83) with both S and K orthogonal. Moreover, if $\mathscr{O}$ is symmetric then $\mathscr{O}$ is orthogonally similar to a direct sum of the types $\mathrm{diag}(1, 1)$ and $\mathrm{diag}(-1, -1)$, and one need only observe (82).

THEOREM 5.13. *Let $\tilde{K}$ be real skew. Then $\tilde{K}$ is a commutator*

$$\tilde{K} = SKS^{-1}K^{-1} \tag{84}$$

of a symmetric S and a skew K if and only if $\tilde{K}$ is orthogonally similar to a direct sum of skew matrices of the type

$$\begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & r^{-1} \\ -r^{-1} & 0 \end{bmatrix}. \tag{85}$$

Here S is never definite and never commutative with $\tilde{K}$, and K is never commutative with $\tilde{K}$. We may, however, make K orthogonal skew.

Proof. These results follow from Theorems 5.7, 5.8, 5.9, and 5.10.

REMARK. Since any skew orthogonal K with degree $K \equiv 0 \pmod 4$ is orthogonally similar to a direct sum of copies of

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

one can apply Theorems 5.12 or 5.13 to K and so build up elaborate iterated commutators of symmetric orthogonal and skew orthogonal matrices.

THEOREM 5.14. *Let S be real symmetric with* $\det S = 1$ *and degree* $S \equiv 0 \pmod 2$. *Then*

$$S = (S_1K_1S_1^{-1}K_1^{-1})(S_2K_2S_2^{-1}K_2^{-1})$$

is a product of two commutators with S_1 and S_2 symmetric and K_1 and K_2 skew orthogonal. If S is positive definite we may also make S_1 and S_2 definite.

Proof. Use Fan's factorization to write S as a product of two symmetric matrices, each of which has its eigenvalues in reciprocal pairs. Apply Theorems 5.7 and 5.10 to the two factors.

We now present a sequence of lemmas which will prepare the way for the proofs of the next theorems.

LEMMA 5.1. *Any 2-square real A with positive determinant can be written as*

$$A = S_1S_2S_3S_4 \tag{86}$$

where S_1, S_2, S_3, S_4 are real symmetric matrices, each with positive determinant.

REMARK. It is known that any real (square) matrix is a product of two real symmetric matrices. However, it will appear below that the two factors cannot always be chosen to have positive determinant.

Proof. From (86) follows

$$RAR^{-1} = (RS_1R^T)(R^{-1T}S_2R^{-1})(RS_3R^T)(R^{-1T}S_4R^{-1}) .$$

Thus it suffices to establish the lemma for some similarity transform (over the reals) of A. If A is scalar, $A = \alpha I$, then take $S_1 = \alpha I$, $S_2 = S_3 = S_4 = I$. If A is not scalar it is nonderogatory, hence we may suppose

$$A = \begin{bmatrix} 0 & 1 \\ -a & 2\rho \end{bmatrix}, \qquad a > 0 .$$

First let $\rho \neq 0$. Put $x = \rho(2a)^{-1/2}$. Put $X = \text{diag}\,(x, x^{-1})$. Then $Y = AX$ has characteristic polynomial $\lambda^2 - 2\rho x^{-1}\lambda + a$, for which the roots are $a^{1/2}(2^{1/2} \pm 1)$. Call these roots δ_1 and δ_2. Both δ_1 and δ_2 are positive, and $\delta_1 \neq \delta_2$. Moreover, $\text{diag}\,(\delta_1, \delta_2)$ is similar to Y. Hence $Y = Q\ \text{diag}\,(\delta_1, \delta_2)Q^{-1}$. Therefore

$$Y = \{Q\,\text{diag}\,(\delta_1, \delta_2)Q^T\}\{Q^{-1T}Q^{-1}\}$$

is a product of two symmetric matrices, each of which has positive determinant. The $A = YX^{-1}I$ is a product of four symmetric matrices, each with positive determinant.

Now let $\rho = 0$. Note that

$$a^{1/2}\begin{bmatrix} 2 & 0 \\ 0 & 2^{-1} \end{bmatrix}\begin{bmatrix} -1 & -5 \\ 1 & 4 \end{bmatrix} = a^{1/2}\begin{bmatrix} -2 & -10 \\ -2^{-1} & 0 \end{bmatrix}.$$

Here $\text{diag}\,(2a^{1/2}, 2^{-1}a^{1/2})$ is a product of two symmetric matrices, each with positive determinant. And

$$B = \begin{bmatrix} -1 & -5 \\ 1 & 4 \end{bmatrix}$$

has characteristic polynomial $\lambda^2 - 3\lambda + 1$, hence is similar to a diagonal matrix B_1 with positive diagonal entries, say $B = RB_1R^{-1} = (RB_1R^T)(R^{-1T}R^{-1})$. Thus B is a product of two symmetric matrices with positive determinant. Finally,

$$a^{1/2}\begin{bmatrix} -2 & -10 \\ 2^{-1} & 2 \end{bmatrix}$$

has characteristic polynomial $\lambda^2 + a$, hence is similar to

$$\begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}.$$

This completes the proof of the lemma.

LEMMA 5.2. *Let $\mathcal{O}$ be proper orthogonal. Then $\mathcal{O} = S_1S_2S_3S_4$ where each S_i is real symmetric and has its eigenvalues in reciprocal pairs*; $i = 1, 2, 3, 4$.

Proof. It suffices to establish this factorization when $\mathscr{O} = F(\varphi)$. By Lemma 5.1, $F(\varphi) = S_1'S_2'S_3'S_4'$ when each S_i' is real symmetric with $\det S_i' > 0$, $i = 1, 2, 3, 4$. Since $\det F(\varphi) = 1$,

$$(\det S_1')(\det S_2')(\det S_3')(\det S_4') = 1 .$$

Let $S_i = (\det S_i')^{-1/2}S_i'$, $1 \leqq i \leqq 4$. Then $\mathscr{O} = S_1S_2S_3S_4$, each S_i is real symmetric and has determinant one, hence its eigenvalues occur in reciprocal pairs; $1 \leqq i \leqq 4$.

LEMMA 5.3. *Let* 2×2 *real* A *satisfy* $\det A = 1$. *Then* A *can be factored as in* (86) *when each* S_i *is real symmetric and* $\det S_i = 1$, $1 \leqq i \leqq 4$.

Proof. Apply Lemma 5.1 to A and insert scalar factors as in the proof of Lemma 5.2.

LEMMA 5.4. *Let* S_1, S_2, S_3 *be real symmetric with positive determinant. Then*

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = S_1S_2S_3 \tag{87}$$

is impossible.

Proof. Suppose (87) holds. Let

$$S_3^{-1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} .$$

Then from (87) we get

$$\begin{bmatrix} b & c \\ -a & -b \end{bmatrix} = S_1S_2 . \tag{88}$$

The left member of (88) has zero trace. On the right side of (88), S_1 and S_2 each are definite (since each has positive determinant). By inserting two factors -1, we may take S_1 positive definite. Then S_1S_2 has the same eigenvalues as $S_1^{1/2}S_2S_1^{1/2}$. Thus, by the law of inertia, both eigenvalues of S_1S_2 are positive, or both are negative. Hence $\operatorname{tr} S_1S_2 = 0$ is impossible.

THEOREM 5.15. *Let* A *be real and* $\det A = 1$.

