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GENERALIZATIONS OF REALCOMPACT SPACES

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Two generalizations of realcompactness are examined; a-realcompactness and c-realcompactness. The first, a-realcompactness, is invariant under perfect maps and is a generalization of almost realcompactness. Spaces that are a-realcompact and cb, almost realcompact and weak cb, or c-realcompact and weak cb, are also realcompact. Using these properties we obtain the following three theorems. If X is weak cb, then the union of two closed realcompact spaces is realcompact. The union of a countable collection of open sets is realcompact, if the closure of each open set is realcompact. If X is cb, the union of a countable collection of realcompact spaces is realcompact. The latter statement has been shown for X normal and the subspaces closed by Mrowka. It is not known if normality implies weak cb. The problem of preserving realcompactness under closed maps is also considered. Using a-realcompactness, we obtain the following special case. Realcompactness is preserved under closed maps if the range is a cb, k-space.

The concept of a maximal cover is introduced. A space is realcompact if and only if every maximal cozero cover has a countable subcover, and a space is topologically complete if and only if every maximal open cover is normal. A space X is almost realcompact [3, p. 128] if for every maximal open filter $\mathscr{A}$ of X with $\cap \overline{\mathscr{A}} = \varnothing$, there exists a countable subfamily $\mathscr{B}$ of $\mathscr{A}$ such that $\cap \overline{\mathscr{B}} = \varnothing$. Every realcompact space is almost realcompact and in [1] it is shown that every completely regular, Hausdorff, weak cb, almost realcompact space is realcompact. If f is a perfect map of X onto Y, then X almost realcompact implies Y is almost realcompact and Y almost realcompact and regular implies X is almost realcompact [3, 8, p. 134].

The reader is referred to [4] for the basic ideas of rings of continuous functions. A map will be used to designate a continuous onto function. A map is said to be *closed* if the image of each closed subset of the domain is closed in the range. If the inverse image of each compact set in the range is compact, then the map is said to be *compact*. *Perfect maps* are those which are closed and compact. A cover $\mathscr{U}$ is said to be *normal* if there exists a locally finite cozero refinement of $\mathscr{U}$. The upper and lower limit functions of a realvalued function f on X are defined as follows:

$$f^*(x) = \inf \{\sup_{y \in N} f(y) \colon N \text{ is a neighborhood of } x\}$$

$$f_*(x) = \sup \{\inf_{y \in N} f(y) \colon N \text{ is neighborhood of } x\} .$$

A real-valued function f is *normal lower semicontinuous* (n.l.s.c.) if f^* is real-valued and $f = (f^*)_*$. A space X is weak *cb*, if for every positive n.l.s.c. function g on X, there exists f in $C(X)$ such that $0 < f(x) \leqq g(x)$ for each x in X. [8, 3.1, p. 237].

Let $\mathscr{C}$ be a collection of subsets of X that are closed under finite intersection. A *$\mathscr{C}$-filter* $\mathscr{M}$ is a subset of $\mathscr{C}$ such that: $\varnothing \notin \mathscr{M}$; if M_1 and M_2 are in $\mathscr{M}$ then $M_1 \cap M_2 \in \mathscr{M}$; and if $G \in \mathscr{C}$, $M \in \mathscr{M}$, and $G \supset M$, then $G \in \mathscr{M}$. Let $G \in \mathscr{C}$ and $H \in \mathscr{C}$. If $G \cup H \in \mathscr{M}$ implies that $G \in \mathscr{M}$ or $H \in \mathscr{M}$, then $\mathscr{M}$ is said to be a *prime filter*. A *$\mathscr{C}$-ultrafilter* is a maximal $\mathscr{C}$-filter, and a filter M is said to be *free* if $\cap \mathscr{M} = \varnothing$ and *convergent* if $\cap \overline{\mathscr{M}} \neq \varnothing$. The particular collections $\mathscr{C}$ that will be used in this paper are the zero-sets, cozero sets, closed sets and the open sets. Realcompactness may also be characterized in terms of zero ultrafilters. A completely regular, Hausdorff space X is *realcompact* if and only if every free zero-ultrafilter $\mathscr{M}$ has a countable subcollection $\{Z_i\}$ such that $\cap Z_i = \varnothing$.

