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VARIOUS m-REPRESENTATIVE DOMAINS IN SEVERAL COMPLEX VARIABLES

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Our main purpose is to introduce several functions which map a bounded domain D onto m-representative domain in several complex variables without the help of the minimum problems or the use of determinantal expressions. We use constructive methods to obtain m-representative functions.

S. Bergman introduced two kinds of canonical domains, *minimal domains and representative domains*, by using the mapping functions which were expressed in terms of the Bergman kernel function and its derivatives (see [1], [2]). Further, M. Maschler introduced two types of canonical domains named *m-minimal* and *m-representative* domains in one variable by using minimum problems. Now, we consider a bounded univalent domain D in C^n, and a vector function $w(z) = (w_1(z), w_2(z), \cdots, w_n(z))'$ in D. If each component $w_i(z)$ is holomorphic, then the function $w(z)$ defines a *holomorphic mapping* of the domain $D \subset C^n$ onto the domain $\varDelta \subset C^n$, and if the mapping $w(z)$ is both holomorphic and locally one-to-one, i.e., $\det dw/dz \neq 0$ (see § 1 and [4], [6]), it is *pseudo-conformal*.

By means of some matrix derivative formulas, the author obtains pseudo-conformal relative invariant matrix systems[1] ${}_\nu T_D(\bar{t}, z)$ and matrix system $\overset{(\nu)}{T}_D(t_0; z)$, $\overset{(\nu)}{S}_D(t_0; z)$. Thus we shall arrive at several types of m-representative functions of D which are constructed by the operators σ_D^ν and δ_D^ν (see § 3, § 4). In general, it is not known if the m-representative functions of a bounded domain are holomorphic or even exist, but we have a holomorphic m-representative function under the condition $K_D(\bar{t}_0, z) \neq 0$ in D (see Theorem 3.2).

1. Preliminaries.

Let $\mathscr{L}^2(D)$ be a class of holomorphic functions $f(z)$ integrable square in the sense of Lebesque in D, namely

$$\int_D |f(z)|^2 dv_z < \infty$$

where dv_z is the volume element in D, and let $\varphi(z) = (\varphi_1(z), \varphi_2(z), \cdots)'$ be a closed system of orthonormal functions in D. The Bergman kernel function of the system $\varphi(z)$ is given by $K_D(\bar{t}, z) = \varphi^*(\bar{t})\varphi(z)$, $z, t \in D$ where the marks $'$ and $*$ denote the transposed and transposed conjugate

[1] Utilizing this matrix, Riemann curvatures were formed in our Seminar, (see Sci. Rep. Tōkyō Kyōiku D. Sec. A, No. 182, 188).

matrices respectively. This function $K_D(\bar{t}, z)$ is characterized by the domain D, and if D be a domain equivalent pseudo-conformally to a bounded domain the Bergman kernel function $K_D(\bar{t}, z)$ exists in D and $K_D(\bar{z}, z) > 0$ for any point $z \in D$. If $\zeta = \zeta(z)$ is a pseudo-conformal mapping of a domain D onto a domain Δ, then we have

$$K_D(\bar{t}, z) = (\overline{\det d\tau(t)/dt})K_\Delta(\bar{\tau}, \zeta)(\det d\zeta(z)/dz) , \tag{1.1}$$

$$T_D(\bar{t}, z) = (d\tau(t)/dt)^* T_\Delta(\bar{\tau}, \zeta)(d\zeta(z)/dz) , \tag{1.2}$$

and we have $T_D(\bar{t}, z) = K_D^{-2}(\bar{t}, z)(K_D(\bar{t}, z)K_{Dt^*z}(\bar{t}, z) - K_{Dt^*}(\bar{t}, z)K_{Dz}(\bar{t}, z))$.

Next, we define a *pseudo-conformal equivalence class of D with respect to a fixed point $t_0(\in D)$*, that is, each domain Δ that belongs to the class is the image of D by a pseudo-conformal transformation $\zeta(z)$ satisfying

$$\zeta(t_0) = 0,\ d\zeta(t_0)/dz = E_n,\ d^2\zeta(t_0)/dz^2 = \cdots = d^m\zeta(t_0)/dz^m = 0 . \tag{1.3}$$

An invariant function of the pseudo-conformal equivalence class satisfying (1.3) is called *m-representative function* of the class, and the image domain by it is called *m-representative domain of the class with center at the origin.* And we define the power of z as follows:

$$z^k \equiv (z_1^k, \cdots, z_2^{k_1} z_2^{k_2} \cdots z_n^{k_n}, \cdots, z_n^k)' , \tag{1.4}$$

where $(k_1, k_2, \cdots, k_n)$ range over all the nonnegative integers such that $k_1 + k_2 + \cdots + k_n = k$ and ${}_nH_k$ monomials of degree k with respect to $z_1, z_2, \cdots, z_n$ are arranged by a certain rule. We define the kth partial derivative of matrix function with respect to z and z^* as

$$\begin{aligned} \partial^k w(\bar{t}, z)/\partial z^k &\equiv \partial^k/\partial z^k \cdot w(\bar{t}, z) \\ &\equiv \left(\frac{\partial^k}{\partial z_1^k}, \cdots, \frac{k!}{k_1!\, k_2! \cdots k_n!} \frac{\partial^k}{\partial z_1^{k_1} \partial z_2^{k_2} \cdots \partial z_n^{k_n}},\right. \\ &\qquad \left. \cdots, \frac{\partial^k}{\partial z_n^k}\right) \times w(\bar{t}, z) , \end{aligned} \tag{1.5}$$

where $\partial^k/\partial z^k$ will be arranged in the same rule as z^k, and the sign $\times$ designates the Kronecker product. If $w(z)$ is a function of z only the kth derivative is denoted by $d^k w(z)/dz^k$, moreover we define

$$\begin{aligned} \partial^2 w/\partial t^* \partial z &= \partial/\partial t^* \times \partial/\partial z \times w = (\partial/\partial t)^* \times (\partial/\partial z) \times w \\ &= \begin{pmatrix} \partial^2 w_1/\partial \bar{t}_1 \partial z_1, & \partial^2 w_1/\partial \bar{t}_1 \partial z_2, & \cdots, & \partial^2 w_1/\partial \bar{t}_1 \partial z_n \\ \partial^2 w_2/\partial \bar{t}_2 \partial z_1, & \partial^2 w_2/\partial \bar{t}_2 \partial z_2, & \cdots, & \partial^2 w_2/\partial \bar{t}_2 \partial z_n \\ & \cdots\cdots\cdots\cdots & & \\ \partial^2 w_n/\partial \bar{t}_n \partial z_1, & \partial^2 w_n/\partial \bar{t}_n \partial z_2, & \cdots, & \partial^2 w_n/\partial \bar{t}_n \partial z_n \end{pmatrix} . \end{aligned} \tag{1.6}$$

We denote the following formulas with respect to the matrix derivatives which will be of use in calculation for demonstration hereafter:

(1.7)
$$\begin{aligned}\partial F^{-1}/\partial z &= -F^{-1}\partial F/\partial z(E_n \times F^{-1}),\ F^{-1}\partial F/\partial z \\ &= -\partial F^{-1}/\partial z(E_n \times F)\ ,\end{aligned}$$

(F is a regular $k \times k$ matrix function, $z = (z_1, \cdots, z_n)'$, and E_n is an $n \times n$ unit matrix)

(1.8)
$$\partial(FG)/\partial z = \partial F/\partial z(E_n \times G) + F\partial G/\partial z\ ,$$

(F, G are $k \times l$, $l \times m$ matrices respectively)

(1.9)
$$\partial F/\partial z = \partial F/\partial \zeta(d\zeta/dz \times E_l) + (d\zeta^*/dz \times E_k)(E_n \times \partial F/\partial \zeta^*)$$

(F is a $k \times l$ matrix)

(1.10)
$$\partial(F \times G)/\partial z = (\partial F/\partial z \times G) + (F \times \partial G/\partial z)(\widetilde{E}_{ln} \times E_\nu)\ ,$$

(F, G are $k \times l$, $\mu \times \nu$ matrices respectively, and

$$\widetilde{E}_{ln} = \begin{pmatrix} e_{11}, \cdots, e_{l1} \\ e_{12}, \cdots, e_{l2} \\ \cdots\cdots \\ e_{1n}, \cdots, e_{ln} \end{pmatrix},$$

where e_{ij} are $l \times n$ matrices in which there is only (i, j) element equal 1, and others 0.)

