PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 3, 1970

ULM'S THEOREM FOR ABELIAN GROUPS MODULO BOUNDED GROUPS

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Let $\hat{A}$ be the category of Abelian groups, $\hat{B}$ the class of bounded Abelian groups. It is shown that if G and H are totally projective p-groups[1], then $G \cong H$ in the quotient category $\hat{A}/\hat{B}$ if and only if there exists an integer $k \geqq 0$ such that for all ordinals α and all integers $r \geqq 0$,

$$\sum_{j=k}^{r+k} f_G(\alpha + j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha + j) \text{ and } \sum_{j=k}^{r+k} f_H(\alpha+j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha + j)\,.$$

This extends a similar result of R. J. Ensey for direct sums of countable reduced p-groups. It is also noted that if G and H are totally projective p-groups, then G is quasi-isomorphic to H if and only if there exists an integer $k \geqq 0$ such that for all integers $n \geqq 0$ and $r \geqq 0$,

$$\sum_{j=k}^{r+k} f_G(n + j) \leqq \sum_{j=0}^{r+2k} f_H(n + j)$$

and

$$\sum_{j=k}^{r+k} f_H(n + j) \leqq \sum_{j=0}^{r+2k} f_G(n + j)\,, \quad \text{and} \quad f_G(\alpha) = f_H(\alpha)$$

for all $\alpha \geqq \omega$. This extends a similar result of R. S. Pierce and R. A. Beaumont for direct sums of countable reduced p-groups.

Preliminaries. Let $\hat{A}$ be the category of groups and $\hat{B}$ the Serre class of bounded groups. Then $\hat{A}/\hat{B}$ is the quotient category as defined by Grothendieck [5]. The objects $\hat{A}/\hat{B}$ are the objects of $\hat{A}$.

$$\mathrm{Hom}_{\hat{A}/\hat{B}}(G, H) = \varinjlim_{(G', H') \in D} \mathrm{Hom}(G', H/H')\,,$$

where $D = \{G', H' \mid G' \subseteqq G, H' \subseteqq H; G/G', H' \in \hat{B}\}$. D is directed by $(G', H') \leqq (G'', H'')$ if and only if $G'' \subseteqq G'$ and $H' \subseteqq H''$. For a thorough discussion of the category $\hat{A}/\hat{B}$, the reader should see either Ensey [4] or E. A. Walker [9]. From Walker's results, it follows that $G \cong H$ in $\hat{A}/\hat{B}$ if and only if there exist subgroups S and A of G, and T and B of H such that $S/A \cong T/B$ and G/S, H/T, A, and B are bounded. Two groups G and H are quasi-isomorphic if there exist isomorphic subgroups S and T of G and H respectively such that G/S and H/T are bounded. Then clearly, quasi-isomorphism implies isomorphism in $\hat{A}/\hat{B}$, and for torsion-free groups, the converse also holds.

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[1] From here on, the word group is used to mean Abelian group.

The following two conditions of Beaumont and Pierce [1] are used in the next section:

(I) there exists an integer $k \geqq 0$ such that for all integers $n \geqq 0$ and $r \geqq 0$

$$\sum_{j=k}^{r+k} f_G(n+j) \leqq \sum_{j=0}^{r+2k} f_H(n+j) \quad \text{and} \quad \sum_{j=k}^{r+k} f_H(n+j) \leqq \sum_{j=0}^{r+2k} f_G(n+j) .$$

(II) $f_G(\alpha) = f_H(\alpha)$ for all $\alpha \geqq \omega$.

Here G and H are groups and the notation $f_G(\beta)$ denotes the β^{th} Ulm invariant of G.

Ensey [4] showed that if G and H are direct sums of countable reduced p-groups, then $G \cong H$ in $\hat{A}/\hat{B}$ if and only if

(III) there exists an integer $k \geqq 0$ such that for all ordinals α and integers $r \geqq 0$

$$\sum_{j=k}^{r+k} f_G(\alpha+j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha+j) \quad \text{and} \quad \sum_{j=k}^{r+k} f_H(\alpha+j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha+j) .$$

In this paper, Ensey's result is extended to the class of totally projective groups. This class may be described as the smallest class $\hat{P}$ of p-groups containing a cyclic group of order p and satisfying the following two conditions:

(a) $G_i \in \hat{P}$ if and only if $\sum_{i \in I} G_i \in \hat{P}$.

