LOCALLY GALOIS ALGEBRAS

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Separable subalgebras of commutative algebras which (a) are the direct limit of separable subalgebras and (b) have sufficiently many automorphisms are shown to be the fixed rings of groups of automorphisms of the algebra. Necessary and sufficient conditions for an arbitrary subalgebra to be the fixed ring of a group are examined.

Also, we show that every element of every separable algebra over a ring is separable if and only if the ring is von Neumann regular.

Given a commutative ring R and a commutative R-algebra S, Villamayor and Zelinsky [10, 3.1] call S a *weakly Galois* R-algebra if S is finitely generated and projective as an R-module, separable as an R-algebra, and if there is a finite group G of R-algebra automorphisms of S such that the subring S^G of G-invariant elements of S is precisely R. Under these hypotheses they achieve the following generalization of the fundamental theorem of Galois theory: Every separable R-subalgebra of S is the fixed ring of a finite group of automorphisms, and conversely.

Our study here is of infinitely generated algebras. We say S is a *locally Galois* R-algebra if every finite subset of S is contained in a weakly Galois subalgebra of S with the following property: there is a finite set of automorphisms of the subalgebra, each of which extends to an automorphism of S, and the fixed ring of the subalgebra under this subset is R. We prove that if S is a locally Galois R-algebra, every separable R-subalgebra of S is the fixed ring of some group of automorphisms of S. In general, the converse of this is false.

Call an R-algebra *locally separable* if every finite subset of the algebra is contained in a separable R-subalgebra. Under special hypotheses on R, which allow infinitely many idempotents, but are otherwise rather restrictive, we characterize those locally separable subalgebras of a locally Galois R-algebra which are the fixed rings of groups of automorphisms.

A major technique used here, as in [10], is to reduce to the case where R has no nontrivial idempotents via Pierce's theory of the Boolean spectrum [7]: Let $X(R)$ denote the quotient space of Spec (R) with connected components identified to points. The quotient map of Spec (R) to $X(R)$ induces a sheaf on $X(R)$, the direct image of the canonical sheaf of local rings on R. The resulting ringed space

$(X(R), \mathscr{F})$ is called the Boolean spectrum: the reason for the name and the fundamental properties of the Boolean spectrum are discussed in [7] and [10]. The principal feature is that each stalk of $\mathscr{F}$ has no nontrivial idempotents and that $(X(R), \mathscr{F})$ is universal with respect to this property.

0. **Preliminaries.** Unless explicitly noted to the contrary, all rings and algebras are assumed commutative with unit. R denotes throughout the fixed, commutative base ring and all unadorned tensors are taken over R.

We shall use the following description of the Boolean spectrum $(X(R), \mathscr{F})$ of R: $X(R)$ is the maximal ideal space of the Boolean algebra $B(R)$ of idempotents of R, topologized by taking the sets $U_e = \{x : 1 - e \in x\}$, for all idempotents e, as basic open sets; and $\mathscr{F}(U_e) = Re$. $U_e \subset U_f$ if and only if $e \leq f$ (that is, $ef = e$). The stalk of $\mathscr{F}$ at x, which we denote R_x, is R/xR. The equivalence of this description and the one given in the introduction above is established in [10, 2.4].

(0.1) If M is an R-module let $M_x = R \otimes R_x$ and for each m in M let m_x be the image of m in M_x.

(0.2) If M and N are R-modules and f is in $\mathrm{Hom}_R(M, N)$, let f_x be the morphism $f \otimes R_x$.

When M is finitely presented the canonical map $\mathrm{Hom}_R(M, N)_x \to \mathrm{Hom}_{R_x}(M_x, N_x)$ is an isomorphism [3, p. 93, 2.8] and thus in this case the two interpretations of f_x for f in $\mathrm{Hom}_R(M, N)$ are compatible.

(0.3) An R-algebra S is *strongly separable* if S is a separable R-algebra which is finitely generated, projective and faithful as an R-module.

(0.4) A separable subalgebra of a strongly separable R-algebra is also strongly separable [6, p. 337, Remark 1].

For the convenience of the reader, we summarize here the results of [10] used in our study ($\mathrm{Aut}_R(S)$ denotes the group of all R-algebra automorphisms of S):

(0.5) Suppose R has no idempotents except 0 and 1. If S is a strongly separable R-algebra and if $S^H = R$ for some subgroup H of $\mathrm{Aut}_R(S)$, then H spans $\mathrm{Hom}_R(S, S)$ as a left S-module [10, 3.5].

(0.6) Let S be an R-algebra and H a subset of $\text{Aut}_R(S)$. The *closure* of H is the set of all automorphisms g such that: for each x in $X(R)$ and each minimal idempotent f in S_x, $f \cdot g_x = f \cdot h_x$ for some h in H [10, 3.7].

(0.7) Let S be a weakly Galois R-algebra. If F is a finite group of automorphisms of S and if $T = S^F$, then $\text{Aut}_T(S)$ is contained in the closure of F. [10, 3.14]

(0.8) Let S be a weakly Galois R-algebra, T an R-subalgebra. Then T is separable over R if and only if $T = S^F$ for some finite subset F of $\text{Aut}_R(S)$. [10, 3.10 and 3.12]

For later use, we will need a stronger version of part of (0.8). We remark that proof given here uses only [5] and (0.5) and hence is independent of (0.8).