2. Relative invariant matrix system. The Riemann mapping theorem does not hold for more than one complex variable, instead various canonical domains have been introduced. In this section, we shall introduce a relative invariant matrix system which is connected with the construction of m-representative functions.

We can easily calculate by virtue of the formulas (1.7), (1.8), and $(A \times B)^* = A^* \times B^*$, $(A \times B)(C \times D) = AC \times BD$, as follows:

(2.1)
$$\begin{aligned}&(E_n \times T_D(\bar{t}, z))\partial/\partial t^*(T_D^{-1}(\bar{t}, z)\partial T_D(\bar{t}, z)/\partial z) \\ &\quad = \partial^2 T_D(\bar{t}, z)/\partial t^*\partial z - \partial T_D(\bar{t}, z)/\partial t^* T_D^{-1}(\bar{t}, z)\partial T_D(\bar{t}, z)/\partial z\ .\end{aligned}$$

Therefore, we introduce

(2.2)
$$\begin{aligned}{}_mT_D(\bar{t}, z) &= \partial^2{}_{m-1}T_D(\bar{t}, z)/\partial t^*\partial z \\ &\quad - \partial_{m-1}T_D(\bar{t}, z)/\partial t^*{}_{m-1}T_D^{-1}(\bar{t}, z)\partial_{m-1}T_D(\bar{t}, z)/\partial z,\ (m \geqq 2)\ ,\end{aligned}$$

where E_n denotes an $n \times n$ unit matrix, and ${}_1T_D(\bar{t}, z) = T_D(\bar{t}, z) = \partial^2 \log K_D(\bar{t}, z)/\partial t^*\partial z$.

THEOREM 2.1. *The square matrix system* ${}_mT_D(\bar{t}, z)$ *is a relative invariant with respect to any pseudo-conformal mapping* $\zeta = \zeta(z)$, *that is,*

$$(2.3) \qquad {}_mT_D(\bar{t}, z) = (d\tau(t)/dt)^{*m}\, {}_mT_\Delta(\bar{\tau}, \zeta)(d\zeta(z)/dz)^m\ ,$$

where $\tau = \zeta(t)$, $\Delta = \zeta(D)$, *and the* mth *power* $(d\zeta/dz)^m$ *of* $d\zeta/dz$ *denotes a suitably contracted* matrix of n *times Kronecker product.*

Proof. If we suppose that the relations (2.3) is established, we may calculate as follows by formulas (1.7) $\sim$ (1.9) and Cauchy-Riemann differential equation $\partial w/\partial z^* = 0$ for the holomorphic mapping,

$$(2.4) \qquad \begin{aligned} \partial_m T_D/\partial z = {} & (d\tau/dt)^{*m}\{\partial_m T_\Delta/\partial\zeta(E_n \times (d\zeta/dz)^m) \\ & + {}_mT_\Delta d(d\zeta/dz)^m/dz(dz/d\zeta \times E_{nm})\}(d\zeta/dz \times E_{nm})\ , \end{aligned}$$

$$(2.5) \qquad \begin{aligned} & \partial_m T_D/\partial t^*\, {}_mT_D^{-1}\partial_m T_D/\partial z \\ & = (d\tau/dt)^{*m+1}\partial_m T_\Delta/\partial\tau^*\, {}_mT_\Delta^{-1}\partial_m T_\Delta/\partial\zeta(d\zeta/dz)^{m+1} \\ & \quad + d(d\tau/dt)^{*m}/dt^*\partial_m T_\Delta/\partial\zeta(d\zeta/dz)^{m+1} \\ & \quad + (d\tau/dt)^{*m+1}\partial_m T_\Delta/\partial\tau^* d(d\zeta/dz)^m/dz \\ & \quad + d(d\tau/dt)^{*m}/dt^*\, {}_mT_\Delta d(d\zeta/dz)^m/dz\ , \end{aligned}$$

$$(2.6) \qquad \begin{aligned} & \partial^2{}_m T_D/\partial t^*\partial z \\ & = (d\tau/dt)^{*m+1}\partial^2{}_m T_\Delta/\partial\tau^*\partial\zeta(d\zeta/dz)^{m+1} \\ & \quad + d(d\tau/dt)^{*m}/dt^*\partial_m T_\Delta/\partial\zeta(d\zeta/dz)^{m+1} \\ & \quad + (d\tau/dt)^{*m+1}\partial_m T_\Delta/\partial\tau^* d(d\zeta/dz)^m/dz \\ & \quad + d(d\tau/dt)^{*m}/dt^*\, {}_mT_\Delta d(d\zeta/dz)^m/dz\ , \end{aligned}$$

whence we have (2.3) with m replaced by $m + 1$.

Now, we may derive some positive definite Hermitian form utilized this result.

LEMMA 2.1.[2] *For the kernel function* $K_D(\bar{t}, z)$ *and* $T_D(\bar{t}, z)$ *of any domain* D, *we have*

$$(2.7) \qquad T_{2D}(\bar{t}, z) \equiv K_D^2(\bar{t}, z)T_D(\bar{t}, z) = \chi^*(\bar{t})\chi(z)\ ,$$

where $\chi(z) = 1/\sqrt{2}(\varphi(z) \times \partial\varphi(z)/\partial z - \partial\varphi(z)/\partial z \times \varphi(z))$.

Here, we shall obtain the relation between $T_{2D}(\bar{t}, z)$ and the author's matrix ${}_2T_D(\bar{t}, z)$ proceeding with our calculations of the matrix derivatives

[2] This lemma is due to S. Katō [**7**].

$$\begin{aligned}\partial^2 T_{2D}(\bar{t}, z)/\partial t^*\partial z - \partial T_{2D}(\bar{t}, z)/\partial t^* T_{2D}^{-1}(\bar{t}, z)\partial T_{2D}(\bar{t}, z)/\partial z \\ = K_D^2(\bar{t}, z)({}_2T_D(\bar{t}, z) + 2T_D(\bar{t}, z) \times T_D(\bar{t}, z)) .\end{aligned} \tag{2.8}$$

In fact, we can derive the following relation by the formula (1.8) and the rule $(A \times B)(C \times D) = AC \times BD$,

$$\partial T_{2D}/\partial t^* = K_D^2 \partial T_D/\partial t^* + \partial K_D^2/\partial t^* \times T_D , \tag{2.9}$$

similarly for $\partial T_{2D}/\partial z$,

$$\begin{aligned}\partial^2 T_{2D}/\partial t^*\partial z = K_D^2 \partial^2 T_D/\partial t^*\partial z + \partial K_D^2/\partial t^* \times \partial T_D/\partial z \\ + \partial^2 K_D^2/\partial t^*\partial z \times T_D + \partial K_D^2/\partial z \times \partial T_D/\partial t^* .\end{aligned} \tag{2.10}$$

Then (2.8) follows. If we call the matrix expression (2.8) ${}_2T_{2D}(\bar{t}, z)$, we can verify that ${}_2T_{2D}(\bar{z}, z)$ is positive definite.