(b) For any ordinal α, $G \in \hat{P}$ if and only if $p^\alpha G$ and $G/p^\alpha G$ are in $\hat{P}$. Full use is made of the fact that P. Hill [7] has recently proved that Ulm's Theorem holds for totally projective groups and that he has shown that there exists a totally projective group G with Ulm invariants $f_G(\alpha)$ if and only if

$$\sum_{\omega\alpha+n \leqq \beta < \omega(\alpha+1)} f_G(\beta) \geqq \sum_{\beta \geqq \omega(\alpha+1)} f_G(\beta)$$

for each $n < \omega$ and each $\alpha < \tau(G)$, where $\tau(G)$ denotes the type of G, defined below. These two results are referred to as "Hill's Uniqueness Theorem" and "Hill's Existence Theorem," respectively.

It should be noted that the results of this paper follow equally well from the uniqueness and existence theorems of P. Crawley and A. Hales [3].

All groups considered are reduced p-groups for a fixed prime p. $\tau(G)$ denotes the type of G, the smallest ordinal τ such that $G^\tau = 0$; $G^\alpha = p^{\omega\alpha}G$; $G_\alpha = G^\alpha/G^{\alpha+1}$, the α^{th} Ulm factor of G. The notation d.s.c., adopted from R. Nunke [8], stands for a direct sum of countable reduced p-groups.

2. Ulm's Theorem in $\hat{A}/\hat{B}$. The following theorem is an extension of a theorem of Beaumont and Pierce [1] from the class of d.s.c. groups to the class of totally projective groups, which follows immediately from results of Nunke [8] and Hill [6], [7].

THEOREM 1. *Let G and H be totally projective. Then G is quasi-isomorphic to H if and only if G and H satisfy conditions* (I) *and* (II).

Proof. It was shown in [1] that if G and H are arbitrary p-groups, then (I) and (II) hold if G is quasi-isomorphic to H. In [6], Hill proved that if G and H are p-groups such that $G/p^\omega G$ and $H/p^\omega H$ are direct sums of cyclic groups then (I) and the condition that $p^\omega G \cong p^\omega H$ are necessary and sufficient conditions in order that G is quasi-isomorphic to H. If G and H are totally projective, then $G/p^\omega G$ and $H/p^\omega H$ are direct sums of cyclic groups and since (II) and Hill's Uniqueness Theorem imply $p^\omega G \cong p^\omega H$, the theorem is proved.

DEFINITION. (*Ensey*, [4].) Let $\{G_\alpha\}_{\alpha \in I}$ and $\{H_\alpha\}_{\alpha \in I}$ be two families of groups indexed by the same set I. These families are uniformly quasi-isomorphic if there exists an integer $k \geqq 0$ and for each $\alpha \in I$, subgroups $S_\alpha \subseteqq G_\alpha$, $T_\alpha \subseteqq H_\alpha$ such that $p^k G_\alpha \subseteqq S_\alpha$, $p^k H_\alpha \subseteqq T_\alpha$, and $S_\alpha \cong T_\alpha$.

The following lemma relates this concept to that of isomorphism in $\hat{A}/\hat{B}$ for totally projective groups.

LEMMA 2. *Let G and H be totally projective. If $G \cong H$ in $\hat{A}/\hat{B}$, then the corresponding Ulm factors of G and H are uniformly quasi-isomorphic.*

Proof. Ensey [4] has shown that for any reduced p-groups G and H such that $G \cong H$ in $\hat{A}/\hat{B}$, (I) holds for G_α and H_α with the k of (I) the same for all α. Since $G_\alpha = p^{\omega\alpha}G/p^{\omega(\alpha+1)}G$, and $p^{\omega\alpha}G$ is totally projective, it follows that G_α and H_α are direct sums of cyclic groups. This being the case, Ensey [4] has shown that the corresponding Ulm factors of G and H are uniformly quasi-isomorphic.

That the converse of Lemma 2 also holds is the content of the next two lemmas and the succeeding theorem. The notation $fr(G)$ is used to denote the final rank of G. Recall that if the rank of G is denoted $r(G)$, then by definition, $fr(G) = \min_{n<\omega} r(p^n G)$. All references to Ensey refer to [4].