THEOREM 0.9. *Let S be a strongly separable R-algebra and F a finite group of R-algebra automorphisms of S. Then S^F is a strongly separable R-algebra.*

Proof. Let $T = S^F$. Clearly S is a separable T-algebra. We first assume that R has no idempotents except 0 and 1 and show that (a) S is T-projective and that (b) $SF = \text{Hom}_T(S, S)$. Since R has no nontrivial idempotents S and hence T have only finitely many idempotents. Let $e_1, \cdots, e_n$ be the minimal idempotents of T. To prove (a) and (b), we shall show that Se_i is Te_i-projective and that

$$Se_iFe_i = \text{Hom}_{Te_i}(Se_i, Se_i)$$

for $i = 1, \cdots, n$; that is, we assume T has no nontrivial idempotents. Let e be a minimal idempotent of S and let H be the subgroup of F fixing e. Since T has no nontrivial idempotents, F acts transitively on the minimal idempotents of S and, exactly as in [9, p. 723, 1.3], we have that T is isomorphic to Te and $(Se)^H = Te$. By [5, p. 18, 1.3] Se is T-projective. Thus S is T-projective. Hence S is weakly Galois over T and, by (0.5), $SF = \text{Hom}_T(S, S)$.

Now let R be arbitrary. For each x in $X(R)$,

$$(SF)_x = S_x F_x \subseteq \text{Hom}_{Rx}(S_x, S_x) \ .$$

By the above, $S_x F_x = \text{Hom}_{S_x F_x}(S_x, S_x)$, since R_x has no nontrivial idempotents. Since also by the above S_x is a projective $S_x^{F_x}$-module, we see that $S_x^{F_x}$ is separable over its center $S_x F_x$ [2, p. 380, 5.1] and that $S_x^{F_x}$ is R_x-separable [1, p. 762, 4.8]. Thus $S_x^{F_x}$ is R_x-separa-

ble [2, p. 374, 2.5]. Since SF is a finitely generated R-algebra we
can now show that SF is R-separable: for SF is finitely presented as
an $(SF)^e$-module, and thus it suffices [3, p. 94, 2.10] to show that for
any maximal ideal m of R $SF \otimes R_m$ is a separable R_m-algebra. Let
x be the set of idempotents in the kernel of $R \to R_m$. Since R_m has
no nontrivial idempotents, x is in $X(R)$. Then

$$SF \otimes R_m = (SF \otimes R_x) \otimes_{R_x} R_m$$

is R_m-separable. So the center T of SF is separable.

1. **Separable subalgebras of locally Galois algebras.** The
principal result of this section is that separable subalgebras of locally
Galois algebras are the fixed rings of groups of automorphisms. We
begin by proving an extension theorem for some special subalgebras
of arbitrary R-algebras; it is this theorem which motivates the de-
finition 'locally Galois.' Our first lemma gives an equivalent definition
of the closure (0.6) of a group of automorphisms.

LEMMA 1.1. *Let S be an R-algebra finitely generated and pro-
jective as an R-module, and H a subset of* $\mathrm{Aut}_R(S)$. *Then an auto-
morphism g is in the closure of H if and only if there are pairwise
orthogonal idempotents $e_1, \cdots, e_n$ of S such that $\Sigma e_j = 1$ and elements
$h_1, \cdots, h_n$ of H, such that $g = \Sigma e_j h_j$.*

Proof. Suppose that g is in the closure of H. Let x be in $X(R)$
and $f_1, \cdots, f_k$ be the minimal idempotents of S_x. By definition of the
closure, there are $h_1, \cdots, h_k$ in H such that

$$f_i g_x = f_i (h_i)_x$$

for $i = 1, \cdots, k$. By [10, 2.12], choose idempotents $E_1, \cdots, E_k$ of S
such that $(E_i)_x = f_i$. These E_i satisfy

$$(*) \qquad \begin{aligned} (E_i)_x (E_j)_x &= (\delta_{ij} E_i)_x \\ \Sigma (E_i)_x &= 1 \\ (E_i g)_x &= (E_i h_i)_x . \end{aligned}$$

By (0.1), there is a basic neighborhood U_e of x on which these
equations are satisfied, i.e.,

$$(**) \qquad \begin{aligned} (e E_i)(e E_j) &= e(\delta_{ij} E_i) \\ e(\Sigma E_i) &= e \\ e(E_i g) &= e(E_i h) . \end{aligned}$$

For each y in $X(R)$ there is a basic neighborhood of y and a set of

idempotents of S and elements of H satisfying (∗) on that neighborhood. By compactness, choose finitely many disjoint such neighborhoods covering $X(R)$; let these be U_{e_q}, $q = 1, \cdots, m$. Since the neighborhoods are disjoint, $e^p e^q = \delta_{p,q} e^p$ and since they cover $X(R)$, $\Sigma e^q = 1$. For each q there is a set of idempotents of $S\{E_i^q\}$ and elements of $H\{h_i^q\}$ satisfying (∗∗). The sets of idempotents $\{e^q E_i^q\}$ and elements of $H\{h_i^q\}$ then satisfy

$$(e^q E_i^q)(e^p E_j^q) = \delta_{p,q}\delta_{i,j}e^q E_i^q$$
(∗∗∗)
$$\Sigma e^q E_i^q = 1$$
$$e^q E_i^q g = e^q E_i^q h_i^q .$$

Now, upon linearly ordering the indices i and q, (∗∗∗) shows that there is a set $e_1, \cdots, e_n$ of pairwise orthogonal idempotents of S such that $\Sigma e_i = 1$ and elements $h_1, \cdots, h_m$ of H such that $e_i g = e_i h_i$, $i = 1, \cdots, n$. Thus $g = \Sigma e_i h_i$.

If conversely $g = \Sigma e_i h_i$ where the e_i and h_i are as in the hypotheses of (1.1) then $e_i g = e_i h_i$ for $i = 1, \cdots, n$ and $U e_i = 1$. Hence by [10, 3.76], g is in the closure of H.

We remark that (1.1) improves [10, 3.76].