(i) *If* A *is* 2-*square then* A *is a product of four commutators*

$$A = \prod_{i=1}^{4} (S_iK_iS_i^{-1}K_i^{-1})$$

where each S_i *is real symmetric and each* K_i *is real skew orthogonal.*

(ii) *If* A *is* $2n$*-square with* $n > 1$, A *is a product of six commutators*

$$A = \prod_{i=1}^{6} (S_i K_i S_i^{-1} K_i^{-1}) \tag{89}$$

where each S_i *is real symmetric and each* K_i *is real skew orthogonal.*

(iii) *It is impossible that*

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \prod_{i=1}^{3} (S_i K_i S_i^{-1} K_i^{-1}) \tag{90}$$

where each S_i *is real symmetric and each* K_i *is real skew.*

Proof. (i) By Lemma 5.3, $A = S_1 S_2 S_3 S_4$ where each S_i is real symmetric with $\det S_i = 1$. By Theorem 5.7 each S_i is a commutator of a real symmetric matrix with a real skew orthogonal matrix.

(ii) Let $A = \mathcal{O} S$ be the polar factorization of A. Then $\det \mathcal{O} = \det S = 1$. By Theorem 5.14,

$$S = (S_1 K_1 S_1^{-1} K_1^{-1})(S_2 K_2 S_2^{-1} K_2^{-1})$$

where S_1 and S_2 are real symmetric and K_1 and K_2 are real skew orthogonal. By Lemma 5.2, $\mathcal{O} = S_3' S_4' S_5' S_6$ where S_3', S_4', S_5', S_6' are each symmetric with eigenvalues occurring in reciprocal pairs. By Theorem 5.7 we have

$$S_i' = S_i K_i S_i^{-1} K_i^{-1}, \qquad 3 \leqq i \leqq 6,$$

with each S_i real symmetric and each K_i real skew orthogonal.

(iii) First note that for 2×2 matrices, if S is symmetric and K is skew, then $SKS^{-1}K^{-1}$ is symmetric. Thus, if (90) were true, the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

would be a product of three symmetric matrices, each with positive determinant. This contradicts Lemma 5.4.

This completes our discussion of commutators of the form $SKS^{-1}K^{-1}$. The next natural question is to discuss commutators of the form $K_1 K_2 K_1^{-1} K_2^{-1}$. This discussion is contained in Part II of this paper.

6. The commutator of a normal and a unitary matrix. In this section we give the following theorem, first proved by Fan.

THEOREM 6.1. *A normal matrix N with* $\det N = 1$ *is a commutator*

$$N = N_1 U N_1^{-1} U^{-1}$$

where U is unitary and N_1 is normal and commutative with N. If N is Hermitian, positive definite Hermitian, or unitary, we may, in addition, choose N_1 to be Hermitian, positive definite Hermitian, or unitary, respectively. If N is real symmetric or symmetric positive definite we may choose U to be real orthogonal and N_1 to be symmetric or symmetric definite, respectively, and still commutative with N.

Proof. Let $N = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Then put

$$N_1^{-1} = \operatorname{diag}(1, \lambda_1, \lambda_1\lambda_2, \cdots, \lambda_1\lambda_2 \cdots \lambda_{n-1}) .$$

Put $U = [1, 1, \cdots, 1]_n$. Then $N_1^{-1} = UN_1^{-1}U^{-1}$. Hence $N = N_1UN_1^{-1}U^{-1}$, and N_1 commutes with N. The other assertions of the theorem follow easily.

7. The commutator of a Hermitian matrix and a unitary matrix.

THEOREM 7.1. *Let N be normal. Then N is a commutator of a Hermitian H and a unitary U,*

$$N = HUH^{-1}U^{-1} \tag{91}$$

if and only if:

(i) *The characteristic polynomial of N is real. Let*

$$\lambda_1, \bar{\lambda}_1, \cdots, \lambda_t, \bar{\lambda}_t$$

be the nonreal eigenvalues of N, and let $\lambda_{t+1}, \cdots, \lambda_k$ be the real eigenvalues.

(ii) *Nonzero real numbers $h_1, h_2, \cdots, h_t, h_{t+1}, \cdots, h_k$ exist such that the numbers*

$$\begin{gathered} |\lambda_1| h_1, -|\lambda_1| h_1, |\lambda_2| h_2, -|\lambda_2| h_2, \cdots, |\lambda_t| h_t, -|\lambda_t| h_t , \\ \lambda_{t+1}h_{t+1}, \lambda_{t+2}h_{t+2}, \cdots, \lambda_k h_k \end{gathered} \tag{92}$$

are the same as the numbers

$$h_1, -h_1, h_2, -h_2, \cdots, h_t, -h_t, h_{t+1}, h_{t+2}, \cdots, h_k , \tag{93}$$

except for order.

If N is real and conditions (i) *and* (ii) *hold, we may take H to be real symmetric and U to be real orthogonal.*

Proof. Suppose (91) holds. Then

$$N^* = H^{-1}NH . \tag{94}$$

Thus if λ is a nonreal eigenvalue of N with a certain multiplicity, $\bar{\lambda}$ is also an eigenvalue of N, with the same multiplicity. Thus after a unitary similarity of (91), we may assume

$$N = \sum_{i=1}^{t}{}^{\cdot} (\gamma_i I_i + \bar{\gamma}_i I_i) + \sum_{i=t+1}^{k}{}^{\cdot} \rho_i I_i , \tag{95}$$

where the γ_i, $\bar{\gamma}_i$ are nonreal and distinct, $1 \leqq i \leqq t$, and the ρ_i are real and distinct, $t < i \leqq k$. Then using (95), $HN^* = NH$ yields

$$H = \sum_{i=1}^{t}{}^{\cdot} \begin{bmatrix} 0 & M_i \\ M_i^* & 0 \end{bmatrix} + \sum_{i=t+1}^{k}{}^{\cdot} H_i . \tag{96}$$

Thus

$$N^{-1}H = \sum_{i=1}^{t}{}^{\cdot} \begin{bmatrix} 0 & \gamma_i^{-1}M_i \\ \bar{\gamma}_i^{-1}M_i^* & 0 \end{bmatrix} + \sum_{i=t+1}^{k}{}^{\cdot} \rho_i^{-1}H_i . \tag{97}$$

Since $N^{-1}H = UHU^{-1}$, $N^{-1}H$ and H have the same eigenvalues. Let $h_{i1}^2, h_{i2}^2, \cdots$ be the eigenvalues of $M_iM_i^*$, $1 \leqq i \leqq t$, and let $h_{i1}, h_{i2}, \cdots$ be the eigenvalues of H_i, $t < i \leqq k$. Using Lemma 3.5 we find that the eigenvalues of $N^{-1}H$ are

$$\begin{gathered} |\gamma_1|^{-1}h_{11},\ -|\gamma_1|^{-1}h_{11},\ |\gamma_1|^{-1}h_{12},\ -|\gamma_1|^{-1}h_{12},\ \cdots , \\ |\gamma_t|^{-1}h_{t1},\ -|\gamma_t|^{-1}h_{t1},\ |\gamma_t|^{-1}h_{t2},\ -|\gamma_t|^{-1}h_{t2},\ \cdots , \\ \rho_{t+1}^{-1}h_{t+1,1},\ \rho_{t+1}^{-1}h_{t+1,2},\ \cdots,\ \rho_k^{-1}h_{k1},\ \rho_k^{-1}h_{k2},\ \cdots , \end{gathered}$$

and the eigenvalues of H are

$$\begin{gathered} h_{11},\ -h_{11},\ h_{12},\ -h_{12},\ \cdots,\ h_{t1},\ -h_{t1},\ h_{t2},\ -h_{t2},\ \cdots , \\ h_{t+1,1},\ h_{t+1,2},\ \cdots,\ h_{k1},\ h_{k2},\ \cdots . \end{gathered}$$

After taking inverses and changing notation, we obtain that the second condition of the theorem is necessary.

