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CONTINUOUS SPECTRA OF SECOND-ORDER DIFFERENTIAL OPERATORS

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We consider the differential operator $l(y) = y'' + qy$, where q is a positive, continuously differentiable function defined on a ray $[a, \infty)$. The operator l determines, with appropriate restrictions, self-adjoint operators defined in the hilbert space $\mathscr{L}_2[a, \infty)$ of quadratically summable, complexvalued functions on $[a, \infty)$. In this note, we prove that if L is such a self-adjoint operator, then the conditions $q(t) \to \infty$ and $q'(t)q(t)^{-1/2} \to 0$ as $t \to \infty$ are sufficient for the continuous spectrum $C(L)$ of L to cover the entire real axis.

Similar results are well-known; however, monotonicity conditions on q and q' are usually required. For example, in [1], p. 116, it is proved that if q tends monotonically to ∞ as $t \to \infty$, preserving the direction of convexity for large t, then the condition $q'(t)q(t)^{-1/2} \to 0$ as $t \to \infty$ is sufficient to imply $C(L) = (-\infty, \infty)$ for every self-adjoint operator L determined by l.

THEOREM. *If $q(t) \to \infty$ as $t \to \infty$, $q'(t)q(t)^{-1/2} \to 0$ as $t \to \infty$, and L is a self-adjoint operator in $\mathscr{L}_2[a, \infty)$ determined by l, then $C(L) = (-\infty, \infty)$.*

Proof. To prove that the real number λ belongs to $C(L)$, it is sufficient to construct a bounded noncompact sequence $y_1, y_2, \cdots$ such that $\|(L - \lambda)y_n\| \to 0$ as $n \to \infty$. The domain of L includes the set $\mathscr{M}$ of all y satisfying (i) y has compact support contained in the open interval (a, ∞), (ii) y' is absolutely continuous, and (iii) $y'' \in \mathscr{L}_2[a, \infty)$ (cf. [3], Chapter V). Hence, it follows that $\lambda \in C(L)$, if we prove that for each $\eta > 0$ and $N > a$, there is a nontrivial $y \in \mathscr{M}$ such that the support of y is contained in $[N, \infty)$ and $\|(L - \lambda)y\| < \eta\|y\|$. To establish this, we recall Lemma 2 of [2]:

Suppose f is a continuously differentiable positive function on $[b, \infty)$, and $f'(t)f(t)^{-1/2} \to 0$ as $t \to \infty$. If ε and K are positive numbers, then there is a number B such that if t and s are $\geqq B$ and $|t - s| \leqq Kf(s)^{1/2}$, then $|f(t)f(s)^{-1} - 1| < \varepsilon$.

We choose $0 < \varepsilon < \eta^2/25$, $K > 6400/\eta^2$ (assume $\eta < 1$), and apply the lemma to $f = q - \lambda$ on an interval $[b, \infty)$ such that $f(t) \geqq \Pi^2$ for $t \geqq b$. Let $s_0 \geqq \max\{N, B\}$ be such that $|f'(t)|f(t)^{-1/2} < \varepsilon$ for $t \geqq s_0$.

Define $s_1, s_2, \cdots$ by

$$s_{i+1} = s_i + \Pi f(s_i)^{-1/2} \qquad (i = 0, 1, \cdots),$$

and denote $f(s_i)^{1/2}$ by α_i. Since for $s_i \leqq s_0 + K\alpha_0$, we have $\alpha_i^2/\alpha_0^2 \leqq 1 + \varepsilon < 4$, it follows that for such s_i,

$$s_i - s_0 = \sum_{j=0}^{i-1} \Pi/\alpha_j \geqq \Pi i/2\alpha_0 ;$$

thus there is an integer p so that $s_p \leqq s_0 + K\alpha_0 < s_{p+1}$. We now construct a $y \in \mathscr{M}$ with support $[s_0, s_p]$.

Since $K > 9$ and $\alpha_i \geqq \Pi$ for each i, there exist $\tau_1, \tau_2 \in \{s_0, \cdots, s_p\}$ such that $s_0 < \tau_1 < \tau_2 < s_p$, $\alpha_0 \leqq \tau_1 - s_0 \leqq 2\alpha_0$, $\alpha_0 \leqq s_p - \tau_2 \leqq 2\alpha_0$, and $\tau_2 - \tau_1 \geqq K\alpha_0/2$. Define h and g on $[a, \infty)$ to be zero exterior to $[s_0, s_p]$ and otherwise by

$$g(t) = (-1)^i\alpha_i^{-1} \sin \alpha_i(t - s_i) \text{ for } s_i \leqq t \leqq s_{i+1}, \qquad (i = 0, \cdots, p - 1)$$

and

$$h(t) = \begin{cases} (t - s_0)/(\tau_1 - s_0), & s_0 \leqq t \leqq \tau_1 , \\ 1, & \tau_1 \leqq t \leqq \tau_2 , \\ (s_p - t)/(s_p - \tau_2), & \tau_2 \leqq t \leqq s_p . \end{cases}$$

If $y = gh$, then a calculation yields that $y \in \mathscr{M}$.