LEMMA 1.2. *Let S be an R-algebra, $e_1, \cdots, e_m$ pairwise orthogonal idempotents of S such that $\Sigma e_i = 1$ and $h_1, \cdots, h_m$ elements of $\mathrm{Aut}_R(S)$. Suppose there is an R-subalgebra T of S such that each e_i is in T, each h_i leaves T setwise invariant, and $g = \Sigma e_i h_i$ restricted to T is in $\mathrm{Aut}_R(T)$. Then g is in $\mathrm{Aut}_R(S)$.*

Proof. Fix j and choose x in T such that $g(x) = e_j$. Since

$$\Sigma e_i h_i(x) = e_j, \ e_j h_j(x) = e_j$$

and $e_i h_i(x) = 0$ for $i \neq j$. Thus $h_j(x) = e_j + f$ where $e_j(f) = 0$ and $x = h_j^{-1}(e_j) + h_j^{-1}(f)$. Then $h_i(x) = h_i h_j^{-1}(e_j) + h_i h_j^{-1}(f)$, for each i. If $i \neq j$, then $0 = e_i h_i(x) = e_i h_i h_j^{-1}(e_j) + e_i h_i h_j^{-1}(f)$. This last expression is a sum of the form $a + b$, where a is idempotent and $ab = 0$; if such a sum is zero, a must be zero. Hence $h_i(h_i^{-1}(e_i)h_j^{-1}(e_j)) = 0$, so $h_i^{-1}(e_i)h_j^{-1}(e_j) = 0$. Thus the $h_i^{-1}(e_i)$ are pairwise orthogonal idempotents. Moreover, their sum is one: for the sum lies in T, and

$$(\Sigma e_i h_i)(\Sigma h_i^{-1}(e_i)) = \Sigma e_i h_i(h_j^{-1}(e_j)) = \Sigma \delta_{ij} e_i = 1 ,$$

and g restricted to T is an automorphism.

Now let $f_i = h_i^{-1}(e_i)$. If s is any element of S, we have:

$$g(\Sigma f_j s) = \Sigma g(f_j s) = \Sigma h_i(t_i)h_i(t_j s) = \Sigma h_i(f_i f_j s)$$
$$= \Sigma h_j(f_j t_j s) = \Sigma e_j h_j(f_j) .$$

Thus if we consider g as a map of $\prod Sf_i$ to $\prod Se_i$ we see that g sends the factor Sf_j to the factor Se_j by the algebra isomorphism h_j. Thus g is an algebra automorphism of S.

THEOREM 1.3. *Let S be an R-algebra and T a strongly separable subalgebra. Suppose there is a finite set F of automorphisms of T such that $T^F = R$ and each element of F extends to an automorphism of S. Then every automorphism of T extends to an automorphism of S.*

Proof. Let F_0 be the subgroup of $\mathrm{Aut}_R(T)$ generated by F. [By [10, 3.14] F_0 is finite.] Clearly each element of F_0 extends to an automorphism of S. Let h be in $\mathrm{Aut}_R(T)$; by (0.7) h is in the closure of F_0. By (1.1) $h = \Sigma e_i f_i$ where the e_i are pairwise orthogonal idempotents of T with $\Sigma e_i = 1$ and each f_i is in F_0. Extend each f_i to an automorphism g_i of S. By (1.2), $g = \Sigma e_i g_i$ is in $\mathrm{Aut}_R(S)$, and g extends h.

Our definition of locally Galois algebra is essentially that every finite subset of the algebra belongs to a subalgebra to which (1.3) applies:

DEFINITION 1.4. An R-algebra S is called *locally Galois* if every finite subset of S is contained in some strongly separable subalgebra T with the following property: there is a finite subset F of $\mathrm{Aut}_R(T)$ such that $T^F = R$ and each element of F extends to an automorphism of S.

LEMMA 1.5. *Let S be a locally Galois R-algebra and T a strongly separable subalgebra. Then $T = S^{\mathrm{Aut}_T(S)}$.*

Proof. Clearly T is contained in $S^{\mathrm{Aut}_T(S)}$. Suppose s is in $S \backslash T$ and choose a strongly separable subalgebra W of S containing T and s such that there is a finite subset F of $\mathrm{Aut}_R(W)$ with $W^F = R$ and each element of F extending to an automorphism of S. By (0.8), there is an h in $\mathrm{Aut}_R(W)$ such that $h(s) \neq s$ but h restricted to T is the identity. Choose, by (1.3), an element g of $\mathrm{Aut}_R(S)$ extending h. Then g is in $\mathrm{Aut}_T(S)$ but $g(s) \neq s$. We conclude that $T = S^{\mathrm{Aut}_T(S)}$.

THEOREM 1.6. *Let S be a locally Galois R-algebra and T a separable subalgebra. Then T is strongly separable and $T = S^{\mathrm{Aut}_T(S)}$.*

Proof. By (1.5), it suffices to show T is strongly separable and by (0.4) it will suffice to show that T is contained in a strongly

separable R-algebra. Since T is separable, there is an idempotent $e = \Sigma x_i \otimes y_i$ in $T \otimes T$ such that $\Sigma x_i y_i = 1$ and for all t in T, $(1 \otimes t)e = (t \otimes 1)e$. We shall also use $e = \Sigma x_i \otimes y_i$ for the image of e under $T \otimes T \to S \otimes S$. Choose a strongly subalgebra W of S containing all the y_i. For each g in $\mathrm{Aut}_W(S)$, apply $1 \otimes g$ to the equation $(1 \otimes t)e = (t \otimes 1)e$ and follow by the multiplication map $S \otimes S \to S$; we get

$$g(t)\Sigma x_i y_i = t\Sigma x_i y_i$$

since $g(y_i) = y_i$; since $\Sigma x_i y_i = 1$ we get $g(t) = t$. Thus T is contained in $S^{\mathrm{Aut}_W(S)}$ and by (1.5) this latter is W.

We observe that the technique of (1.6) also gives the following:

PROPOSITION 1.7. *Let S be an R-algebra and C a class of subalgebras such that:*
  (a) *Any finite subset of S is contained in some element of C.*
  (b) *For all T in C, $T = S^{\mathrm{Aut}_T(S)}$.*
Then every separable subalgebra of S is contained in some element of C.