Conversely, the conditions of the theorem imply that nonzero real numbers $h_1, h_2, \cdots, h_t, \cdots, h_k$ exist such that the numbers

$$\pm|\lambda_1|^{-1}h_1,\ \pm|\lambda_2|^{-1}h_2,\ \cdots,\ \pm|\lambda_t|^{-1}h_t,\ \lambda_{t+1}^{-1}h_{t+1},\ \lambda_{t+2}^{-1}h_{t+2},\ \cdots,\ \lambda_k^{-1}h_k \tag{98}$$

are a rearrangement of the numbers

$$\pm h_1,\ \pm h_2,\ \cdots,\ \pm h_t,\ h_{t+1},\ h_{t+2},\ \cdots,\ h_k . \tag{99}$$

Let $\lambda_j = r_j \exp(-i\varphi_j)$, $1 \leqq j \leqq t$. After a unitary similarity we may assume

$$N = \sum_{i=1}^{t}{}^{\cdot}\, r_i F(\varphi_i) \dotplus \operatorname{diag}(\lambda_{t+1}, \cdots, \lambda_k) .$$

Let

$$H = \sum_{i=1}^{t}{}^{\cdot} \begin{bmatrix} 0 & h_i \\ h_i & 0 \end{bmatrix} \dotplus \operatorname{diag}(h_{t+1}, \cdots, h_k) .$$

The eigenvalues of H are the numbers (99). We find that

$$N^{-1}H = \sum_{i=1}^{t}{}^{\cdot}\, r_i^{-1} h_i G(-\varphi_i) \dotplus \operatorname{diag}(\lambda_{t+1}^{-1} h_{t+1}, \cdots, \lambda_k^{-1} h_k) .$$

The eigenvalues of $N^{-1}H$ are the numbers (98). Since $N^{-1}H$ and H are two real symmetric matrices with the same eigenvalues, an orthogonal $\mathcal{O}$ exists such that $N^{-1}H = \mathcal{O} H \mathcal{O}^{-1}$. Hence $N = H\mathcal{O}H^{-1}\mathcal{O}^{-1}$, as required.

THEOREM 7.2. *Suppose normal N is a commutator* (91) *of a Hermitian H and a unitary U, such that*

$$NH = HN . \tag{100}$$

Then N is Hermitian and $\det N = 1$. *The converse assertion is contained in Theorem* 6.1.

Proof. From (94) and (100) follows $N^* = N$.

THEOREM 7.3. (i) *Let N be normal. If N is a commutator* (91) *of a Hermitian H and a unitary U such that*

$$NU = UN , \tag{101}$$

then N is unitary, N has real characteristic polynomial, and $\det N = 1$. *Conversely, if N is unitary with real characteristic polynomial and* $\det N = 1$, *then N is a commutator* (91) *with H Hermitian unitary and U unitary and commutative with N.*

(ii) *Let $\mathcal{O}$ be proper orthogonal. Then $\mathcal{O}$ is a commutator $\mathcal{O} = S\mathcal{O}_1 S^{-1}\mathcal{O}_1^{-1}$ of a symmetric orthogonal S and a proper orthogonal $\mathcal{O}_1$ with $\mathcal{O}_1$ commutative with $\mathcal{O}$.*

REMARK. For unitary matrices, Theorem 7.3 improves Theorem 7.1.

Proof. Suppose (91) and (101) hold. Then, as in the proof of Theorem 7.1 we obtain (95) and (96). Because of (101),

$$U = \sum_{i=1}^{t}{}^{\cdot} \operatorname{diag}(U_i, \tilde{U}_i) \dotplus \sum_{i=t+1}^{k}{}^{\cdot}\, U_i .$$

Then $NUH = HU$ yields

$$\sum_{i=1}^{t}{}^{\cdot} \begin{bmatrix} 0 & \gamma_i U_i M_i \\ \bar{\gamma}_i \tilde{U}_i M_i^* & 0 \end{bmatrix} + \sum_{i=t+1}^{k}{}^{\cdot} \rho_i U_i H_i$$

$$(102) \qquad = \sum_{i=1}^{t}{}^{\cdot} \begin{bmatrix} 0 & M_i \tilde{U}_i \\ M_i^* U_i & 0 \end{bmatrix} + \sum_{i=t+1}^{k}{}^{\cdot} H_i U_i \,.$$

Comparing the two sides of (102), we obtain

$$|\gamma_i|^2 M_i^{-1} U_i M_i = M_i^* U_i M_i^{*-1} \,, \qquad 1 \leqq i \leqq t \,,$$

hence, by taking determinants, $|\gamma_i| = 1$; $1 \leqq i \leqq t$. We also get $\rho_i U_i H_i = H_i U_i$, hence by taking determinants, we find $\rho_i = \pm 1$. This proves that N is unitary. By Theorem 7.1 we already know that N has real characteristic polynomial.

To establish the converse we notice that if N is unitary with real characteristic polynomial and $\det N = 1$, then N is unitarily similar to a direct sum of copies of $F(\varphi)$ and an identity matrix. We therefore need only notice that by Lemma 3.3

$$F(\varphi) = G(\theta)F(-\varphi/2)G(\theta)^{-1}F(-\varphi/2)^{-1}$$

for any choice of θ, and $F(\varphi)$ and $F(-\varphi/2)$ commute. Here $G(\theta)$ is, of course, symmetric orthogonal.

THEOREM 7.4. *Let N be normal. If N is a commutator* (91) *of a definite H and a unitary U then N is positive definite Hermitian and* $\det N = 1$. *The converse assertion is contained in Theorem* 6.1.

Proof. Since $N = H(UH^{-1}U^{-1})$ is a product of the two positive definite Hermitian matrices H and UHU^*, it follows from Lemma 3.1 (iii) that N has all eigenvalues positive. Therefore N is positive definite Hermitian.

THEOREM 7.5. *Let normal N be a commutator* (91) *of a definite H and a unitary U such that* (101) *holds. Then $N = I$.*

Proof. By Theorem 7.4 N is positive definite. By Theorem 7.3 N is unitary. Hence $N = I$. Theorem 7.5 is a special case of Theorem 1 of [4].

THEOREM 7.6. *Let K be real skew with* $\det K = 1$. *Then K is a commutator,*

$$(103) \qquad K = S\mathcal{O}S^{-1}\mathcal{O}^{-1}$$

with S real symmetric and $\mathscr{O}$ orthogonal. Moreover S is never definite and never commutative with K. Ω can be chosen to be commutative with K if and only if K is also orthogonal.

Proof. Let $\pm r_1 i, \pm r_2 i, \cdots, \pm r_t i$ be the eigenvalues of K, with $r_1, r_2, \cdots, r_t$ each positive. Then $\det K = 1$ implies $r_1 r_2 \cdots r_t = 1$. Let $h_1 = 1$, $h_2 = r_1$, $h_3 = r_1 r_2, \cdots, h_t = r_1 r_2 \cdots r_{t-1}$. Then the numbers $\pm r_1 h_1, \pm r_2 h_2, \cdots, \pm r_t h_t$ are a rearrangement of the numbers $\pm h_1, \pm h_2, \cdots, \pm h_t$. Apply Theorems 7.1–7.5.

THEOREM 7.7. *Let θ be a nonreal number with $|\theta| = 1$. Let H be Hermitian. If*

$$\theta H = H_1 U H_1^{-1} U^{-1} \tag{104}$$

is a commutator of Hermitian H_1 and unitary U then $\theta = \pm i$ and iH is unitarily similar to a real skew symmetric K for which $\det K = 1$.