LEMMA 3. *Let G be totally projective and $\tau(G) = \tau$. For $\alpha < \tau$, let $G_\alpha = S_\alpha \oplus T_\alpha$ where $fr(T_\alpha) = fr(G_\alpha)$. Let H be totally projective* *such that for all* $\alpha < \tau$, $H_\alpha = p^k S_\alpha \oplus T_\alpha$ *for some fixed integer* $k \geqq 0$. *Then* $G \cong H$ *in* $\hat{A}/\hat{B}$.

Proof. Since Hill's Existence Theorem is used several times in this proof, it seems advisable to translate his condition from the setting of Ulm invariants to that of Ulm factors, their ranks and final ranks. The Ulm invariants of G_α are the Ulm invariants of $p^{\omega\alpha}G$ for $n < \omega$. But $f_{p^{\omega\alpha}G}(n) = f_G(\omega\alpha + n)$. Thus $f_{G_\alpha}(n) = f_G(\omega\alpha + n)$, $n < \omega$. Therefore $r(G_\alpha) = \sum_{\omega\alpha \leqq \beta < \omega(\alpha+1)} f_G(\beta)$ and $fr(G_\alpha) = \min_{n<\omega} \sum_{\omega\alpha+n \leqq \beta < \omega(\alpha+1)} f_G(\beta)$. Hence Hill's condition can be written $fr(G_\alpha) \geqq \sum_{\alpha<\beta<\tau} r(G_\beta)$. Thus such an H exists and since $fr(T_\alpha) = fr(G_\alpha)$, T_α can be broken up in the following way: $T_\alpha = \sum_{\alpha<\beta<\tau} G_{\alpha\beta}$ where $fr(G_{\alpha\beta}) = fr(G_\alpha)$. Again by Hill's Existence Theorem, there exist totally projective groups $\{L_\alpha\}_{\alpha<\tau}$ with Ulm factors $(L_\alpha)_\beta = G_{\beta\alpha}$ for all $\beta < \alpha$; $(L_\alpha)_\alpha = S_\alpha$; and $(L_\alpha)_\beta = 0$ for all $\beta > \alpha$. Since the class of totally projective groups is closed under taking arbitrary direct sums, $\sum_{\alpha<\tau} L_\alpha$ is totally projective, and thus $G \cong \sum_{\alpha<\tau} L_\alpha$ by Hill's Uniqueness Theorem since they have the same Ulm factors. For all $\alpha < \tau$, let $M_\alpha = L_\alpha / L_\alpha^\alpha[p^k]$. Ensey has shown that for any reduced p-group G, $(G/A)^\beta = G^\beta/A$ for all $\beta \leqq \gamma$ whenever $A \subseteqq G^\gamma$. Hence $p^{\omega\alpha}M_\alpha = (L_\alpha/L_\alpha^\alpha[p^k])^\alpha = L_\alpha^\alpha/L_\alpha^\alpha[p^k] \cong p^k L_\alpha^\alpha$, and since L_α is totally projective, $p^{\omega\alpha}M_\alpha$ is totally projective. Using the same result of Ensey's, $M_\alpha/p^{\omega\alpha}M_\alpha = (L_\alpha/L_\alpha^\alpha[p^k])/(L_\alpha^\alpha/L_\alpha^\alpha[p^k]) \cong L_\alpha/L_\alpha^\alpha$ and hence $M_\alpha/p^{\omega\alpha}M_\alpha$ is totally projective. Therefore M_α is totally projective. Ensey has also shown that for any reduced p-group G, ordinal α and integer $k \geqq 0$, $G/G^\alpha[p^k]$ has Ulm factors G_β for $\beta \neq \alpha$ and $p^k G^\alpha/G^{\alpha+1}$ for $\beta = \alpha$. Thus M_α has Ulm factors $(L_\alpha)_\beta$ for $\beta \neq \alpha$ and $(M_\alpha)_\alpha \cong p^k L_\alpha^\alpha / L_\alpha^{\alpha+1} \cong p^k(L_\alpha)_\alpha = p^k S_\alpha$. Let $M = \sum_{\alpha<\tau} M_\alpha$ and $A = \sum_{\alpha<\tau} L_\alpha^\alpha[p^k]$. Then M is totally projective and since $G \cong \sum_{\alpha<\tau} L_\alpha$, $G/A \cong M$. Ensey has shown that $K \cong L$ in $\hat{A}/\hat{B}$ if and only if there exists a bounded subgroup $B \subseteqq L$ such that K is quasi-isomorphic to L/B. Therefore $G \cong M$ in $\hat{A}/\hat{B}$ since A is bounded. But since M has the same Ulm factors as H, $M \cong H$ in $\hat{A}/\hat{B}$ by Hill's Uniqueness Theorem. Therefore $G \cong H$ in $\hat{A}/\hat{B}$.