In light of (1.6), it is natural to try to describe all subalgebras of S of the form S^H for some subgroup H of $\mathrm{Aut}_R(S)$. There are examples showing that such subalgebras need not even be a union of separable subalgebras. Something can be said, however, if H is of a special type.

DEFSNITION 1.8. Let S be an R-algebra and H a group of R-algebra automorphisms of S. H is *almost finite* if it has only finitely many restrictions to each finitely generated subalgebra of S.

Equivalent forms of (1.8) are given below in (1.12).

PROPOSITION 1.9. Let S be an R-algebra, H an almost finite subgroup of $\mathrm{Aut}_R(S)$ and suppose that every finite subset of S is contained in a strongly separable subalgebra of S. Then every finite subset of S is contained in a strongly separable subalgebra left setwise invariant under H.

Proof. Let F be a finite subset of S and let T_0 be a strongly separable subalgebra containing F. Let $h_1, \cdots, h_k$ be the restrictions of H to T_0 and let $T = h_1(T_0) \cdot h_2(T_0), \cdots, h_k(T_0)$. Clearly T is left setwise invariant under H. T is an epimorphic image of $h_1(T_0) \otimes \cdots \otimes h_k(T_0)$ hence finitely generated as an R-module and separable as an R-algebra. Under our hypotheses, T is contained in a strongly separable subalgebra of S, hence by (0.4) is itself strongly separable.

THEOREM 1.10. *Let S be a locally Galois R-algebra, T, T_0 separable subalgebras of S and H an almost finite subgroup of $\mathrm{Aut}_R(S)$. Then:*

(a) $T \cap T_0$ *is strongly separable.*

(b) $T \cap S^H$ *is strongly separable.*

(c) *Every finite subset of S^H is contained in a separable subalgebra of S^H.*

Proof. (a) By (1.6), T and T_0 are finitely generated. Thus, since S is locally Galois, there is a strongly separable subalgebra S_0, containing both T and T_0, for which there is a finite subset F of $\mathrm{Aut}_R(S_0)$ with $S^F = R$. By (0.8), then, there are finite subsets K and L of $\mathrm{Aut}_R(S_0)$ such that $T = S_0^K$ and $T_0 = S_0^L$. We can assume, of course, that both K and L contain the identity. Then $T \cap T_0 = S_0^{KL}$, where KL is the set of products kl for k in K, l in L. Thus by (0.8) again, $T \cap T_0$ is strongly separable.

(b) Since by (1.6) T is finitely generated, by (1.9) there is a strongly separable subalgebra W of S containing T and left setwise invariant under H. By (1.6) also, $W = S^J$ where $J = \mathrm{Aut}_W(S)$. Since W is setwise invariant under H, H is contained in the normalizer of J so J is a normal subgroup of JH. Also, $W \cap S^H = (S^J)^{JH/J}$. Now $JH/J = H/J \cap H$ and this latter group is finite, since it consists of of the distinct restrictions of H to W. By (0.9), then, applied to the subalgebra $W \cap S^H$ of $W = S^J$, $W \cap S^H$ is R-separable. Now by (a), $T \cap S^H = T \cap (W \cap S^H)$ is also separable.

(c) Let F be a finite subset of S^H. Choose a strongly separable subalgebra V of S containing F. Then by (b), $V \cap S^H$ is a separable subalgebra of S^H containing F.

We formalize the property of S^H established in (1.10) (c):

DEFINITION 1.11. An R-algebra is said to be *locally separable* if every finite subset of the algebra is contained in a separable subalgebra.

There are examples showing that, without further hypotheses on R it is neither necessary nor sufficient for a subalgebra of a locally Galois algebra to be locally separable to be the fixed ring of a group of automorphisms. Moreover, a locally separable subalgebra of a locally Galois algebra which is the fixed ring of a group need not be the fixed ring of an almost finite group.

The remainder of this section studies a special class of locally Galois algebras. We begin with equivalent characterizations of almost finite groups of automorphisms.

PROPOSITION 1.12. *Let S be an R-algebra, G a subgroup of $\mathrm{Aut}_R(S)$.*

The following are equivalent:

(a) G *is almost finite.*

(b) *For each s in S the orbit Gs is finite.*

(c) *For each s in S there is a normal subgroup H of finite index in G such that s is in S^H.*

(d) *For any finite subset F of S there is a normal subgroup H of finite index in G such that F is in S^H.*

(e) *G is dense in a profinite group G_0 of algebra automorphisms of S such that when S carries the discrete topology, the action $G_0 \times S \to S$ is continuous.*

These equivalences are well known, and we omit the standard proof.

We now cosider flat extensions of the base ring.

LEMMA 1.13. *Let S, B, T be R-algebras, with T a flat R-module, and G an almost finite group of R-algebra automorphisms of S such that $S^G = B$. Then $(S \otimes T)^{G \otimes 1} = B \otimes T$.*

Proof. Let x belong to $(S \otimes T)^{G \otimes 1}$, say $x = \Sigma s_i \otimes t_i$. Choose a normal subgroup H of finite index in G such that each s_i is in S^H. Let $g_1, \cdots, g_m$ be coset representatives of H in G. Then

$$0 \longrightarrow B \longrightarrow S^H \longrightarrow \prod_{i=1}^{m} S^H$$

is exact, where the last map sends s to $(g_1(s) - s, \cdots, g_m(s) - s)$. Since by hypothesis T is flat over R,

$$0 \longrightarrow B \otimes T \longrightarrow S^H \otimes T \longrightarrow \prod (S^H \otimes T)$$

is exact, where $(\prod S^H) \otimes T$ is identified with $\prod (S^H \otimes T)$. The jth component of the last map of the sequence sends y to $(g_j \otimes 1)(y) - y$ for any y in $S^H \otimes T$. Thus since x is in the kernel of this map, x is in $B \otimes T$. Thus $(S \otimes T)^{G \otimes 1} \subseteqq B \otimes T$; the reverse conclusion is obvious and the result.