Proof. Suppose (104) holds. Then, by Theorem 7.1, for certain eigenvalues λ_1 and λ_2 of H, we have $\theta\lambda_1 = re^{i\varphi}$, $\theta\lambda_2 = re^{-i\varphi}$. Then $\theta(\lambda_1 + \lambda_2) = 2r\cos\varphi$. This implies θ is real unless it happens that $\lambda_2 = -\lambda_1$ and $\varphi = \pm\pi/2$. Then it must be true that $\theta = \pm i$. Moreover it follows that if λ_1 is an eigenvalue of H with a certain multiplicity, $-\lambda_1$ is also an eigenvalue with the same multiplicity. Thus, after a change of notation, iH is unitarily similar to

$$\operatorname{diag}(r_1 i, -r_1 i, r_2 i, -r_2 i, \cdots, r_t i, -r_t i),$$

which in turn is unitarily similar to

$$K = \sum_{j=1}^{t}\cdot \begin{bmatrix} 0 & r_j \\ -r_j & 0 \end{bmatrix}.$$

THEOREM 7.8. *Let U be unitary with* $\det U = 1$. *Then*

$$U = (H_1 U_1 H_1^{-1} U_1^{-1})(H_2 U_2 H_2^{-1} U_2^{-1}) \tag{105}$$

is a product of two commutators, with H_1 and H_2 Hermitian unitary and U_1 and U_2 unitary. If U is 2-square, one commutator suffices in (105).

Proof. By Fan's factorization $U = VW$ where the eigenvalues of V and W occur in reciprocal pairs. Apply Theorem 7.3 to V and W. If U is 2-square the eigenvalues of U must appear in reciprocal pairs.

THEOREM 7.9. *Let A be complex with* $\det A = 1$. *Then*

$$A = (H_1 U_1 H_1^{-1} U_1^{-1})(H_2 U_2 H_2^{-1} U_2^{-1})(H_3 U_3 H_3^{-1} U_3^{-1}) \tag{106}$$

where H_1 is positive definite Hermitian, H_2 and H_3 are Hermitian unitary, and U_1, U_2, U_3 are unitary. If A is 2-square, (106) *may be improved to*

$$A = (H_1 U_1 H_1^{-1} U_1^{-1})(H_2 U_2 H_2^{-1} U_2^{-1}) \tag{107}$$

where H_1, U_1, H_2, U_2 *are as just stated. If A is real,* (106) *may be improved to*

$$A = (S_1 \mathcal{O}_1 S_1^{-1} \mathcal{O}_1^{-1})(S_2 \mathcal{O}_2 S_2^{-1} \mathcal{O}_2^{-1}) \ , \tag{108}$$

where S_1 is positive definite symmetric, S_2 is orthogonal symmetric, and $\mathcal{O}_1$ and $\mathcal{O}_2$ are orthogonal.

Proof. Let $A = HU$ be the polar factorization of A. Apply Theorem 6.1 to H and Theorem 7.8 to U. If A is real, write $A = S\mathcal{O}$ and apply Theorem 6.1 to S and Theorem 7.3 to $\mathcal{O}$.

We next investigate commutators of the form $K\mathcal{O}K^{-1}\mathcal{O}^{-1}$.

THEOREM 7.10. *Let N be real and normal. Then N is a commutator,*

$$N = K\mathcal{O}K^{-1}\mathcal{O}^{-1} \tag{109}$$

of a skew K and an orthogonal $\mathcal{O}$ if and only if:

(i) *Each eigenvalue of N has even multiplicity. Let*

$$\lambda_1, \lambda_1, \bar{\lambda}_1, \bar{\lambda}_1, \lambda_2, \lambda_2, \bar{\lambda}_2, \bar{\lambda}_2, \cdots, \lambda_u, \lambda_u, \bar{\lambda}_u, \bar{\lambda}_u \tag{110}$$

be the nonreal eigenvalues of N, and let

$$\lambda_{u+1}, \lambda_{u+1}, \lambda_{u+2}, \lambda_{u+2}, \cdots, \lambda_k, \lambda_k \tag{111}$$

be the real eigenvalues of N.

(ii) *Positive real numbers $h_1, h_2, \cdots, h_u, h_{u+1}, h_{u+2}, \cdots, h_k$ exist such that the numbers*

$$\begin{gathered} |\lambda_1| h_1, |\lambda_1| h_1, |\lambda_2| h_2, |\lambda_2| h_2, \cdots, |\lambda_u| h_u, |\lambda_u| h_u \ , \\ |\lambda_{u+1}| h_{u+1}, |\lambda_{u+2}| h_{u+2}, \cdots, |\lambda_k| h_k \end{gathered} \tag{112}$$

are the same as the numbers

$$h_1, h_1, h_2, h_2, \cdots, h_u, h_u, h_{u+1}, h_{u+2}, \cdots, h_k \ , \tag{113}$$

except for order.

Proof. Suppose that (109) holds. After an orthogonal similarity of N, K, $\mathcal{O}$, we may assume that

$$N = \sum_{i=1}^{n}{}^{\cdot}\, r_i \Phi_{2m_i}(\varphi_i) \dot{+} \sum_{i=u+1}^{v}{}^{\cdot}\, R_i I_{\alpha_i} \dot{+} \sum_{i=v+1}^{w}{}^{\cdot}\, -R_i I_{\alpha_i} \,. \tag{114}$$

Here in (114), and throughout this proof, a subscript on a matrix denotes the degree of the matrix. In (114) the R_i and r_i are positive, distinct direct summands do not have any common eigenvalue, and each $\Phi_{2m_i}(\varphi_i)$ has no real eigenvalue.

From (109) we obtain

$$KN^T = NK \,. \tag{115}$$

From (114) and (115) we obtain a partitioning of K, as follows:

$$K = \operatorname{diag}(K_{2m_1}, K_{2m_2}, \cdots, K_{2m_u}, K_{\alpha_{u+1}}, \cdots, K_{\alpha_w}) \,. \tag{116}$$

In (116) each direct summand is a nonsingular skew matrix; hence in particular $\alpha_{u+1}, \cdots, \alpha_w$ are each even. Thus each R_i and each $-R_i$ has even multiplicity.

We now fix our attention on K_{2m_1}. From (115) K_{2m_1} satisfies

$$K_{2m_1}\Phi_{2m_1}(-\varphi_1) = \Phi_{2m_1}(\varphi_1)K_{2m_1} \,.$$

To relieve the notation, let us drop the subscript 1, and write

$$K_{2m}\Phi_{2m}(-\varphi) = \Phi_{2m}(\varphi)K_{2m} \,. \tag{117}$$

Partition $K_{2m} = (M_{\mu\nu})_{1 \leqq \mu, \nu \leqq m}$ into 2×2 submatrices $M_{\mu\nu}$. Fix momentarily μ and ν, and let

$$M_{\mu\nu} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

Then (117) yields $M_{\mu\nu}F(-\varphi) = F(\varphi)M_{\mu\nu}$, hence $b \sin\varphi = c \sin\varphi$ and $-a \sin\varphi = d \sin\varphi$. Since $\sin\varphi \neq 0$ (because $\Phi_{2m}(\varphi)$ does not have real eigenvalues), we obtain that $b = c$ and $d = -a$. Restoring μ and ν, we thus have

$$M_{\mu\nu} = \begin{bmatrix} a_{\mu\nu} & b_{\mu\nu} \\ b_{\mu\nu} & -a_{\mu\nu} \end{bmatrix} , \qquad 1 \leqq \mu,\ \nu \leqq m \,.$$