THEOREM 2.1. *The matrix function*

$${}_2T_D(\bar{t}, z) + mT_D(\bar{t}, z) \times T_D(\bar{t},, z), \ (m > 2)$$

is relative invariant under any pseudo-conformal mapping $\zeta = \zeta(z)$, *and positive definite for* $t = z$.

Proof. By using $\chi(z)$ in Lemma 2.1, we have

$${}_2T_{2D}(\bar{z}, z) = \chi_{z^*}^*(\bar{z})\chi_z(z) - \chi_{z^*}^*(\bar{z})\chi(z)T_{2D}^{-1}(\bar{z}, z)\chi^*(\bar{z})\chi_z(z) ,$$

therefore we obtain for any n^2-dimensional column vector u,

$$\begin{aligned}\begin{pmatrix} E_n & , T_{2D}^{-1/2}\partial T_{2D}/\partial z u \\ u^*\partial T_{2D}/\partial z^* T_{2D}^{-1/2}, & u^*\partial^2 T_{2D}/\partial z^*\partial z u \end{pmatrix} \\ = (\chi(z)T_{2D}^{-1/2}, \partial\chi(z)/\partial z u)^*(\chi(z)T_{2D}^{-1/2}, \partial\chi(z)/\partial z u) .\end{aligned} \tag{2.11}$$

Then we have

$$\begin{aligned}&\det(\chi T_{2D}^{-1/2}, \partial\chi/\partial z u)^*(\chi T_{2D}^{-1/2}, \partial\chi/\partial z u) \\ &= u^*\partial^2 T_{2D}/\partial z^*\partial z u - u^*\partial T_{2D}/\partial z^* T_{2D}^{-1}\partial T_{2D}/\partial z u = u^*{}_2T_{2D}u \geqq 0 .\end{aligned}$$

Therefore, ${}_2T_D + 2\cdot T_D \times T_D$ is nonnegative definite, then ${}_2T_D + m\cdot T_D \times T_D \ (m > 2)$ is positive definite.

Next, we state the following symbol,

$$\tau_D F_D(\bar{t}, z) = \partial^2 F_D/\partial t^*\partial z - \partial F_D/\partial t^* F_D^{-1}\partial F_D/\partial z , \tag{2.12}$$

then we have ${}_mT_D(\bar{t}, z) = (\tau_D)^{m-1}T_D(\bar{t}, z)$.

THEOREM 2.2. *For any matrix function* $F_D(\bar{t}, z)$ *which transforms by relation* $F_D(\bar{t}, z) = (d\tau(t)/dt)^* F_\Delta(\bar{\tau}, \zeta)(d\zeta(z)/dz)$ *under pseudo-conformal mapping* $\zeta = \zeta(z)$, *we have*

$$(2.13)\qquad (\tau_D)^m F_D(\bar{t}, z) = (d\tau(t)/dt)^{*m+1}(\tau_\Delta)^m F_\Delta(\bar{\tau}, \zeta)(d\zeta(z)/dz)^{m+1} .$$

COROLLARY 2.1. *If we construct the matrix functions*

$$(2.14)\qquad F_D^\mu(\bar{t}, z) \equiv \partial^2 \log\det (K_D^\mu(\bar{t}, z) T_D(\bar{t}, z))/\partial t^* \partial z ,$$

we obtain the following transformation expression

$$(2.15)\qquad \begin{aligned} {}^\mu\overset{(m)}{G}_D(\bar{t}, z) &\equiv (\tau_D)^{m-1} F_D^\mu(\bar{t}, z) \\ &= (d\tau(t)/dt)^{*m}(\tau_\Delta)^{m-1} F_\Delta^\mu(\bar{\tau}, \zeta)(d\zeta(z)/dz)^m , \end{aligned}$$

where μ is an arbitrary real number.

3. m-representative domains derived by operators ${}^i\sigma_D^\nu$. First, we define matrix functions ${}_{(\nu)}T_D(\bar{t}, z)$ (not ${}_\nu T_D(\bar{t}, z)$) with respect to both z and $t^*(z, t \in D)$ with a fixed point t_0 of D as follows.

$$(3.1)\qquad \begin{aligned} {}_{(\nu)}T_D(\bar{t}, z) &= \partial^2{}_{(\nu-1)}T_D(\bar{t}, z)/\partial t^* \partial z \\ &\quad - \partial_{(\nu-1)}T_D(\bar{t}, t_0)/\partial t^* ({}_{(\nu-1)}T_D)^{-1} \partial_{(\nu-1)}T_D(\bar{t}_0, z)/\partial z, \ (\nu \geqq 2) , \end{aligned}$$

where ${}_{(1)}T_D(\bar{t}, z) = T_D(\bar{t}, z)$, ${}_{(\nu-1)}T_D = {}_{(\nu-1}T_D(\bar{t}_0, t_0)$, and by putting $t = t_0$, we have

$$(3.2)\qquad \begin{aligned} {}_{(\nu)}T_D(\bar{t}_0, z) &= \partial^2{}_{(\nu-1)}T_D(\bar{t}_0, z)/\partial t^* \partial z \\ &\quad - \partial_{(\nu-1)}T_D/\partial t^* ({}_{(\nu-1)}T_D)^{-1}{}_{\partial(\nu-1)}T_D(\bar{t}_0, z)/\partial z . \end{aligned}$$

where $\partial_{(\nu-1)}T_D/\partial t^* \equiv [\partial_{(\nu-1)}T_D(\bar{t}, z)/\partial t^*]_{z=t_0, t=t_0}$. The definite integral of a matrix $A(z)$ is

$$(3.3)\qquad \int_{t_0}^z A(z)dz = B(z) - B(t_0) ,$$

where $dB(z)/dz = A(z)$, then we have

$$(3.4)\qquad \int_{t_0}^z {}_{(2)}T_D(\bar{t}_0, z)dz = \partial T_D(\bar{t}_0, z)/\partial t^* - \partial T_D/\partial t^* (T_D)^{-1} T_D(\bar{t}_0, z) ,$$

$$(3.5)\qquad \begin{aligned} &\int_{t_0}^z \int_{t_0}^z {}_{(3)}T_D(\bar{t}_0, z)dz)^2 \\ &= \int_{t_0}^z (\partial_{(2)}T_D(\bar{t}_0, z)/\partial t^* - \partial_{(2)}T_D/\partial t^* ({}_{(2)}T_D)^{-1}{}_{(2)}T_D(\bar{t}_0, z))dz \\ &= \partial^2 T_D(\bar{t}_0, z)/\partial t^{*2} - \partial^2 T_D/\partial t^{*2}(T_D)^{-1} T_D(\bar{t}_0, z) \\ &\quad - \partial_{(2)}T_D/\partial t^* ({}_{(2)}T_D)^{-1}(\partial T_D(\bar{t}_0, z)/\partial t^* - \partial T_D/\partial t^* (T_D)^{-1} T_D(\bar{t}_0, z)) . \end{aligned}$$