LEMMA 4. *Let* G *be totally projective,* $\tau(G) = \tau$, *and for* $\alpha < \tau$, *let* $G_\alpha = S_\alpha \oplus T_\alpha$. *Let* H *be totally projective such that for all* $\alpha < \tau$, $H_\alpha = p^k S_\alpha \oplus T_\alpha$ *for some fixed integer* $k \geqq 0$. *Then* $G \cong H$ *in* $\hat{A}/\hat{B}$.

Proof. Such an H exists by Hill's Existence Theorem. Let $I_1 = \{\alpha < \tau \mid fr(T_\alpha) = fr(G_\alpha)\}$, $I_2 = \{\alpha < \tau \mid fr(T_\alpha) < fr(G_\alpha)\}$. For $\alpha \in I_2$, $fr(S_\alpha) = fr(G_\alpha)$. Thus, let $S_\alpha = S'_\alpha \oplus S''_\alpha$ where $fr(S'_\alpha) = fr(S''_\alpha) = fr(S_\alpha)$. Let H' be totally projective with the following Ulm factors: $H'_\alpha =$

$p^k S_\alpha \oplus T_\alpha$ for all $\alpha \in I_1$, $H'_\alpha = p^k S'_\alpha \oplus S''_\alpha \oplus T_\alpha$ for all $\alpha \in I_2$. Such an H' exists by Hill's Existence Theorem. By Lemma 3, $G \cong H'$ in $\hat{A}/\hat{B}$ and $H' \cong H$ in $\hat{A}/\hat{B}$. Therefore $G \cong H$ in $\hat{A}/\hat{B}$.

THEOREM 5. *Let G and H be totally projective. Then $G \cong H$ in $\hat{A}/\hat{B}$ if and only if the corresponding Ulm factors of G and H are uniformly quasi-isomorphic.*

Proof. Suppose the corresponding Ulm factors of G and H are uniformly quasi-isomorphic. Without loss of generality, it can be assumed that $\tau(G) = \tau(H) = \tau$. Suppose $\tau(G) = \tau$, $\tau(H) \geqq \tau + 1$. Since H_τ is quasi-isomorphic to $G_\tau = 0$, H_τ is bounded and thus $\tau(H) = \tau + 1$. Since H_τ is bounded, $G \cong H$ in $\hat{A}/\hat{B}$ if and only if $G \cong H/H_\tau$ in $\hat{A}/\hat{B}$. H/H_τ has type τ and is totally projective by the description given previously. Moreover, the corresponding Ulm factors of G and H/H_τ are uniformly quasi-isomorphic since the Ulm factors of H and H/H_τ agree except at the τ^{th} place. By uniform quasi-isomorphism, there exists an integer $k \geqq 0$ and for each $\alpha < \tau$, subgroups $S_\alpha \subseteqq G_\alpha$, $T_\alpha \subseteqq H_\alpha$ such that $p^k G_\alpha \subseteqq S_\alpha$, $p^k H_\alpha \subseteqq T_\alpha$, and $S_\alpha \cong T_\alpha$. Thus for each $\alpha < \tau$, (I) holds for G_α and H_α. D. Bertholf [2] has shown that if this is the case, then for all $\alpha < \tau$, $G_\alpha = G_{\alpha,0} \oplus \cdots \oplus G_{\alpha,2k}$ and $H_\alpha = H_{\alpha,0} \oplus \cdots \oplus H_{\alpha,2k}$ where $p^k G_{\alpha,0} \cong H_{\alpha,0}$, $p^{k-1} G_{\alpha,1} \cong H_{\alpha,1}, \cdots, G_{\alpha,k} \cong H_{\alpha,k}$, $G_{\alpha,k+1} \cong pH_{\alpha,k+1}, \cdots, G_{\alpha,2k} \cong p^k H_{\alpha,2k}$. By applying Lemma 4.2$k$ times, $G \cong H$ in $\hat{A}/\hat{B}$ is verified. The converse is Lemma 2.