Let V be the 2-square unitary matrix

$$V = 2^{-1/2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} .$$

Note that $V^*F(\varphi)V = \operatorname{diag}(e^{i\varphi}, e^{-i\varphi})$, and that

$$V^* M_{\mu\nu} V = \begin{bmatrix} 0 & z_{\mu\nu} \\ \bar{z}_{\mu\nu} & 0 \end{bmatrix}, \qquad 1 \leqq \mu,\ \nu \leqq m\ ,$$

where $z_{\mu\nu} = a_{\mu\nu} - ib_{\mu\nu}$. Let V_{2m} be the direct sum of m copies of V. Then we have

$$V_{2m}^* K_{2m} V_{2m} = \left[\begin{bmatrix} 0 & z_{\mu\nu} \\ \bar{z}_{\mu\nu} & 0 \end{bmatrix}\right]_{1 \leqq \mu, \nu \leqq m}, \tag{118}$$

$$V_{2m}^* \Phi_{2m}(\varphi) V_{2m} = \operatorname{diag}(e^{i\varphi}, e^{-i\varphi}, e^{i\varphi}, e^{-i\varphi}, \cdots, e^{i\varphi}, e^{-i\varphi})\ . \tag{119}$$

Let W_{2m} be the $2m$-square permutation matrix such that for any $2m$-square matrix M, the rows of $W_{2m}^T M W_{2m}$ are the rows of M in the order

$$1, 3, 5, \cdots, 2m - 1, 2, 4, 6, \cdots, 2m\ , \tag{120}$$

and the columns of $W_{2m}^T M W_{2m}$ are the columns of M in the order (120). Then

$$W_{2m}^* V_{2m}^* K_{2m} V_{2m} W_{2m} = \begin{bmatrix} 0 & Z_m \\ \bar{Z}_m & 0 \end{bmatrix}, \tag{121}$$

where $Z_m = (z_{\mu\nu})_{1 \leqq \mu, \nu \leqq m}$, and

$$W_{2m}^* V_{2m}^* \Phi_{2m}(\varphi) V_{2m} W_{2m} = e^{i\varphi} I_m \dotplus e^{-i\varphi} I_m\ . \tag{122}$$

In (121) because K_{2m} is skew symmetric,

$$\begin{bmatrix} 0 & Z_m \\ \bar{Z}_m & 0 \end{bmatrix}$$

is skew Hermitian. Hence, $Z_m^T = -Z_m$, that is Z_m is complex skew symmetric.

Returning now to (109), (114), we let

$$U = \sum_{t=1}^{u}{}^{\cdot}\ V_{2m_t} W_{2m_t} \dotplus \sum_{t=u+1}^{w}{}^{\cdot}\ I_{\alpha_t}\ .$$

From (109), (114), (121), (122) we get

$$\begin{aligned} U^* N^{-1} U = & \sum_{t=1}^{u}{}^{\cdot} \operatorname{diag}(r_t^{-1} e^{-i\varphi_t} I_{m_t}, r_t^{-1} e^{i\varphi_t} I_{m_t}) \\ & \dotplus \sum_{t=u+1}^{v}{}^{\cdot}\ R_t^{-1} I_{\alpha_t} \dotplus \sum_{t=v+1}^{w}{}^{\cdot}\ -R_t^{-1} I_{\alpha_t}\ , \end{aligned}$$

$$U^* K U = \sum_{t=1}^{u}{}^{\cdot} \begin{bmatrix} 0 & Z_{m_t} \\ \bar{Z}_{m_t} & 0 \end{bmatrix} \dotplus \sum_{t=u+1}^{v}{}^{\cdot}\ K_{\alpha_t} \dotplus \sum_{t=v+1}^{w}{}^{\cdot}\ K_{\alpha_t}\ , \tag{123}$$

$$U^*N^{-1}KU = \sum_{t=1}^{u}{}^{\cdot} \begin{bmatrix} 0 & r_t^{-1}e^{-i\varphi_t}Z_{m_t} \\ r_t^{-1}e^{i\varphi_t}\bar{Z}_{m_t} & 0 \end{bmatrix} + \sum_{t=u+1}^{v}{}^{\cdot}\ R_t^{-1}K_{\alpha_t} + \sum_{t=v+1}^{w}{}^{\cdot}\ -R_t^{-1}K_{\alpha_t}\ . \tag{124}$$

Since $N^{-1}K = \mathcal{O}K\mathcal{O}^{-1}$, (123) and (124) have the same eigenvalues. We therefore proceed to evaluate the eigenvalues of a matrix of the form

$$\begin{bmatrix} 0 & \gamma Z \\ \bar{\gamma}\bar{Z} & 0 \end{bmatrix} \tag{125}$$

where Z is complex skew symmetric, m-square, nonsingular, and $\gamma \neq 0$. By Lemma 3.7 a unitary T exists such that

$$TZT^T = \sum_{i=1}^{r}{}^{\cdot} \begin{bmatrix} 0 & \rho_i \\ -\rho_i & 0 \end{bmatrix} + 0, \qquad \rho_i > 0 \text{ for } 1 \leqq i \leqq r\ .$$

Since Z is nonsingular, m must be even, and

$$TZT^T = \sum_{i=1}^{m/2}{}^{\cdot} \begin{bmatrix} 0 & \rho_i \\ -\rho_i & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \gamma Z \end{bmatrix} \begin{bmatrix} 0 & \gamma Z \\ \bar{\gamma}\bar{Z} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \gamma Z \end{bmatrix}^{-1} \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}^{*}$$
$$= \begin{bmatrix} 0 & I \\ |\gamma|^2(TZT^T)\overline{(TZT^T)} & 0 \end{bmatrix}.$$

Thus (125) has the same eigenvalues as

$$\begin{bmatrix} 0 & I \\ |\gamma|^2 \operatorname{diag}(-\rho_1^2, -\rho_1^2, -\rho_2^2, -\rho_2^2, \cdots, -\rho_{m/2}^2, -\rho_{m/2}^2) & 0 \end{bmatrix}. \tag{126}$$

The eigenvalues of (126) are

$$i\,|\gamma|\,\rho_t,\ -i\,|\gamma|\,\rho_t,\ i\,|\gamma|\,\rho_t,\ -i\,|\gamma|\,\rho_t\ , \qquad 1 \leqq t \leqq m/2\ .$$

Returning now to (123) and (124), let the eigenvalues of

$$\begin{bmatrix} 0 & Z_{m_t} \\ \bar{Z}_{m_t} & 0 \end{bmatrix}$$

be

$$i\rho_{tj},\ -i\rho_{tj},\ i\rho_{tj},\ -i\rho_{tj}\ ; \quad 1 \leqq j \leqq m_t/2;\ 1 \leqq t \leqq u\ , \tag{127}$$

and let the eigenvalues of K_{α_t} be

$$(128)\qquad i\rho_{tj},\ -i\rho_{tj}\ ;\qquad 1 \leqq j \leqq \alpha_t/2;\ u < t \leqq w\ .$$