Therefore, if we introduce a matrix function as follows

$$(3.6)\qquad \begin{aligned} \overset{(2)}{M}{}_D^1(t_0; z) &\equiv {}^1\sigma_D^1 T_D(\bar{t}_0, z) \\ &\equiv T_D(\bar{t}_0, z) - \partial T_D/\partial z_{(2)} T_D^{-1} \int_{t_0}^z {}_{(2)}T_D(\bar{t}_0, z)dz , \end{aligned}$$

we have an invariant holomorphic function $\overset{(2)}{\zeta}{}^1_D(z; t_0)$ under any pseudo-conformal mapping $\zeta = \zeta(z)$ which satisfies the conditions

$$\zeta(t_0) = 0,\ d\zeta(t_0)/dz = E,\ d^2\zeta(t_0)/dz^2 = 0\ , \tag{3.7}$$

and the invariant function also satisfies (3.7):

$$\overset{(2)}{\zeta}{}^1_D(z; t_0) \equiv T_D^{-1}\int_{t_0}^{z} \overset{(2)}{M}{}^1_D(t_0; z)dz\ . \tag{3.8}$$

Because, in general, for any pseudo-conformal mapping $\zeta = \zeta(z)$ satisfying (1.3) we have $\partial^{p+q}T_D(\bar{t}_0, t_0)/\partial t^{*p}\partial z^q = \partial^{p+q}T_\Delta(\bar{0}, 0)/\partial\tau^{*p}\partial\zeta^q$, $(0 \leqq p, q \leqq m-1)$, and we have $\partial^p T_D(\bar{t}_0, z)/\partial t^{*p} = \partial^p T_\Delta(0, \zeta)/\partial\tau^{*p} d\zeta(\bar{z})/dz$ only if $q = 0$. (See (2.4), (2.6) and [7]).

By this function $\overset{(2)}{\zeta}{}^1_D$, D and $\Delta(=\zeta(D))$ generate the some domain R. We call this unique domain R *2-representative domain of the pseudo-conformal equivalence class of D with center at the origin*, and the function $\overset{(2)}{\zeta}{}^1_D(z; t_0)$ will be called *2-representative function*. Moreover if we define a matrix

$$\begin{aligned}\overset{(3)}{M}{}^1_D(t_0; z) &= {}^1\sigma^2_D{}^1\sigma^1_D T_D(\bar{t}_0, z) = {}^1\sigma^2_D({}^1\sigma^1_D T_D(\bar{t}_0, z)) = \overset{(2)}{M}{}^1_D(t_0; z)\\ &\quad - \partial^2\overset{(2)}{M}{}^1_D/\partial z^2{}_{(3)}T_D^{-1}\int_{t_0}^{z}\int_{t_0}^{z}{}_{(3)}T_D(\bar{t}_0, z)(dz)^2\ ,\end{aligned} \tag{3.9}$$

we obtain a *3-representative function* $\overset{(3)}{\zeta}{}^1_D(z; t_0)$ of the pseudo-conformal equivalence class of D which satisfies the conditions $\zeta(t_0) = 0$, $d\zeta(t_0)/dz = E$, $d^2\zeta(t_0)/dz^2 = d^3\zeta(t_0)/dz^3 = 0$:

$$\overset{(3)}{\zeta}{}^1_D(z; t_0) \equiv T_D^{-1}\int_{t_0}^{z} \overset{(3)}{M}{}^1_D(t_0; z)dz\ . \tag{3.10}$$

Now, we have the following relation:

$$\begin{aligned}\overset{(3)}{N}(t_0, z) &\equiv (E, 0, 0)\begin{pmatrix} T_D & T_z & T_{z^2}\\ T_{t^*} & T_{t^*z} & T_{t^*z^2}\\ T_{t^{*2}} & T_{t^{*2}z} & T_{t^{*2}z^2}\end{pmatrix}^{-1}\begin{pmatrix} T_D(\bar{t}_0, z)\\ \partial T_D(\bar{t}_0, z)/\partial t^*\\ \partial^2 T_D(\bar{t}_0, z)/\partial t^{*2}\end{pmatrix}\\ &= T_D^{-1}\overset{(3)}{M}{}^1_D(\bar{t}_0; z)\ ,\end{aligned} \tag{3.11}$$

where $T_{t^{*p}z^q} = \partial^{p+q}T_D(\bar{t}_0, t_0)/\partial t^{*p}\partial z^q$. It is proved by means of the well-known formula

$$\begin{aligned}&\begin{pmatrix} K & L\\ M & N\end{pmatrix}^{-1}\\ &= \begin{pmatrix} K^{-1} + K^{-1}L(N-MK^{-1}L)^{-1}MK^{-1}, & -K^{-1}L(N-MK^{-1}L)^{-1}\\ -(N-MK^{-1}L)^{-1}MK^{-1}, & (N-MK^{-1}L)^{-1}\end{pmatrix},\end{aligned} \tag{3.12}$$

(see [5]) .

In general, if we introduce the matrix functions as follows

$$\overset{(m)}{M}{}^1_D(t_0; z) = {}^1\sigma_D^{m-1}{}^1\sigma_D^{m-2}\cdots{}^1\sigma_D^1 T_D(\bar{t}_0, z),\ (m \geqq 2)\ , \tag{3.13}$$

where

$$\begin{aligned}{}^1\sigma_D^{\nu-1}F(t_0; z) &= F(t_0; z)\\ &\quad - (\partial^{\nu-1}F(t_0; z)/\partial z^{\nu-1})_{z=t_0(\nu)} T_D^{-1}\int_{t_0}^z\cdots\int_{t_0}^z {}_{(\nu)}T_D(\bar{t}_0, z)\\ &\quad (dz)^{\nu-1}\ ,\end{aligned} \tag{3.14}$$

for any matrix function $F(t_0; z)$, then we have an m-representative function of the pseudo-conformal equivalence class of D with respect to a fixed point t_0:

$$\overset{(m)}{\zeta}{}^1_D(z; t_0) \equiv T_D^{-1}\int_{t_0}^z \overset{(m)}{M}{}^1_D(t_0; z)dz\ . \tag{3.15}$$

Similarly, if we construct the matrix functions

$$\overset{(m)}{M}{}^{1'}_D(t_0; z) = {}^{1'}\sigma_D^{m-1}\,{}^{1'}\sigma_D^{m-2}\cdots{}^{1'}\sigma_D^1 T_D(\bar{t}_0, z),\ (m \geqq 2)\ , \tag{3.16}$$

by ${}^{1'}\sigma_D^{\nu}$ replaced ${}_{(\nu)}T_D(\bar{t}_0, z)$ with ${}_{(\nu)'}T_D(\bar{t}_0, z)$, i.e.,

$$\begin{aligned}{}_{(\nu)'}T_D(\bar{t}_0, z) &= \partial^{2(\nu-1)}T_D(\bar{t}_0, z)/\partial t^{*\nu-1}\partial z^{\nu-1}\\ &\quad - (T_{t^{*\nu-1}}, T_{t^{*\nu-1}z}, \cdots, T_{t^{*\nu-1}z^{\nu-2}})\\ &\quad \begin{pmatrix} T_D & T_z & \cdots & T_{z^{\nu-2}}\\ T_{t^*} & T_{t^*z} & \cdots & T_{t^*z^{\nu-2}}\\ \cdots & \cdots & \cdots & \cdots\\ T_{t^{*\nu-2}} & T_{t^{*\nu-2}z} & \cdots & T_{t^{*\nu-2}z^{\nu-2}}\end{pmatrix}^{-1}\begin{pmatrix} T_{z^{\nu-1}}(\bar{t}_0, z)\\ T_{t^*z^{\nu-1}}(\bar{t}_0, z)\\ \vdots\\ T_{t^{*\nu-2}z^{\nu-1}}(\bar{t}_0, z)\end{pmatrix},\end{aligned} \tag{3.17}$$