PROPOSITION 1.14. *Let S be an R-algebra such that*:

(a) *Every finite subset of S is contained in a strongly separable subalgebra of S.*

(b) *There is an almost finite subgroup G of $\text{Aut}_R(S)$ such that $S^R = R$. Then S is a locally Galois R-algebra.*

Proof. By (1.9), every finite subset of S is contained in a strongly separable subalgebra left setwise invariant under G. If T is such a

subalgebra, F the group of restrictions of G to T then F is finite, $T^F = R$, and each element of F extends to an automorphism of S. Thus S is locally Galois.

PROPOSITION 1.15. *Let S be a locally Galois R-algebra and T a flat R-algebra. The $S \otimes T$ is a locally Galois T-algebra. If G is an almost finite group with $S^G = R$, $G \otimes 1$ is an almost finite group with $(S \otimes T)^{G \otimes 1} = T$.*

Proof. The first assertion is obvious and the second is an immediate consequence of (1.13).

2. **Locally separable subalgbras of locally Galois algebras.** This section characterizes, under suitable hypotheses on the base ring R, those locally separable subalgebras of locally Galois algebras which are the fixed rings of automorphism groups. The concept of normality plays an important role in this.

DEFINITION 2.1. Let S be an R-algebra. An R-subalgebra T of S is said to be *normal* if every automorphism of S sends T into itself. S is said to be a *normally Galois R-algebra* if every finite subset of S is contained in a normal strongly separable subalgebra T such that there is a finite subset F of $\operatorname{Aut}_R(T)$ with $T^F = R$ and each element of F extends to an automorphism of S. (Note that normally Galois implies locally Galois.)

LEMMA 2.2. *Let S be an R-algebra and T a subalgebra.*
(a) *If T is normal, $\operatorname{Aut}_T(S)$ is a normal subgroup of $\operatorname{Aut}_R(S)$.*
(b) *If H is a normal subgroup of $\operatorname{Aut}_R(S)$, S^H is a normal subalgebra.*

PROPOSITION 2.3. *Let S be a weakly Galois R-algebra, T a normal, separable subalgebra and suppose R is connected (i.e., has no nontrivial idempotents). Then either T is connected or contains all idempotents of S.*

Proof. Each element of $\operatorname{Aut}_R(S)$ restricted to the set E of minimal idempotents of S induces a permutation of E. If E has n elements, the restriction map r induces a homomorphism of $\operatorname{Aut}_R(S)$ onto S_n, the symmetric group on n symbols. $T = S^{\operatorname{Aut}_T(S)}$ and $\operatorname{Aut}_T(S)$ is normal by (2.2) (a). Thus $r(\operatorname{Aut}_T(S))$ is a normal subgroup of S_n; if it is not the identity, it acts transitively on E. In the first case each element of E is in T; in the second case, the only idempotents left fixed under $\operatorname{Aut}_T(S)$ are 0 and 1, so T is connected.

PROPOSITION 2.4. *Let R be connected and S a locally Galois R-algebra. Then the following are equivalent:*

(a) *There is a collection $\mathscr{S}$ of weakly Galois subalgebras of S such that every finite subset of S is contained in some element of $\mathscr{S}$ and if T_1, T_2 are in $\mathscr{S}$ with T_1 contained in T_2, T_1 is normal in T_2.*

(b) *S has finitely many idempotents.*

(c) *$\mathrm{Aut}_R(S)$ is almost finite.*

(d) *S is normally Galois.*

Proof. (a)$\Rightarrow$(b). Suppose e is a nontrivial idempotent of S and let T be an element of $\mathscr{S}$ containing e. Let f be any other idempotent of S and T_1 an element of $\mathscr{S}$ containing both T and f. T is not connected and hence by (2.3) contains every idempotent of T_1, in particular f. Thus T contains all idempotents of S; since T is a finitely generated projective R-module these idempotents are finite in number.

(b)$\Rightarrow$(c). Let $E = \{e_1, \cdots, e_n\}$ be the set of minimal idempotents of S and H the kernel of the restriction map of $\mathrm{Aut}_R(S)$ to the permutations of S. H is the product of the groups $H_i = \mathrm{Aut}_{Re_i}(Se_i)$. Each Se_i is connected and $(Se_i)^{H_i} = Re_i$. Since each Se_i is locally separable, by [6, p. 335, 1] each H_i is almost finite. Then H is almost finite and since H is of finite index in $\mathrm{Aut}_R(S)$ this latter is almost finite.

(c)$\Rightarrow$(d). Since $S^{\mathrm{Aut}_R(S)} = R$, the implication follows from (1.9).

(d)$\Rightarrow$(a). Take for $\mathscr{S}$ the set of subalgebras satisfying the condition of (2.1).

In [9, p. 723, 1.3], it was shown that, over a connected ring, the full automorphism group of a weakly Galois algebra is finite.

EXAMPLE 2.5. The full automorphism group of a locally Galois algebra over a field need not be almost finite.

Let R be a field and S a Galois extension field of infinite degree over R. Clearly S is a locally Galois R-algebra. By (1.15), $S \otimes S$ is a locally Galois S-algebra. Since S is locally separable over R, S is $S \otimes S$ flat so by [8, p. 94, 1] $S \otimes S$ is a von Neumann regular ring. If $\mathrm{Aut}_S(S \otimes S)$ is almost finite, by (2.4) $S \otimes S$ has only finitely many idempotents. A regular ring with only finitely many idempotents is a finite product of fields; it follows that the multiplication map $S \otimes S \to S$ splits, so S is R-separable. But since S is R-projective, by [9, p. 722, 1.1] we also have S finitely generated over R, contrary to hypothesis.

We remark that the Example (2.5) also shows that normally

Galois is not preserved under flat base ring extension. In case the extension is by a stalk over a point in the Boolean spectrum, however, the property is stable.