Since $\varepsilon < 1/4$, from the lemma above we conclude that

$$f(t)/f(s) = \{f(t)/f(s_0)\}/\{f(s)/f(s_0)\} < (5/4)/3/4) < 2 \tag{1}$$

for all $t, s \in [s_0, s_p]$. Applying the mean value theorem, it follows that for $t \in [s_i, s_{i+1}]$,

$$\begin{aligned} |f(t) - f(s_i)| &= |f'(t^*)(t - s_i)| \\ &\leqq \{|f'(t^*)|f(t^*)^{-1/2}\}\{\Pi f(t^*)^{1/2}f(s_i)^{-1/2}\} \\ &< \Pi(2)^{1/2}\varepsilon < 5\varepsilon . \end{aligned} \tag{2}$$

For $t \in [s_i, s_{i+1}] \subset [s_0, \tau_1]$, we have by application of (1), (2), and $\tau_1 - s_0 \geqq \alpha_0$ that

$$\begin{aligned} |y''(t) + f(t)y(t)| &= |2(\tau_1 - s_0)^{-1}(-1)^i \cos \alpha_i(t - s_i) \\ &\quad + (t - s_0)(\tau_1 - s_0)^{-1}[f(t) - f(s_i)]g(t)| \\ &\leqq 2(\tau_1 - s_0)^{-1} + 5\varepsilon\alpha_i^{-1} \\ &< 2/\alpha_0 + 5(2)^{1/2}/\alpha_0 < 10/\alpha_0 . \end{aligned}$$

From this inequality and $\tau_1 - s_0 \leqq 2\alpha_0$, it follows that

$$\int_{s_0}^{\tau_1} |y'' + fy|^2 dt \leqq (100/\alpha_0^2)(\tau_1 - s_0) \leqq 200/\alpha_0 . \tag{3}$$

Similarly, we have

$$\int_{\tau_2}^{s_p} |y'' + fy|^2 dt \leqq 200/\alpha_0 . \tag{4}$$

For $[s_i, s_{i+1}] \subset [\tau_1, \tau_2]$, the definition of y and (1) yield

$$\int_{s_i}^{s_{i+1}} y^2 dt = (s_{i+1} - s_i)/2\alpha_i^2 \geqq (s_{i+1} - s_i)/4\alpha_0^2 ,$$

hence, this inequality and $(\tau_2 - \tau_1) \geqq K\alpha_0/2$ imply that

$$\int_{\tau_1}^{\tau_2} y^2 dt \geqq (\tau_2 - \tau_1)/4\alpha_0^2 \geqq K/8\alpha_0 . \tag{5}$$

Since on $[s_i, s_{i+1}]$,

$$|y''(t) + f(t)y(t)| = |[f(t) - f(s_i)]y(t)| \leqq 5 \in |y(t)| ,$$

we have

$$\begin{gathered} \int_{s_i}^{s_{i+1}} |y'' + fy|^2 dt < 25\varepsilon^2 \int_{s_i}^{s_{i+1}} y^2 dt; \textit{ thus} \\ \left\{\int_{\tau_1}^{\tau_2} |y'' + fy|^2 dt\right\}\left\{\int_{\tau_1}^{\tau_2} y^2 dt\right\}^{-1} < 25\varepsilon^2 < \varepsilon . \end{gathered} \tag{6}$$

From the definition of ε and K, (3), (4), (5), and (6), we obtain

$$\left\{\int_{s_0}^{s_p} |y'' + fy|^2 dt\right\}\left\{\int_{s_0}^{s_p} y^2 dt\right\}^{-1} < \{3200/K\} + \varepsilon < \eta^2;$$

therefore the proof is complete.

In [3], p. 235, asymptotic methods are used to obtain criteria for $C(L) = (-\infty, \infty)$. In this development much of the argument depends on the divergent integral $\int_a^\infty q^{-1/2} dt = \infty$. The condition $q'(t)q(t)^{-1/2} \to 0$ as $t \to \infty$ implies the divergence of this integral. We raise the following question for a class $C^{(1)}$ function q: Are the conditions $q(t) \to \infty$ as $t \to \infty$ (perhaps monotonically) and $\int_a^\infty q^{1/2} dt = \infty$ sufficient to imply $C(L) = (-\infty, \infty)$?

References

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