then we have another m-representative function

$$\overset{(m)}{\zeta}{}^{1'}_D(z; t_0) \equiv T_D^{-1}\int_{t_0}^z \overset{(m)}{M}{}^{1'}_D(t_0; z)dz = \int_{t_0}^z N_D^{E_n,0,\cdots 0}(z, t_0)dz\ , \tag{3.18}$$

where

$$N_D^{E_n,0\cdots 0}(z, t_0) = (E, 0, \cdots, 0)\begin{pmatrix} T_D & \cdots & T_{z^{m-1}}\\ \cdots & \cdots & \cdots\\ T_{t^{*m-1}} & \cdots & T_{t^{*m-1}z^{m-1}}\end{pmatrix}^{-1}\begin{pmatrix} T_D(\bar{t}_0, z)\\ \vdots\\ T_{t^{*m-1}}(\bar{t}_0, z)\end{pmatrix},$$

because we can compute

$$\overset{(m)}{N}(z) \equiv (E, 0, \cdots, 0)\begin{pmatrix} T_D & \cdots & T_{z^{m-2}} & T_{z^{m-1}}\\ \cdots & \cdots & \cdots & \cdots\\ T_{t^{*m-2}} & & T_{t^{*m-2}z^{m-2}} & T_{t^{*m-2}z^{m-1}}\\ T_{t^{*m-1}} & \cdots & T_{t^{*m-1}z^{m-2}} & T_{t^{*m-1}z^{m-1}}\end{pmatrix}^{-1}\begin{pmatrix} T_D(z)\\ \vdots\\ T_{t^{*m-1}}(z)\end{pmatrix} \tag{3.19}$$

$$= \overset{(m-1)}{N}(z) - \partial^{m-1}\overset{(m-1)}{N}/\partial z^{m-1}{}_{(m)'}T_D^{-1}\int_{t_0}^{z}\cdots\int_{t_0}^{z}{}_{(m)'}T_D(\bar{t}_0, z)(dz)^{m-1}$$

$$= {}^{1'}\sigma_D^{m-1}\overset{(m-1)}{N}(z)\,. \qquad \text{(See [7])}\,.$$

THEOREM 3.1. *If* $\det{}_{(\nu)}T_D(\bar{t}_0, t_0) \neq 0$, *and* $\det{}_{(\nu)'}T_D(\bar{t}_0, t_0) \neq 0$, $(2 \leqq \nu \leqq m)$ *at a fixed point* t_0 *of* D, *then we have m-representative domains of the pseudo-conformal equivalence class of* D *mapped by the m-representative (holomorphic) functions* (3.15) *and* (3.18) *respectively.*

Next, by the property of Kronecker product we can calculate formally

$$(T(\bar{t}_0, z))^{\nu}(dz)^{\nu} = (T_D(\bar{t}_0, z)dz)^{\nu}\,,$$

therefore we define

$$\text{(3.20)} \qquad \int_{t_0}^{z}\cdots\int_{t_0}^{z}(T_D(\bar{t}_0, z))^{\nu}(dz)^{\nu} = \left(\int_{t_0}^{z}T_D(\bar{t}_0, z)dz\right)^{\nu}.$$

Then we have the following m-representative function

$$\text{(3.21)} \qquad \begin{aligned} \overset{(m)}{\zeta}{}_D^2(z; t_0) &= T_D^{-1}\int_{t_0}^{z}\overset{(m)}{M}{}_D^2(t_0; z)dz \\ &= \overset{(m-1)}{\zeta}{}_D^2(z; t_0) - 1/m!\, d^m\overset{(m-1)}{\zeta}{}_D^2/dz^m(T_D^{-1})^m \\ &\qquad \left(\int_{t_0}^{z}T_D(\bar{t}_0, z)dz\right)^m, \ (m \geqq 2)\,, \end{aligned}$$

where

$$\overset{(1)}{\zeta}{}_D^2(z; t_0) \equiv T_D^{-1}\int_{t_0}^{z}T_D(\bar{t}_0, z)dz\,,$$

and

$$\begin{aligned} \overset{(m)}{M}{}_D^2(t_0; z) &\equiv {}^2\sigma_D^{m-1}\cdots{}^2\sigma_D^1 T_D(\bar{t}_0, z) = {}^2\sigma_D^{m-1}\overset{(m-1)}{M}{}_D^2(t_0; z) \\ &= \overset{(m-1)}{M}{}_D^2(t_0; z) - 1/m!\,\partial^{m-1}\overset{(m-1)}{M}{}_D^2/\partial z^{m-1}(T_D^{-1})^m \\ &\qquad \int_{t_0}^{z}\cdots\int_{t_0}^{z}(T_D(\bar{t}_0, z))^m(dz)^{m-1}\,. \end{aligned}$$

Firstly, we introduce a 2-representative domain of the pseudo-conformal equivalence class of a domain D in this case. We can compute as follows by the above-mentioned formulas (1.7) $\sim$ (1.10):

$$\begin{aligned} d/dz\left(\int_{t_0}^{z}T_D(\bar{t}_0, z)dz\right)^2 &= T \times (\quad) + ((\quad) \times T)(\tilde{E}_{1n} \times 1) \\ &= T \times (\quad) + (\quad) \times T\,, \end{aligned}$$

$$d^2/dz^2\left(\int_{t_0}^{z} T_D(\bar{t}_0, z)dz\right)^2 = T_z \times (\quad) + (T \times T)(\tilde{E}_{nn} \times 1) + T \times T$$
$$+ ((\quad) \times T_z)(\tilde{E}_{1n} \times E)$$
$$= T_z \times (\quad) + T^2\tilde{E}_{nn} + T^2 + (\quad) \times T_z\,,$$

where

$$(\quad) \equiv \int_{t_0}^{z} T_D(\bar{t}_0, z)dz,\ T \equiv T_D(\bar{t}_0, z),\ T_z \equiv \partial T_D(\bar{t}_0, z)/\partial z\,.$$

Then we have

$$(d^2(\quad)^2/dz^2)_{z=t_0} = T_D^2(\tilde{E}_{nn} + E^2)\,. \tag{3.22}$$

Further, we have following results.

LEMMA 3.1. *For any n row vector $x = (x_1, x_2, \cdots, x_n)$, we have*

$$x^2\tilde{E}_{nn} = x^2\,, \tag{3.23}$$

and, in general, for arbitrary positive integers p, q

$$x^{2+p+q}(E^p \times \tilde{E}_{nn} + E^q) = x^{2+p+q}\,. \tag{3.24}$$

Thus we have

$$d^2\overset{(1)}{\zeta}{}^2_D/dz^2(\tilde{E}_{nn} + E^2) = 2d^2\overset{(1)}{\zeta}{}^2_D/dz^2\,,$$

for any n column vector $\overset{(1)}{\zeta}{}^2_D$.

Therefore we have a 2-representative function

$$\begin{aligned}\overset{(2)}{\zeta}{}^2_D(z; t_0) &\equiv T_D^{-1}\int_{t_0}^{z} \overset{(m)}{M}{}^2_D(t_0; z)dz \\ &= \overset{(1)}{\zeta}{}^2_D(z; t_0) - 1/2!\ d^2\overset{(1)}{\zeta}{}^2_D/dz^2(T_D^{-1})^2\left(\int_{t_0}^{z} T_D(\bar{t}_0, z)dz\right)^2.\end{aligned} \tag{3.25}$$

where

$$\overset{(2)}{M}{}^2_D(t_0; z) \equiv {}^2\sigma^1_D T_D(\bar{t}_0, z) = T_D(\bar{t}_0, z)$$
$$- 1/2!\ \partial T_D/\partial z(T_D^{-1})^2\int_{t_0}^{z} (T_D(\bar{t}_0, z))^2dz\,.$$

In fact, $\overset{(2)}{\zeta}{}^2_D(t_0; t_0) = 0$, $d\overset{(2)}{\zeta}{}^2_D(t_0; t_0)/dz = E$, $d^2\overset{(2)}{\zeta}{}^2_D(t_0; t_0)/dz^2 = d^2\overset{\langle 1\rangle}{\zeta}{}^2_D/dz^2 - 1/2!$ $d^2\overset{(2)}{\zeta}{}^2_D/dz^2(\tilde{E}_{nn} + E^2) = 0$, and clearly $\overset{(2)}{\zeta}{}^2_D(z; t_0)$ is invariant under any pseudo-conformal mapping $\zeta = \zeta(z)$ which satisfies the normalization conditions (3.7).