PROPOSITION 2.6. *Let S be a normally Galois R-algebra, x a point in $X(R)$. Then S_x is a normally Galois R_x-algebra.*

Proof. By (1.15), S_x is a locally Galois R_x-algebra. Let $\mathscr{S}$ be the collection of all normal subalgebras of S satisfying the condition of (2.1). Let $\mathscr{S}_x = \{T_x : T \in \mathscr{S}\}$. Each element of S_x is weakly Galois over R_x and every finite subset of S_x is contained in some element of $\mathscr{S}_x$. Let T_0, T_1 be elements of $\mathscr{S}_x$ with $T_0 \subseteqq T_1$ and let T, T' be elements of $\mathscr{S}$ with $T \subseteqq T'$ and $T_x = T_0$, $T'_x = T_1$. Suppose h is in $\mathrm{Aut}_{R_x}(T_1)$. By [10, 2.14] choose g in $\mathrm{Aut}_R(T')$ with $g_x = h$. Since $g(T) \subseteqq T$, $h(T_0) \subseteqq T_0$. Hence T_0 is a normal subalgebra T_1. By (2.4) (a)$\Rightarrow$(d), applied to $\mathscr{S}_x$, we conclude that S_x is a normally Galois R_x-algebra.

PROPOSITION 2.7. *Let R be connected, S a normally Galois R-algebra and T a locally separable subalgebra. Then $T = S^H$ where H is the subgroup of $\mathrm{Aut}_R(S)$ fixing T.*

Proof. Give $\mathrm{Hom}_R(S, S)$ the weakest topology such that for every finitely generated subalgebra S_0 of S the restriction map $\mathrm{Hom}_R(S, S) \to \mathrm{Hom}_R(S_0, S)$, induced by the inclusion of S_0 in S, is continuous. This is called the finite topology on $\mathrm{Hom}_R(S, S)$. We claim that $\mathrm{Aut}_R(S)$ is compact in this topology. For by (2.4) (d)$\Rightarrow$(c), $\mathrm{Aut}_R(S)$ is almost finite, and by (1.12) (a)$\Rightarrow$(e), $\mathrm{Aut}_R(S)$ can be embedded in a compact group G_0 of automorphisms of S such that the action $G_0 \times S \to S$ is continuous. Thus $G_0 = \mathrm{Aut}_R(S)$ is compact in the finite topology. Now the proof proceeds exactly as in the classical situation: let s be in $S \backslash T$ and for each finitely generated separable subalgebra T' of T let $N(T') = \{g \in \mathrm{Aut}_R(S) : g \mid T' = id$ and $g(s) \neq s\}$. $N(T')$ is a nonempty closed subset of $\mathrm{Aut}_R(S)$ and the collection of all $N(T')$, as T' ranges over the finitely generated separable subalgebras of T, has the finite intersection property. Then since $\mathrm{Aut}_R(S)$ is compact, there is a g in $\mathrm{Aut}_R(S)$ which belongs to each $N(T')$. We have $g \mid T = id$ and $g(s) \neq s$, and hence conclude that $T \supseteqq S^H$. Since the reverse inclusion is obvious, we have the result.

Now suppose that R is arbitrary, S is a normally Galois R-algebra, and T is an almost separable subalgebra of S. Let s be in $S \backslash T$ and x a point in $X(R)$ such that s_x is not in T_x. By (2.7), there is an h in $\mathrm{Aut}_{R_x}(S_x)$ such that $h \mid T_x = id$ and $h(s_x) \neq s_x$. We would like to find a g in $\mathrm{Aut}_R(S)$ such that $g \mid T = id$ and $g_x = h_x$, so $g(s) \neq s$; that

is, we want to show first that $\mathrm{Aut}_R(S) \to \mathrm{Aut}_{R_x}(S_x)$ is onto and second that if f and g are in $\mathrm{Hom}_R(T, S)$ with $f_x = g_x$ in $\mathrm{Hom}_{Rx}(T_x, S_x)$ (i.e., equal as maps) then $f = g$ on a neighborhood of x. In order to prove the second assertion, however, we will need to assume that x is an isolated point of $X(R)$, and in this case the first assertion is trivial. For completeness, however, we include a more general result.

LEMMA 2.8. *Let S be a normally Galois R-algebra, G a profinite group such that $G \times S \to S$ is continuous and $S^G = R$, and x a point in $X(R)$. Then $\mathrm{Aut}_R(S) \to \mathrm{Aut}_{Rx}(S_x)$ is onto.*

Proof. Let $\mathrm{Hom}_{Rx}(S_x, S_x)$ carry the finite topology; note that G_x is compact in this topology. Let $H = \mathrm{Aut}_{Rx}(S_x) \cap (e_1 G_x + \cdots + e_n G_x)$ where $e_1, \cdots, e_n$ are the minimal idempotents of S. H, being the intersection of compact sets, is closed in $\mathrm{Aut}_{Rx}(S_x)$. We claim that H is also dense in $\mathrm{Aut}_{Rx}(S_x)$: for the family of normal weakly Galois subalgebras of S_x which contain all the e_i, $i = 1, \cdots, n$, is cofinal, and hence to establish the claim it suffices to show that if T is a subalgebra in the above mentioned cofinal family and g in $\mathrm{Aut}_{Rx}(S_x)$ then there is an h in H *such* that $g \,|\, T = h \,|\, T$. But this is precisely what was shown in (1.3). Hence $H = \mathrm{Aut}_{Rx}(S_x)$.

Let g be any element of $\mathrm{Aut}_{Rx}(S_x)$; by the above, $g = e_1 g_1 + \cdots + e_n g_n$, where g_i is in G_x. Choose, by [10, 2.12], orthogonal idempotents $E_1, \cdots, E_n$ of S such that $(E_i)_x = e_i$. Let f be the sum of the E_i; we can assume f is in R. Let T be a normal weakly Galois subalgebra of Sf containing $E_1, \cdots, E_n$. Choose $h_1, \cdots, h_n$ in Gf such that $(h_i)_x = g_i$, and let $h = E_1 g_1 + \cdots + E_n g_n$. Since $(h \,|\, T)_x = g_x \,|\, T_x$ is an automorphism of T_x, there is an idempotent e in Rf such that $he \,|\, Te$ is an automorphism of Te. Hence by (1.2), he is an automorphism of Se. Then $h' = (1 - e) + he$ is an automorphism of S such that $h'_x = g$.