In (127) and (128) we can choose the notation so that each $\rho_{tj} > 0$. One now finds that the eigenvalues of (123) are (127) and (128), whereas the eigenvalues of (124) are

$$(129)\qquad ir_t^{-1}\rho_{tj},\ -ir_t^{-1}\rho_{tj},\ ir_t^{-1}\rho_{tj},\ -ir_t^{-1}\rho_{tj}\ ;\quad 1 \leqq j \leqq m_t/2\ ;\ 1 \leqq t \leqq u\ ;$$

together with

$$(130)\qquad iR_t^{-1}\rho_{tj},\ -iR_t^{-1}\rho_{tj}\ ;\qquad 1 \leqq j \leqq \alpha_t/2\ ;\ u < t \leqq w\ ;$$

the numbers (129) and (130) must be a rearrangement of (127) and (128). Throughout (127)-(130) we may discard the common factor of i. After discarding the i, the positive numbers in (127), (128) ,

$$(131)\qquad \begin{array}{ll} \rho_{tj},\ \rho_{tj}\ , & 1 \leqq j \leqq m_t/2\ ,\ 1 \leqq t \leqq u\ , \\ \rho_{tj}\ , & 1 \leqq j \leqq \alpha_t/2\ ,\ u < t \leqq w\ , \end{array}$$

must be a rearrangement of the positive numbers in (129), (130):

$$(132)\qquad \begin{array}{ll} r_t^{-1}\rho_{tj},\ r_t^{-1}\rho_{tj}\ , & 1 \leqq j \leqq m_t/2\ ,\ 1 \leqq t \leqq u\ , \\ R_t^{-1}\rho_{tj}\ , & 1 \leqq j \leqq \alpha_t/2,\ u < t \leqq w\ . \end{array}$$

After taking inverses in (131) and (132), and making some notational changes, we find that the conditions of the theorem must hold.

Suppose now that the conditions of the theorem are satisfied. Let the nonreal eigenvalues of N be

$$r_te^{i\varphi_t},\ r_te^{i\varphi_t},\ r_te^{-i\varphi_t},\ r_te^{-i\varphi_t}\ ,\qquad 1 \leqq t \leqq s\ ,$$

and let the real eigenvalues be

$$R_t,\ R_t \qquad s < t \leqq k\ .$$

Then N is orthogonally similar to

$$(133)\qquad \sum_{t=1}^{s}{}^{\cdot}\,(r_tF(\varphi_t) + r_tF(\varphi_t)^T) + \sum_{t=s+1}^{k}{}^{\cdot}\,\mathrm{diag}\,(R_t,\ R_t)\ .$$

We may assume N is given by (133). The conditions of the theorem imply the existence of positive numbers $h_1, \cdots, h_k$ such that

$$(134)\qquad \begin{array}{c} r_1^{-1}h_1,\ r_1^{-1}h_1,\ r_2^{-1}h_2,\ r_2^{-1}h_2,\ \cdots,\ r_s^{-1}h_s,\ r_s^{-1}h_s\ , \\ |R_{s+1}|^{-1}h_{s+1},\ |R_{s+2}|^{-1}h_{s+2},\ \cdots,\ |R_k|^{-1}h_k \end{array}$$

are a rearrangement of

$$(135)\qquad h_1,\ h_1,\ h_2,\ h_2,\ \cdots,\ h_s,\ h_s,\ h_{s+1},\ h_{s+2},\ \cdots,\ h_k\ .$$

Put

$$K = \sum_{t=1}^{s}{}^{\cdot} \begin{bmatrix} 0 & 0 & h_t & 0 \\ 0 & 0 & 0 & h_t \\ -h_t & 0 & 0 & 0 \\ 0 & -h_t & 0 & 0 \end{bmatrix} \dot{+} \sum_{t=s+1}^{k}{}^{\cdot} \begin{bmatrix} 0 & h_t \\ -h_t & 0 \end{bmatrix} .$$

Matrix K is real and skew and has eigenvalues

$$(136) \qquad \begin{gathered} \pm ih_1, \pm ih_1, \pm ih_2, \pm ih_2, \cdots, \pm ih_s, \pm ih_s , \\ \pm ih_{s+1}, \pm ih_{s+2}, \cdots, \pm ih_k . \end{gathered}$$

We compute that

$$N^{-1}K = \sum_{t=1}^{s}{}^{\cdot} \begin{bmatrix} 0 & r_t^{-1}h_tF(\varphi_t)^T \\ -r_t^{-1}h_tF(\varphi_t) & 0 \end{bmatrix} \dot{+} \sum_{t=s+1}^{k}{}^{\cdot} \begin{bmatrix} 0 & R_t^{-1}h_t \\ -R_t^{-1}h_t & 0 \end{bmatrix} .$$

The matrix $N^{-1}K$ is skew symmetric. Using Lemma 3.4, one can compute the eigenvalues of $N^{-1}K$. Then turn out to be

$$(137) \qquad \begin{gathered} \pm ir_1^{-1}h_1, \pm ir_1^{-1}h_1, \pm ir_2^{-1}h_2, \pm ir_2^{-1}h_2, \cdots, \pm ir_s^{-1}h_s, \pm ir_s^{-1}h_s , \\ \pm i\,|R_{s+1}|^{-1}h_{s+1}, \pm i\,|R_{s+2}|^{-1}h_{s+2}, \cdots, \pm i\,|R_k|^{-1}h_k . \end{gathered}$$

Because (135) is a rearrangement of (134), (137) is a rearrangement of (136). Thus $N^{-1}K$ and K are real skew matrices with the same eigenvalues, hence $N^{-1}K = \mathcal{O}K\mathcal{O}^{-1}$ for some orthogonal $\mathcal{O}$. Hence $N = K\mathcal{O}K^{-1}\mathcal{O}^{-1}$, as required. Note that if $s = 0$ (that is, if all eigenvalues of N are real) the construction just given produces a K commutative with N.

THEOREM 7.11. *Let N be real and normal. Then N is a commutator* (109) *of a skew K and an orthogonal $\mathcal{O}$ such that*

$$(138) \qquad NK = KN$$

holds, if and only if: (i) *N is symmetric*; (ii) *each eigenvalue of N has even multiplicity*; (iii) $\det N = 1$.

Proof. Suppose (109) and (138) hold. Then $N = (iK)\mathcal{O}(iK)^{-1}\mathcal{O}^{-1}$ and N commutes with iK, hence by Theorem 7.2 N is symmetric. By Theorem 7.10 each eigenvalue of N has even multiplicity. Clearly $\det N = 1$. Conversely if $\lambda_1, \lambda_1, \lambda_2, \lambda_2, \cdots, \lambda_k, \lambda_k$ are the eigenvalues of N, then $\det N = 1$ implies $|\lambda_1| \cdots |\lambda_k| = 1$. Put

$$h_1 = 1, h_2 = |\lambda_1|, h_3 = |\lambda_1||\lambda_2|, \cdots, h_k = |\lambda_1||\lambda_2| \cdots |\lambda_{k-1}| .$$

Then the numbers $|\lambda_1| h_1, \cdots, |\lambda_k| h_k$ are just a rearrangement of $h_1, \cdots, h_k$, and the proof of Theorem 7.10 showed how to construct skew K commutative with N such that (109) holds.