Similarly, we have a 3-representative function

$$\begin{aligned}\overset{(3)}{\zeta}{}^{2}_{D}(z;t_0) &\equiv T_D^{-1}\int_{t_0}^{z}\overset{(3)}{M}{}^{2}_{D}(t_0;z)dz \\ &= \overset{(2)}{\zeta}{}^{2}_{D}(z;t_0) - 1/3!d^3\overset{(2)}{\zeta}{}^{2}_{D}/dz^3(T_D^{-1})^3\Big(\int_{t_0}^{z}T_D(\bar t_0,z)dz\Big)^3,\end{aligned}\tag{3.26}$$

where

$$\begin{aligned}\overset{(3)}{M}{}^{2}_{D}(t_0;z) &= {}^2\sigma^2_D\overset{(2)}{M}{}^{2}_{D}(t_0;z) = \overset{(2)}{M}{}^{2}_{D}(t_0;z) \\ &\quad - 1/3!\partial^2\overset{(2)}{M}{}^{2}_{D}/\partial z^2(T_D^{-1})^3\int_{t_0}^{z}\int_{t_0}^{z}(T_D(\bar t_0,z))^3(dz)^2 .\end{aligned}$$

Clearly it is invariant and

$$\begin{aligned}&\overset{(3)}{\zeta}{}^{2}_{D}(t_0;t_0) = 0,\ d\overset{(3)}{\zeta}{}^{2}_{D}(t_0;t_0)/dz = E,\ d^2\overset{(3)}{\zeta}{}^{2}_{D}(t_0;t_0)/dz^2 = 0 , \\ &d^3\overset{(3)}{\zeta}{}^{2}_{D}(t_0;t_0)/dz^3 = d^3\overset{(2)}{\zeta}{}^{2}_{D}/dz^3 - 1/3!\ d^3\overset{(2)}{\zeta}{}^{2}_{D}/dz^3(T_D^{-1})^3T_D^3(E\times(\widetilde E_{nn}+E^2)) \\ &\qquad\cdot((\widetilde E_{nn}\times E)(E\times\widetilde E_{nn}) + (\widetilde E_{nn}\times E) + E^3) \\ &\quad = d^3\overset{(2)}{\zeta}{}^{2}_{D}/dz^3 - 1/3!\ (3!\ d^3\overset{(2)}{\zeta}{}^{2}_{D}/dz^3) = 0 .\end{aligned}$$

This result from the following calculation:

$$\begin{aligned}d^3/dz^3(\ \ )^3 = T_{z^2}&\times(\ \ )^2 + (T_z\times d/dz(\ \ )^2)\widetilde E_{n^2n} \\ &+ \{T_z\times d/dz(\ \ )^2 + (T\times d^2/dz^2(\ \ )^2)(\widetilde E_{nn}\times E)\}(E\times\widetilde E_{nn}) \\ &+ T_z\times d/dz(\ \ )^2 + (T\times d^2/dz^2(\ \ )^2)(\widetilde E_{nn}\times E) \\ &+ T\times d^2/dz^2(\ \ )^2 + (\ \ )\times d^3/dz^3(\ \ )^2 .\end{aligned}$$

In general, we have

THEOREM 3.2. *If $K_D(\bar t_0, z)\neq 0$ in a bounded domain D, we have an m-representative (holomorphic) function $\overset{(m)}{\zeta}{}^{2}_{D}(z;t_0)$ (see (3.21)) of the pseudo-conformal equivalence class of D with respect to a point t_0.*

REMARK 1. $\overset{(1)}{\zeta}{}^{i}_{D}(z;t_0) = T_D^{-1}\int_{t_0}^{z}T_D(\bar t_0,z)dz = M_D^{0E_n}(t_0,z)/m^1_D(t_0,z)$, $(i = 1, 1', 2)$, because $d(M_D^{0E_n}(t_0,z)/m^1_D(t_0,z))/dz = T_D^{-1}T_D(\bar t_0,z)$, where

$$M_D^{0E_n}(t_0,z) = (0, E)\begin{pmatrix}K_D & K_z\\ K_{t^*} & K_{t^*z}\end{pmatrix}^{-1}\begin{pmatrix}K_D(\bar t_0,z)\\ \partial K_D(\bar t_0,z)/\partial t^*\end{pmatrix},$$

$m^1_D(t_0,z) = K_D(\bar t_0,z)/K_D(\bar t_0,t_0)$. (This result was obtained by Tsuboi [5]).

REMARK 2. In the case of one variable, our 2-representative functions of an unit disk with respect to t_0 become $\overset{(2)}{\zeta}{}^{i}_{D}(z;t_0) = (1-|t_0|^2)(1-\bar t_0 u)u$, $(i = 1, 1', 2)$, where $u = (z-t_0)/(1-\bar t_0 z)$.

REMARK 3. The function $\overset{(m)}{\zeta}{}^{2}_{D}(z;t_0)$ is expressed as follows:

$$\overset{(m)}{\zeta}{}^2_D(z;\, t_0) = \overset{(1)}{\zeta}{}^2_D(z; t_0) - \sum_{\nu=2}^{m} 1/\nu!\, d^\nu \overset{(\nu-1)}{\zeta}{}^2_D /dz^\nu (\overset{(1)}{\zeta}{}^2_D(z;\, t_0))^\nu \,. \tag{3.27}$$

4. **m-representative domain by the operator δ^ν_D.** As § 3, we shall start with the case $m = 2$. We construct the matrix function $\overset{(2)}{T}_D(t_0;\, z) = \delta^1_D T_D(\bar{t}_0,\, z)$, (see (4.6)) as follows:

$$\begin{aligned}\overset{(2)}{T}_D(t_0;\, z) = {} & T_D(\bar{t}_0,\, z) \\ & - \partial T_D(\bar{t}_0,\, t_0)/\partial z(\partial^2 T_D(\bar{t}_0,\, t_0)/\partial t^* \partial z)^{-1} \partial T_D(\bar{t}_0,\, z)/\partial t^* \,.\end{aligned} \tag{4.1}$$

Under any pseudo-conformal mapping which satisfies the normalization conditions (3.7) at a point t_0 of D, we have

$$\overset{(2)}{T}_D(t_0;\, z) = \overset{(2)}{T}_\Delta(0;\, \zeta) d\zeta/dz \,. \tag{4.2}$$

Then we have an invariant function which satisfies (3.7):

$$\overset{(2)}{\eta}_D(z;\, t_0) = (\overset{(2)}{T}_D(t_0;\, t_0))^{-1} \int_{t_0}^{z} \overset{(2)}{T}_D(t_0;\, z) dz \,. \tag{4.3}$$

This function is a 2-representative function of the pseudo-conformal equivalence class of D.