One can show by example that, under the hypotheses of (2.8), $\mathrm{Aut}_R(S)_x \to \mathrm{Aut}_{Rx}(S_x)$ need not be an isomorphism.

LEMMA 2.9. *Let R be any commutative ring, x an element of $X(R)$. Then $\mathrm{Hom}_R(M, N)_x \to \mathrm{Hom}_{Rx}(M_x, N_x)$ is one to one for all R-modules M and N if and only if R_x is R-projective.*

Proof. Suppose M is free, i.e., $M = \oplus R$. We have the following commutative diagram:

$$\mathrm{Hom}_R(\oplus R, N)_x \longrightarrow \mathrm{Hom}_{Rx}(\oplus R_x, N_x)$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$(\textstyle\prod N)_x \longrightarrow \textstyle\prod(N_x) .$$

If the top map is one-to-one, so is the bottom, the vertical maps being isomorphisms. Now let $N = R$ and suppose the rank of M is equal to the cardinality of x. Then if $\{f\}$ is a sequence in $\prod R$ consisting of all the elements of x, we have $\{f\}_x \to \{f_x\} = \{0\}$. If the map is one-to-one, then, we have $\{f\}_x = 0$; thus there is an idempotent h of R such that $(1 - h) \in x$ and $h\{f\} = 0$. Thus $(1 - h)f = f$ for all f in x, and $Rx = R(1 - h)$, so $R_x = Rh$. Thus R_x is R-projective.

If conversely R_x is projective, we have $R_x = Re$ for some idempotent e, and if M, N are R-modules, clearly

$$\mathrm{Hom}_R(M, N) \otimes Re = \mathrm{Hom}_{Re}(Me, Ne) .$$

LEMMA 2.10. *Let R be any commutative ring. Then x in $X(R)$ is an isolated point if and only if $Rx = R(1 - f)$ for some (necessarily minimal) idempotent f of R.*

Proof. If x is isolated, then $x = U_f$ for some idempotent f. Then $1 - e$ is in x if and only if $U_f \subseteq U_e$, i.e., if and only if $fe = f$, so $(1 - f)(1 - e) = 1 - e$. Thus $Rx = R(1 - f)$. Now suppose, conversely, that $Rx = R(1 - f)$ and y is any element of U_f. Then $1 - f \in y$, so $Rx \leq Ry$. Let P be a prime ideal of R containing Ry; $P \cap B(R)$ is a maximal ideal of $B(R)$ containing both x and y, hence equal to both x and y. It follows that $U_f = \{x\}$, and thus x is isolated.

We remark that by (2.9) and (2.10), if x is an isolated point of $X(R)$ then both the properties discussed immediately before (2.8) hold for x: by (2.10), $R_x = Rf$, and hence

$$\mathrm{Aut}_R(S) = \mathrm{Aut}_{Rx}(S_x) \times \mathrm{Aut}_{R(1-f)}(S(1 - f)) .$$

Also, we many view (2.9) and (2.10) as evidence that the second property will not hold unless x is isolated.

DEFINITION 2.11. Let S be a commutative R-algebra. A subalgebra T of S is called *separated* if given s in S but not in T there is a minimal idempotent e of R such that se in not in Te.

THEOREM 2.12. *Let S be a normally Galois R-algebra, T a separated locally separable subalgebra. Then $T = S^{\mathrm{Aut}T(S)}$.*

Proof. We need to show that if s is an element of $S \backslash T$ there is a g in $\mathrm{Aut}_R(S)$ such that $g(s) \neq s$ but $g \,|\, T = id$. Since T is separated, there is an isolated point x of $X(R)$ such that s_x is not in T_x. By (2.7), there is an h in $\mathrm{Aut}_{Rx}(S_x)$ such that $h \,|\, T_x = id$ and $h(s_x) \neq s_x$. Since x is isolated, there is a g in $\mathrm{Aut}_R(S)$ with $g_x = h$ (as maps).

Considering $g \mid T$, then, as an element of $\mathrm{Hom}_R(T, S)$ we have, by (2.9), that $(g \mid T)_x = g_x \mid T_x = id_x$. By (2.10), there is a minimal idempotent f of R such that $R_x = Rf$; hence $gf \mid Tf = id$. Replacing g by $(1 - f) + gf$ then gives the desired automorphism.

Theorem (2.12) gives a sufficient condition for a locally separable subalgebra of a normally Galois R-algebra to be the fixed ring of a subgroup of $\mathrm{Aut}_R(S)$. We now consider hypotheses on R under which the condition becomes sufficient.

Recall that a ring is said to be *semi-hereditary* if every finitely generated submodule of a free module is projective. A ring is semi-hereditary if and only if every finitely generated ideal is projective, and hence a von Neumann regular ring is semi-hereditary.

THEOREM 2.13. *Let R be a semi-hereditary ring and suppose that every idempotent of R contains a minimal idempotent. Let S be a normally Galois R-algebra and T a locally separable subalgebra of S. Then T is the fixed ring of a subgroup of $\mathrm{Aut}_R(S)$ if and only if T is separated.*

Proof. By (2.12), we need only show that if $T = S^H$ for some subgroup H of $\mathrm{Aut}_R(S)$ then T is separated. Let s be an element of S and T_0 a normal weakly Galois subalgebra of S containing s. Let g be in H and consider the R-submodule $R(g(s) - s)$ of T_0; since R is semi-hereditary, this submodule is projective. For any x in $X(R)$, $(R(g(s) - s))_x = R_x(g_x(s_x) - s_x)$. Thus if $g_x(s_x) = s_x$ for all x in a dense subset of $X(R)$, $R(g(s) - s)$ would be a projective R-module of rank zero on a dense subset of $X(R)$. Then since the rank function of a projective module is continuous on $X(R)$, $R(g(s) - s) = 0$ and $g(s) = s$.