THEOREM 6. *Let G and H be totally projective. Then $G \cong H$ in $\hat{A}/\hat{B}$ if and only if* (III) *is satisfied.*

Proof. Suppose (III) holds. Then there exists a $k \geqq 0$ such that for each α and integers $r \geqq 0$ and $n \geqq 0$,

$$\sum_{j=k}^{r+k} f_{G_\alpha}(n+j) = \sum_{j=k}^{r+k} f_G(\omega\alpha + n + j) \leqq \sum_{j=0}^{r+2k} f_H(\omega\alpha + n + j)$$
$$= \sum_{j=0}^{r+2k} f_{H_\alpha}(n+j) ,$$

and similarly, $\sum_{j=k}^{r+k} f_{H_\alpha}(n+j) \leqq \sum_{j=0}^{r+2k} f_{G_\alpha}(n+j)$. Therefore G_α and H_α satisfy (I) for all α and a fixed k. If this is the case, Ensey has shown that the corresponding Ulm factors of G and H are uniformly quasi-isomorphic. Therefore, by Theorem 5, $G \cong H$ in $\hat{A}/\hat{B}$. Ensey has shown that the converse is true for arbitrary reduced p-groups.

COROLLARY 7. *Let G and H be totally projective, α any ordinal.*

Then $G \cong H$ *in* $\hat{A}/\hat{B}$ *if and only if* $p^\alpha G \cong p^\alpha H$ *in* $\hat{A}/\hat{B}$ *and* $G/p^\alpha G \cong H/p^\alpha H$ *in* $\hat{A}/\hat{B}$.

Proof. Ensey has shown that if G and H are any reduced p-groups and if $G \cong H$ in $\hat{A}/\hat{B}$, then $p^\alpha G \cong p^\alpha H$ in $\hat{A}/\hat{B}$ and $G/p^\alpha G \cong H/p^\alpha H$ in $\hat{A}/\hat{B}$. Now suppose $p^\alpha G \cong p^\alpha H$ in $\hat{A}/\hat{B}$ and $G/p^\alpha G \cong H/p^\alpha H$ in $\hat{A}/\hat{B}$. Without loss of generality, it can be assumed that α is a limit ordinal. If not, then $\alpha = \beta + n$ where β is a limit. But if $p^\alpha G \cong p^\alpha H$ in $\hat{A}/\hat{B}$, then $p^\beta G \cong p^\beta H$ in $\hat{A}/\hat{B}$, and if $G/p^\alpha G \cong H/p^\alpha H$ in $\hat{A}/\hat{B}$, then $G/p^\beta G \cong H/p^\beta H$ in $\hat{A}/\hat{B}$. This follows since α and β differ by a finite ordinal and hence $p^\beta G/p^\alpha G$ is bounded. Now assuming α is a limit ordinal, the Ulm sequence of G is precisely that of $G/p^\alpha G$ followed by that of $p^\alpha G$; that is, the Ulm sequence of G is $f_{G/p\alpha G}(0)$, $f_{G/p\alpha G}(1), \cdots, f_{G/p\alpha G}(\beta), \cdots$ for all $\beta < \alpha$ followed by $f_{p^\alpha G}(0), f_{p^\alpha G}(1), \cdots$. Now $p^\alpha G$, $p^\alpha H$, $G/p^\alpha G$, and $H/p^\alpha H$ are all totally projective, so by Theorem 6, there exist $k_1 \geqq 0$ and $k_2 \geqq 0$ such that the inequalities in condition (III) hold for $p^\alpha G$ and $p^\alpha H$, and $G/p^\alpha G$ and $H/p^\alpha H$. Therefore, letting $k = \max(k_1, k_2)$, condition (III) holds for G and H and hence $G \cong H$ in $\hat{A}/\hat{B}$ by Theorem 6.

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Received December 4, 1969. The results of this paper were part of a doctoral dissertation completed in May 1969 under Professor Elbert A. Walker at New Mexico State University. The work was partially supported by NSF Grant GP 6564.

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