In general, we define as follows:

$$\overset{(m)}{T}_D(t_0;\, z) = \delta^{m-1}_D \cdots \delta^1_D T_D(\bar{t}_0,\, z),\ (m \geqq 2)\,, \tag{4.4}$$

$$\overset{(\lambda)}{S}_D(t_0;\, z) = \delta^{\lambda-1}_D \cdots \delta^1_D \partial^\lambda T_D(\bar{t}_0,\, z)/\partial t^{*\lambda},\ \overset{(1)}{S}_D(t_0;\, z) = \partial T_D(\bar{t}_0,\, z)/\partial t^* \,, \tag{4.5}$$

where

$$\delta^\nu_D F(t_0;\, z) = F(t_0;\, z) - (\partial^\nu F(t_0;\, z)/\partial z^\nu)_{z=t_0} (\partial^\nu \overset{(\nu)}{S}_D(t_0;\, t_0)/\partial z^\nu)^{-1} \overset{(\nu)}{S}_D(t_0;\, z)\,, \tag{4.6}$$

$$\delta^\nu_D \cdots \delta^1_D F(t_0;\, z) = \delta^\nu_D (\cdots (\delta^2_D(\delta^1_D F(t_0;\, z)) \cdots)\,, \tag{4.7}$$

for any matrix function $F(t_0;\, z)$. Then we have

$$\overset{(m)}{T}_D(t_0;\, z) = \overset{(m)}{T}_\Delta(0;\, \zeta) d\zeta(z)/dz\,, \tag{4.8}$$

$$\overset{(\lambda)}{S}_D(t_0;\, z) = \overset{(\lambda)}{S}_\Delta(0;\, \zeta) d\zeta(z)/dz,\ (\lambda \leqq m-1)\,, \tag{4.9}$$

because

$$\begin{aligned}& \delta^\nu_D \cdots \delta^1_D \partial^\mu T_D(\bar{t}_0,\, z)/\partial t^{*\mu} \\ & = \delta^\nu_\Delta \cdots \delta^1_\Delta \partial^\mu T_\Delta(0,\, \zeta)/\partial \tau^{*\mu} (d\zeta(z)/dz)\,,\end{aligned} \tag{4.10}$$

under any pseudo-conformal mapping $\zeta = \zeta(z)$ which satisfies (1.3).

On the other hand, we can calculate instantly

$$\partial \overset{(m)}{T}_D(t_0;\, t_0)/\partial z = \cdots = \partial^{m-1}\overset{(m)}{T}_D(t_0;\, t_0)/\partial z^{m-1} = 0\ , \tag{4.11}$$

$$\partial \overset{(m-1)}{S}_D(t_0;\, t_0)/\partial z = \cdots = \partial^{m-2}\overset{(m-1)}{S}_D(t_0;\, t_0)/\partial z^{m-2} = 0\ , \tag{4.12}$$

because $(d^\nu(\delta_D^\nu F(t_0;\, z))/dz^\nu)_{z=t_0} = 0$.

THEOREM 4.1. *If* $\overset{(m)}{T}_D(t_0;\, z)$ *exists and* $\det \overset{(m)}{T}_D(t_0;\, t_0) \neq 0$ *at a fixed point* t_0 *of* D, *then we have an* m*-representative* (*holomorphic*) *function of the pseudo-conformal equivalence class of* D:

$$\overset{(m)}{\eta}_D(z;\, t_0) = (\overset{(m)}{T}_D(t_0;\, t_0))^{-1}\int_{t_0}^{z} \overset{(m)}{T}_D(t_0;\, z)dz\ . \tag{4.13}$$

Further, we have

THEOREM 4.2. *We obtain several* m*-representative functions of the pseudo-conformal equivalence class of* D *with respect to the fixed point* t_0 *of* D:

$$\overset{(m)}{\rho}{}^i_D(z;\, t_0) = (\delta_D^{m-1}\overset{(m-1)}{M}{}^i_D(t_0;\, t_0))^{-1}\int_{t_0}^{z} \delta_D^{m-1}\overset{(m-1)}{M}{}^i_D(t_0;\, z)dz,\ (i = 1, 1')\ , \tag{4.14}$$

$$\overset{(m)}{\chi}{}^i_D(z;\, t_0) = (\overset{(m-1)}{T}_D(t_0;\, t_0))^{-1}\int_{t_0}^{z} {}^i\sigma_D^{m-1}\overset{(m-1)}{T}_D(t_0;\, z)dz,\ (i = 1, 1')\ , \tag{4.15}$$

$$\overset{(m)}{\mu}{}^1_D(z;\, t_0) = T_D^{-1}\int_{t_0}^{z} {}^1\sigma_D^{m-1}\overset{(m-1)}{M}{}^{1'}_D(t_0;\, z)dz\ , \tag{4.16}$$

$$\overset{(m)}{\mu}{}^2_D(z;\, t_0) = T_D^{-1}\int_{t_0}^{z} {}^{1'}\sigma_D^{m-1}\overset{(m-1)}{M}{}^1_D(t_0;\, z)dz\ , \tag{4.17}$$

$$\overset{(m)}{\mu}{}^3_D(z;\, t_0) = \overset{(m-1)}{\varepsilon}_D(z;\, t_0) - 1/m!\ \partial^m \overset{(m-1)}{\varepsilon}_D/\partial z^m(\overset{(1)}{\zeta}{}^2_D(z;\, t_0))^m\ , \tag{4.18}$$

where $\overset{(m-1)}{\varepsilon}_D(z;\, t_0)$ is an arbitrary holomorphic $(m-1)$-representative function.

REMARK 1. We can obtain other m-representative functions

$$\begin{aligned} \overset{(m)}{\nu}{}^1_D(z;\, t_0) &= (\delta_D^{m-1}N_{\mu D}^{E_n\cdots 0}(t_0,\, t_0))^{-1}\int_{t_0}^{z} \delta_D^{m-1}N_{\mu D}^{E_n\cdots 0}(z,\, t_0)dz\ , \\ {}^i\overset{(m)}{\nu}{}^2_D(z;\, t_0) &= \int_{t_0}^{z} {}^i\sigma_D^{m-1}N_{\mu D}^{E_n\cdots 0}(z,\, t_0)dz,\ (i = 1, 1') \end{aligned} \tag{4.19}$$

where

$$N_{\mu D}^{E_n \cdots 0}(z, t_0) = (E_n, 0, \cdots, 0) \begin{pmatrix} T_{\mu D}, \cdots, & \partial^{m-2} T_{\mu D}/\partial z^{m-2} \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \partial^{m-2} T_{\mu D}/\partial t^{*m-2}, \cdots, & \partial^{2(m-2)} T_{\mu D}/\partial t^{*m-2}\partial z^{m-2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} T_{\mu D}(z_0, \bar{t}) \\ \vdots \\ \partial^{m-2} T_{\mu D}(z, \bar{t}_0)/\partial t^{*m-2} \end{pmatrix}, \quad \text{(see [7])} .$$

REMARK 2. $\overset{(m)}{\eta}_D(z; t_0)$ was published temporarily in Mathematical Seminar of Tōkyō University of Education [8], and the author showed $\overset{(2)}{\eta}_D(z; t_0) = (1 - |t_0|^2)(1 - \bar{t}_0 u)u$ where $u = (z - t_0)/(1 - \bar{t}_0 z)$, and D is an unit disk in one variable.

We shall further proceed with our studies. First, we shall substitute the auxiliary conditions

$$\text{(4.20)} \qquad \zeta(t_0) = 0,\ d\zeta(t_0)/dzA = A,\ d^2\zeta(t_0)/dz^2A^2 = \cdots = d^m\zeta(t_0)/dz^mA^m = 0 ,$$

for the normalization conditions (1.3), where A is an $n \times \nu$ matrix ($\nu \leqq n$). (The case of conditions $\zeta(t_0) = 0, d\zeta(t_0)/dzA = A$ was first studied by Y. Michiwaki, Nagaoka Technical College.)