The hypothesis on the idempotents of R means that the isolated points of $X(R)$ are dense. Thus if s is not in T, there is a g in H with $g(s) \neq s$ and hence, by the above, an isolated point x of $X(R)$ such that $g_x(s_x) \neq s_x$. If e is the minimal idempotent of R such that $R_x = Re$, $Te = (S^H)e = (Se)^{He} = S_x^{Hx}$, so $s_x = se$ is not in Te. It follows that T is separated.

The hypothesis that R be semi-hereditary was employed only to show that if $g_x(s_x) = s_x$ on a dense subset of $X(R)$, $g(s) = s$. A similar result holds in general without hypotheses on R if we assume that s is a separable element.

PROPOSITION 2.14. *Let S be a normally Galois R-algebra, g an element of $\mathrm{Aut}_R(S)$ and s a separable element of S (i.e., $R[s]$ is a separable R-algebra). Then $g(s) = s$ if and only if $g_x(s_x) = s_x$ for all x on a dense subset of $X(R)$.*

Proof. Let T_0 be a normal weakly Galois subalgebra containing s. Let H be the subgroup of $\mathrm{Aut}_R(T_0)$ generated by $g \mid T_0$, and let $T = T_0^H$. H is finite [10, 2.16] and hence T is a strongly separable R-algebra. Let $T' = T[s]$. T' is an epimorphic image of $T \otimes R[s]$, hence separable, and T' is contained in T_0, hence is strongly separable. Since $T \subseteq T'$, T is a direct summand of T' and $M = T'/T$ is a finitely generated projective R-module. For each x in $X(R)$, $T_x = (T_0)_x^{Hx}$, $T'_x = T_x[s_x]$ and $M_x = T'_x/T_x$. Thus $g_x(s_x) = s_x$ if and only if $M_x = 0$. Thus if $g_x(s_x) = s_x$ on a dense subset of $X(R)$, M_x has zero rank on a dense subset of $X(R)$; hence $M = 0$, $T' = T$ and $g(s) = s$. The converse is trivial.

With (2.14) available, it is clear that (2.13) can be proved with the hypothesis of semi-hereditary replaced by the condition that any element of any strongly separable R-algebra is a separable element. In fact, however, this condition on R is much stronger than semi-hereditary. We shall show that it holds if and only if R is a von Neumann regular ring.

PROPOSITION 2.15. *Let R be a von Neumann regular ring, S a strongly separable R-algebra. Then every element of s is separable.*

Proof. Let s be in S. $R[s]$ is finitely generated as an R-algebra, hence is separable if for each prime p of R $(R[s])_P$ is separable over R_p. But since R is regular, $\mathrm{Spec}(R) = X(R)$ and hence we need to show that each $(R[s])_x$ is separable to conclude the result. We have $(R[s])_x = R_x[s_x] \subseteq S_x$, and hence, since R_x is a field and S_x is R_x-separable, $R[s]_x$ is R_x-separable.

LEMMA 2.16. *Let R be a connected ring and S a strongly separable R algebra not equal to R. If every element of S is separable, every nonunit of R is nilpotent.*

Proof. First observe that every subalgebra of S is separable: for by hypothesis every finitely generated subalgebra is separable, and hence every subalgebra is locally separable. Let T be a subalgebra of S, and write T as the union of an ascending chain $\{T_i\}$ of separable subalgebras. Since S is strongly separable, each of these subalgebras is a direct summand of S. It follows that the ranks of the S/T_i must properly decrease as i increases, and hence is constant for $i \geqq j$, some j. But then since T_j is a summand of T_i for all $i \geqq j$, and T_i and T_j have the same rank, $T_i = T_j$ for all $i \geqq j$.

Now let b be a nonunit of R which is not nilpotent. Let m be a maximal ideal of R containing b. We consider the subalgebra $T = R + mS$ of S. We will show that $T \neq R$ and $T \neq S$. Thus T is a

strongly separable R-algebra, not equal to R, such that every element of T is R-separable, and $[T : R] < [S : R]$. By induction, then, we have a contradiction. If $T = S$, $S/mS = R + mS/mS = R/R \cap mS$ is of rank $\leqq 1$ over R/m. This is impossible since $S \neq R$ and hence $[S : R] > 1$. If $T = R$, $Sb \leqq R$ so b annihilates S/R, which is a finitely generated projective R-module of rank $\geqq 1$. Let p be a prime of R. If b is not in p, S_p/R_p is zero since it is annihilated by a unit of R_p; on the other hand $[S_p/R_p : R_p] \geqq 1$. Hence b must be in p. But then b is in every prime of R, hence nilpotent, contrary to assumption.

PROPOSITION 2.17. *If every element of every strongly separable R-algebra is separable, then R is a von Neumann regular ring.*

Proof. Let x be a point of $X(R)$; since $R \times R$ is a strongly separable R-algebra, every element of $R_x \times R_x$ is separable over R_x. Thus, by (2.16) every nonunit of R_x is nilpotent. Let r be in R_x with $r^2 = 0$. If $r \neq 0$, $R_x[(r, 0)]$ is a separable subalgebra of $R_x \times R_x$ not equal to R_x, hence of rank $\geqq 2$. Thus $R_x[(r, 0)] = R_x \times R_x$. But every element of $R_x[(r, 0)]$ may be written as $t(1, 1) + s(r, 0) = (t + sr, t)$ for some s, t in R_x. Hence there are s, t such that $(t + sr, t) = (0, 1)$, i.e., $1 = -sr$. Thus r is a unit, contradiction. It follows that R_x has no nilpotents, hence is a field, and since each R_x is a field, R is von Neumann regular.

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