THEOREM 7.12. *Let N be real and normal. Then N is a commutator* (109) *of a skew K and an orthogonal $\mathcal{O}$ such that*

$$N\mathcal{O} = \mathcal{O}N \tag{139}$$

if and only if: (i) *N is proper orthogonal,* (ii) *each eigenvalue of N has even multiplicity. If these conditions hold, we may in fact make K skew orthogonal.*

Proof. Suppose (109) and (139) hold. Then from

$$N = (iK)\mathcal{O}(iK)^{-1}\mathcal{O}^{-1}$$

and Theorem 7.3 we deduce that N is proper orthogonal. From Theorem 7.10 we deduce that each eigenvalue of N has even multiplicity. For the converse we need only consider two cases: $N = F(\varphi) + F(\varphi)^T$, and $N = \operatorname{diag}(-1, -1)$. In the first possibility let

$$K = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \mathcal{O} = \operatorname{diag}(1, 1, F(\varphi)) .$$

Then $N\mathcal{O} = \mathcal{O}N$ and (109) holds. Moreover K is orthogonal and $\mathcal{O}$ is orthogonal. For the second case observe

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} . \tag{140}$$

THEOREM 7.13. *Suppose N is real normal but that N has no real eigenvalues. Then N is a commutator* (109) *of a skew K and an orthogonal $\mathcal{O}$ if and only if each eigenvalue of N has even multiplicity and* $\det N = 1$. *It cannot happen that K commutes with N and $\mathcal{O}$ can commute with N if and only if N is also orthogonal.*

Proof. That the conditions are necessary follows from Theorem 7.10. Conversely, let $\lambda_1, \lambda_1, \bar{\lambda}_1, \bar{\lambda}_1, \cdots, \lambda_k, \lambda_k, \bar{\lambda}_k, \bar{\lambda}_k,$ be the eigenvalues of N. Then $|\lambda_1| \cdots |\lambda_k| = 1$. Put

$$h_1 = 1, h_2 = |\lambda_1|, \cdots, h_k = |\lambda_1| \cdots |\lambda_{k-1}| .$$

Then the conditions of Theorem 7.10 are satisfied.

Theorem 7.13, of course, applies when N is skew symmetric. When the eigenvalues of N are all real, Theorem 7.11 provides a strengthened form of Theorem 7.10.

THEOREM 7.14. *Let $\mathcal{O}$ be proper orthogonal and n-square with* $n \equiv 0 \pmod 4$. *Then*

$$\mathcal{O} = (K_1\mathcal{O}_1K_1^{-1}\mathcal{O}_1^{-1})(K_2\mathcal{O}_2K_2^{-1}\mathcal{O}_2^{-1})$$

is a product of two commutators, with K_1 and K_2 skew orthogonal, and $\mathcal{O}_1$ and $\mathcal{O}_2$ orthogonal.

Proof. As $n \equiv 0 \pmod 4$, $\mathcal{O}$ is orthogonally similar to a direct sum of 4-square blocks of the form $F(\varphi_1) \dotplus F(\varphi_2)$. Now

$$\begin{aligned}&\operatorname{diag}(F(\varphi_1), F(\varphi_2))\\ &= \operatorname{diag}(1, -1, 1, -1)\operatorname{diag}(G(\pi/2 - \varphi_1), G(\pi/2 - \varphi_2)) .\end{aligned}$$

Here $\operatorname{diag}(1, -1, 1, -1)$ satisfies the conditions of Theorem 7.12, hence is a commutator of a skew orthogonal matrix with an orthogonal matrix. Moreover $\operatorname{diag}(G(\pi/2 - \varphi_1), G(\pi/2 - \varphi_2))$ also is orthogonal with eigenvalues $+1$ (twice) and -1 (twice), hence is orthogonally similar to $\operatorname{diag}(1, -1, 1, -1)$. This completes the proof.

THEOREM 7.15. *Let S be positive definite symmetric and n-square, with* $n \equiv 0 \pmod 4)$, *and* $\det S = 1$. *Then*

$$S = \prod_{i=1}^{4}(K_i\mathcal{O}_iK_i^{-1}\mathcal{O}_i^{-1})$$

is a product of four commutator, with each K_i skew and each $\mathcal{O}_i$ orthogonal.

Proof. First use Fan's factorization to express S as a product $S = S_1S_2$ where the eigenvalues of S_1 and of S_2 occur in reciprocal pairs. Now note that $\operatorname{diag}(\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}) = PQ$, where

$$P = \operatorname{diag}(\lambda_1^{1/2}\lambda_2^{-1/2}, \lambda_2^{1/2}\lambda_1^{-1/2}, \lambda_2^{1/2}\lambda_1^{-1/2}, \lambda_1^{1/2}\lambda_2^{-1/2})$$

and

$$Q = \operatorname{diag}(\lambda_1^{1/2}\lambda_2^{1/2}, \lambda_1^{-1/2}\lambda_2^{-1/2}, \lambda_1^{1/2}\lambda_2^{1/2}, \lambda_1^{-1/2}\lambda_2^{-1/2}) .$$

Thus S_1 and S_2 are each a product of two symmetric matrices to each of which Theorem 7.11 may be applied. This yields the result.

THEOREM 7.16. *Let real A be n-square with* $n \equiv 0 \pmod 4$ *and* $\det A = 1$. *Then*

$$A = \prod_{i=1}^{6} K_i\mathcal{O}_iK_i^{-1}\mathcal{O}_i^{-1}$$

where K_1, K_2 are skew orthogonal, K_3, K_4, K_5, K_6 are skew, and

$\mathcal{O}_1, \cdots, \mathcal{O}_6$ *are orthogonal.*

Proof. Let $A = \mathcal{O}S$. Use the two previous theorems.

One can show, at least for $n = 2$, that no counterpart of Theorem 7.16 can hold when $n \equiv 2 \pmod 4$. For if K is any 2-square skew, and $\mathcal{O}$ is any 2-square orthogonal, then a direct computation reveals and $K\mathcal{O}K^{-1}\mathcal{O}^{-1} = \pm I$. Thus any product $\prod_i K_i\mathcal{O}_iK_i^{-1}\mathcal{O}_i^{-1} = \pm I$.

8. On the commutator of a Hermitian matrix with a unitary Hermitian matrix. In § 4, certain normal matrices were seen to be the commutator of a Hermitian and a unitary Hermitian matrix. We ask: When can this happen?

THEOREM 8.1. *Let N be normal. Then N is a commutator* (91) *with H Hermitian and U unitary Hermitian if and only if N is unitarily similar to a direct sum of types* (7), (8), (10), (11) *and the following special form of type* (9):

$$\operatorname{diag}(re^{i\varphi}, r^{-1}e^{i\varphi}, re^{-i\varphi}, r^{-1}e^{-i\varphi}), \qquad r > 0,\ \varphi \text{ real}. \tag{141}$$

Proof. Suppose (91) holds with H Hermitian and U unitary Hermitian. By Theorem 4.1 N is unitarily similar to a direct sum of types (7)-(11). From the forms of types (7)-(11) and the fact (Theorem 7.1) that the eigenvalues of N come in conjugate pairs, it is clear that the totality of diagonal elements of type (9) is composed of conjugate pairs. Without loss of generality we may assume no $e^{i\varphi}$ in type (9) is real, since otherwise type (9) may be reclassified under types (7) or (8). If, in (9), we have $r_2 = r_1$ or $r_2 = r_1^{-1}$ then type (9) is already in the form (141). Then the totality of the remaining blocks of type (9) must have their diagonal elements in conjugate pairs. If $r_1 \neq r_2$, $r_1 \neq r_2^{-1}$, then in addition to (9) we must have a block

$$\operatorname{diag}(r_3e^{i\varphi}, r_3^{-1}e^{i\varphi}, r_1e^{-i\varphi}, r_1^{-1}e^{-i\varphi}). \tag{142}$$

We may recombine the blocks (9) and (142) as

$$\operatorname{diag}(r_1e^{i\varphi}, r_1^{-1}e^{i\varphi}, r_1e^{-i\varphi}, r_1^{-}e^{-i\varphi}), \tag{143}$$

$$\operatorname{diag}(r_3e^{i\varphi}, r_3^{-1}e^{i\varphi}, r_2e^{-i\varphi}, r_2^{-1}e^{-i\varphi}). \tag{144}$$

The block (143) has the form (141); and now the remaining blocks of type (9) not yet considered together with (144) retain the property that their diagonal elements come in conjugate pairs. By repetition of this argument, we see that the condition of the theorem is necessary.