In the case of $m = 2$, we construct the following matrix function

$$\text{(4.21)} \qquad {}_A\overset{(2)}{T}_D(t_0; z) = {}_A\delta_D^1 T_D(\bar{t}_0, z) = T_D(\bar{t}_0, z) - \partial T_D(\bar{t}_0, t_0)/\partial zA^2(A^{*2}\partial^2 T_D(\bar{t}_0, t_0)/\partial t^*\partial zA^2)^{-1} A^{*2}\partial T_D(\bar{t}_0, z)/\partial t^* ,$$

then we can calculate easily

$$\text{(4.22)} \qquad {}_A\overset{(2)}{T}_D(t_0; z) = (d\tau(t_0)/dt)^*\, {}_A\overset{(2)}{T}_\Delta(0; \zeta)(d\zeta(z)/dz) ,$$

under any pseudo-conformal mapping $\zeta = \zeta(z)$ which satisfies the conditions

$$\text{(4.23)} \qquad \zeta(t_0) = 0,\ d\zeta(t_0)/dzA = A,\ d^2\zeta(t_0)/dz^2A^2 = 0 ,$$

because, from (2.4) and (2.6) we have

$$\text{(4.24)} \qquad \partial T_D(\bar{t}, z)/\partial zA^2 = (d\tau(t)/dt)^*\partial T_\Delta/\partial\zeta(d\zeta(z)/dzA)^2 + (d\tau(t)/dt)^* T_\Delta(d^2\zeta(z)/dz^2)A^2 ,$$

$$\text{(4.25)} \qquad \begin{aligned} A^{*2}\partial^2 T_D(\bar{t}, z)/\partial t^*\partial zA^2 = {} & (d\tau(t)/dtA)^{*2}\partial^2 T_\Delta/\partial\tau^*\partial\zeta(d\zeta(z)/dzA)^2 \\ & + (d\tau(t)/dtA)^{*2}\partial T_\Delta/\partial\tau^* d^2\zeta(z)/dz^2A^2 \\ & + (d^2\tau(t)/dt^2A^2)^*\partial T_\Delta/\partial\zeta(d\zeta(z)/dzA)^2 \\ & + (d^2\tau(t)/dt^2A^2)^* T_\Delta d^2\zeta(z)/dz^2A^2 . \end{aligned}$$

Therefore, we have an invariant (holomorphic) function which satisfies the conditions (4.23):

$$(4.26)\qquad {}_A\overset{(2)}{\eta}_D(z;\, t_0) = A(A^*{}_A\overset{(2)}{T}_D(t_0;\, t_0)A)^{-1}\int_{t_0}^{z} A^*{}_A\overset{(2)}{T}_D(t_0;\, z)dz\ .$$

We shall call this function an $A-2$-representative function of the pseudo-conformal equivalence class of D with respect to $t_0 \in D$.

Next, we shall define as follows:

$$(4.27)\qquad {}_A\overset{(m)}{T}_D(t_0;\, z) = {}_A\delta_D^{m-1}\cdots{}_A\delta_D^{1}T_D(\bar{t}_0,\, z)\ ,$$

$$(4.28)\qquad {}_A\overset{(\lambda)}{S}_D(t_0;\, z) = {}_A\delta_D^{\lambda-1}\cdots{}_A\delta_D^{1}\partial^{\lambda}T_D(\bar{t},\, z)/\partial t^{*\lambda}\ ,$$

where

$$\begin{aligned}{}_A\delta_D^{\nu}F(t_0;\, z) = {} & F(t_0;\, z)\\ & -(\partial^{\nu}F(t_0;\, z)/\partial z^{\nu})_{z=t_0}A^{\nu+1}(A^{*\nu+1}\partial^{\nu}{}_A\overset{(\nu)}{S}_D(t_0;\, t_0)/\partial z^{\nu}A^{\nu+1})^{-1}\\ & \cdot A^{*\nu+1}{}_A\overset{(\nu)}{S}_D(t_0;\, z)\ .\end{aligned}$$

Then we have

$$(4.29)\qquad {}_A\overset{(m)}{T}_D(t_0;\, z) = (d\tau(t_0)/dt)^*{}_A\overset{(m)}{T}_\Delta(0;\, \zeta)(d\zeta(z)/dz)\ ,$$

$$(4.30)\qquad {}_A\overset{(\lambda)}{S}_D(t_0;\, z) = (d\tau(t_0)/dt)^{*\lambda+1}{}_A\overset{(\lambda)}{S}_D(0;\, \zeta)(d\zeta(z)/dz),\ (\lambda \leqq m-1)\ ,$$

because

$$A^{*\mu+1}\partial^{\mu+\nu}T_D(\bar{t}_0,\, t_0)/\partial t^{*\mu}\partial z^{\nu}A^{\nu+1} = A^{*\mu+1}\partial^{\mu+\nu}T_\Delta(\bar{0},\, 0)/\partial\tau^{*\mu}\partial\zeta^{\nu}A^{\nu+1}\ ,$$

under any pseudo-conformal mapping $\zeta = \zeta(z)$ which satisfies (4.20).

THEOREM 4.3. *We have an invariant function which satisfies* (4.20):

$$(4.31)\qquad {}_A\overset{(m)}{\eta}_D(z;t_0) = A(A^*{}_A\overset{(m)}{T}_D(t_0;\, t_0)A)^{-1}\int_{t_0}^{z} A^*{}_A\overset{(m)}{T}_D(t_0;\, z)dz\ .$$

We call this function an $A-m$*-representative function of the pseudo-conformal equivalence class of* D, and the image domain by it is called an $A-m$*-representative domain of the class with senter at the origin.*

Next, we shall substitute the auxiliary conditions

$$(4.32)\quad \zeta(t_0) = 0,\ \det d\zeta(t_0)/dz \neq 0,\ d^2\zeta(t_0)/dz^2 = \cdots = d^m\zeta(t_0)/dz^m = 0\ ,$$

for the normalization conditions (1.3).

Then, we can easily verify the following relation

$$dz^* \overset{(m)}{T}{}^*_D(t_0; z) T_D^{-1}(\bar{t}_0, t_0) \overset{(m)}{T}_D(t_0; z) dz = d\zeta^* \overset{(m)}{T}{}^*_\Delta(0; \zeta) T_\Delta^{-1}(\bar{0}, 0) \overset{(m)}{T}_\Delta(0; \zeta) d\zeta \,, \tag{4.33}$$

under any pseudo-conformal mapping $\zeta = \zeta(z)$ which satisfies (4.32). Therefore, we have

$$T_D^{-1/2}(\bar{t}_0, t_0) \overset{(m)}{T}_D(t_0; z) dz = U T_\Delta^{-1/2}(\bar{0}, 0) \overset{(m)}{T}_\Delta(0; \zeta) d\zeta \,. \tag{4.34}$$

THEOREM 4.4. *We have a following function which is invariant except only unitary transformation under any pseudo-conformal mapping $\zeta = \zeta(z)$ satisfying* (4.32):

$$\overset{(m)}{{}_N\eta_D}(z; t_0) = T_D^{-1/2}(\bar{t}_0, t_0) \int_{t_0}^{z} \overset{(m)}{T}_D(t_0; z) dz \,. \tag{4.35}$$

We call this function an m-normal function of the pseudo-conformal equivalence class with the conditions (4.32).

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