For the converse, we need only refer to the last part of the proof of Theorem 4.1, noticing that H_2 defined in (31) is symmetric orthogonal when $r_1 = r_2$.

The results corresponding to Theorem 8.1 when N is real, when N commutes with H or with U, and when H is definite, are all contained in the theorem of §§ 4, 5, 7 and so no further discussion is needed here.

9. The commutator of two normal matrices when it is normal and commutes with both factors. Recently several papers have appeared studying the system of matrix equations

$$C = ABA^{-1}B^{-1}\,, \qquad CA = AC\,, \qquad CB = BC\,. \tag{145}$$

It turns out to be easy to show that C has roots of unity as eigenvalues, and it is possible, though more difficult, to obtain the necessary and sufficient conditions that the elementary divisors of C must satisfy in order for C to be representable in the form (145). Here we shall study (145) when C, A, and B are normal. We shall obtain a result analogous to one obtained by I. Sinha [6, 8]. In this § 9, I_α is to denote the α-square identity matrix.

THEOREM 9.1. *Let N, A, B be normal matrices such that*

$$N = ABA^{-1}B^{-1}\,, \qquad NA = AN\,, \qquad NB = BN\,. \tag{146}$$

Then N is unitary and after a simultaneous unitary similarity of N, A, B we have

$$N = \sum_{i=1}^{r}{}^{\cdot}\, \gamma_i I_{n_i}\,, \tag{147}$$

$$A = \sum_{i=1}^{r}{}^{\cdot}\, [H_i, H_i, \cdots, H_i, U_i H_i]_{k_i}\,, \tag{148}$$

$$B = \sum_{i=1}^{r}{}^{\cdot}\, \operatorname{diag}(I_{\sigma_i}, \gamma_i I_{\sigma_i}, \gamma_i^2 I_{\sigma_i}, \cdots, \gamma_i^{k_i-1} I_{\sigma_i})\,. \tag{149}$$

Here γ_i is a primitive k_i^{th} root of unity for some k_i dividing n_i, and $\sigma_i = n_i/k_i$. Furthermore, H_i is a σ_i-square positive definite Hermitian matrix and U_i is a σ_i-square unitary matrix commutative with H_i; $1 \leqq i \leqq r$. Conversely, if N, A, B are as just described, then N, A, B are each normal and (146) *holds.*

Proof. We may begin with N diagonal, as in (147), where $\gamma_1, \gamma_2, \cdots, \gamma_r$ are the distinct eigenvalues of N. Then $NA = AN$ and $NB = BN$ force A and B to decompose into direct sums conformally with the direct sum decomposition (147) of N. To simplify the notation we may now consider

$$\gamma I_n = ABA^{-1}B^{-1} . \tag{150}$$

Taking determinants, $\gamma^n = 1$. Thus γ is a root of unity, say a primitive k^{th} root of unity, so that k divides n. Making a unitary similarity of (150) we can get B diagonal. From $\gamma B = ABA^{-1}$ it follows that if β is an eigenvalue of B with a certain multiplicity, $\gamma\beta$ is also an eigenvalue of B, with the same multiplicity. So we can let $B = B_1 \dotplus \cdots \dotplus B_s$ where $B_i = \beta_i \operatorname{diag}(I_{\sigma_i}, \gamma I_{\sigma_i}, \cdots, \gamma^{k-1} I_{\sigma_i})$ for some σ_i, with $\beta_i^k \neq \beta_j^k$ if $i \neq j$. Then $\gamma BA = AB$ forces A to partition as $A = A_1 \dotplus \cdots \dotplus A_s$, with $A_i = [A_{i1}, A_{i2}, \cdots, A_{ik}]_k$, $1 \leqq i \leqq s$. Again to simplify notation we consider each direct summand individually, so let us examine

$$\gamma I_{\sigma k} = ABA^{-1}B^{-1} ,$$

with

$$B = \operatorname{diag}(\beta I_\sigma, \gamma\beta I_\sigma, \cdots, \gamma^{k-1}\beta I_\sigma) ,$$
$$A = [A_1, A_2, \cdots, A_k]_k .$$

Let $A_i = U_i \widetilde{H}_i$ be the polar factorization of A_i, $1 \leqq i \leqq k$. Let $W = W_1 \dotplus \cdots \dotplus W_k$ where $W_1 = I_\sigma$, $W_2 = U_1$, $W_3 = U_1 U_2, \cdots, W_k = U_1 \cdots U_{k-1}$. Then $WBW^* = B$ and $WAW^* = [H_1, H_2, \cdots, H_{k-1}, UH_k]_k$ for certain positive definite $H_1, \cdots, H_k$ and unitary U. So change notation and let $A = [H_1, \cdots, H_{k-1}, UH_k]_k$. Then $AA^* = A^*A$ yields $H_1^2 = H_2^2 = \cdots = H_k^2$ and $UH_k^2U^* = H_{k-1}^2$. As the H_i are positive definite these equations imply $H_1 = H_2 = \cdots = H_k = H$ (say) and $UHU^* = H$. Thus $A = [H, H, \cdots, H, UH]_k$ as claimed, with U unitary and commutative with H. The converse is direct.

THEOREM 9.2. *The necessary and sufficient condition that a normal matrix N be representable as a commutator* (146) *of normal matrices are*: (i) *N is unitary*; (ii) *each eigenvalue γ of N is a root of unity satisfying*

$$(\textit{multiplicity of } \gamma) \equiv 0 \ (\operatorname{mod} (\textit{order of } \gamma)) .$$

If these conditions are satisfied we may take both A and B unitary, and also both real if N is real.

Proof. It is clear from the formulas (147), (148), (149) how to choose A and B unitary if N is unitary (take H_i to be the identity.) Suppose N is real. Then N is orthogonally similar to a direct sum of blocks of the form $\Phi_{2k}(\varphi)$ and $\operatorname{diag}(-1, -1)$ where angle φ has the form $\varphi = 2\pi j/k$. Set $B = F(0) \dotplus F(\varphi) \dotplus F(2\varphi) \dotplus \cdots \dotplus F((k-1)\varphi)$, and put $A = [I_2, I_2, \cdots, I_2]_k$, where I_2 is the 2-square identity. Then $\Phi_{2k}(\varphi)BA = AB$ and $\Phi_{2k}(\varphi)$ commutes with both A and B. Moreover

$$\text{diag}\,(-1, -1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} . \tag{151}$$

This completes the proof.

THEOREM 9.3. *Suppose normal matrices N, A, B satisfy* (146). *If one of A or B is Hermitian then N is Hermitian unitary, and* $\det N = 1$. *Conversely, if N is Hermitian unitary then* (146) *holds where A and B can both be chosen to be Hermitian unitary and also real if N is real.*

Proof. If (146) holds with A Hermitian then one easily sees that each $k_i \leqq 2$. Thus N is Hermitian, and clearly $\det N = 1$. For the converse one need only note (151).

THEOREM 9.4. *If A or B is positive definite in* (146) *then* $N = I$.

THEOREM 9.5. *Suppose N, A, B are real and normal, and* (146) *holds with A skew. Then N is symmetric, proper orthogonal, and degree N is even. Conversely, if N is symmetric and proper orthogonal with even degree then*

$$N = KSK^{-1}S^{-1}\,, \qquad NK = KN\,, \qquad NS = SN$$

with K skew orthogonal and S symmetric orthogonal.

Proof. For the first assertion use Theorem 9.3 and $N = (iK)S(iK)^{-1}S^{-1}$. For the converse note (140).

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