If A is a subspace of X, the symbol $\bar{A}$ will be used to denote the closure of A in X. If it is not clear that the closure is being taken in X, the notation $cl_X A$ will be used. Let $\mathscr{R}$ be a collection of subsets of X. Then $\overline{\mathscr{R}} = \{\bar{R} \colon R \in \mathscr{R}\}$ and $X - \mathscr{R} = \{X \backslash R \colon R \in \mathscr{R}\}$.

1. Maximal covers and α-realcompactness. The topological spaces in this paper are not assumed to be completely regular, Hausdorff unless explicitly stated. We dualize the notion of free $\mathscr{C}$-ultrafilters with the following definition. Let $\mathscr{R}$ be a collection of subsets of X which is closed under finite unions. $\mathscr{U}$ is called a *maximal $\mathscr{R}$-cover* of X if $\mathscr{U}$ covers X and does not have a finite subcover, $\mathscr{U} \subset \mathscr{R}$ and if for each set G in $\mathscr{R} \backslash \mathscr{U}$, $\mathscr{U} \cup \{G\}$ has a finite subcover of X. The following lemmas are easily verified.

LEMMA 1.1. *$\mathscr{U}$ is a maximal $\mathscr{R}$-cover if and only if $\mathscr{M} = \{X \backslash U \colon U \in \mathscr{U}\}$ is a free $X - \mathscr{R}$ ultrafilter.*

LEMMA 1.2. *If $\mathscr{U}$ is an $\mathscr{R}$-cover with no finite subcover, then $\mathscr{U}$ is contained in a maximal $\mathscr{R}$-cover.*

LEMMA 1.3. *A space X is compact if and only if there are no maximal open covers.*

LEMMA 1.4. *A completely regular, Hausdorff space X is realcompact if and only if every maximal cozero cover has a countable subcover.*

We now make the following definition. A space X is said to be *a-realcompact* if every maximal open cover has a countable subcover. Clearly closed subsets of a-realcompact spaces are a-realcompact. It is doubtful that a-realcompactness is a productive property.

THEOREM 1.5. *Let ϕ be a perfect map of X onto Y. Then X is a-realcompact if and onlf if Y is a-realcompact.*

Proof. Assume that X is a-realcompact and let $\mathscr{V}$ be a maximal open cover of Y. Then $\phi^{-1}(\mathscr{V}) = \{\phi^{-1}(V)\colon V \in \mathscr{V}\}$ is contained in a maximal open cover $\mathscr{U}$ of X. Let $U \in \mathscr{U}$. If $Y\backslash\phi(X\backslash U) \notin \mathscr{V}$, there exists a $V \in \mathscr{V}$ such that $Y\backslash\phi(X\backslash U) \cup V = Y$. It follows that $U \cup \phi^{-1}(V) = X$, which is impossible. Hence if $U \in \mathscr{U}$, then $Y\backslash\phi(X\backslash U) \in \mathscr{V}$. Since X is a-realcompact, there exists an increasing sequence $\{U_i\}$ in $\mathscr{U}$, such that $\cup U_i = X$ and, since ϕ is a compact map, for each $y \in Y$ there exists a k such that $\phi^{-1}(y) \subset U_k$. Thus $\{Y\backslash\phi(X\backslash U_i)\}$ is a countable cover of Y and Y is a-realcompact.

To prove the converse, let $\mathscr{U}$ be a maximal open cover of X and set $\mathscr{V} = \{Y\backslash\phi(X\backslash U)\colon U \in \mathscr{U}\}$. Let V be an open set such that $V \notin \mathscr{V}$. Then $\phi^{-1}(V) \notin \mathscr{U}$, and there exists a $U \in \mathscr{U}$ such that $U \cup \phi^{-1}(V) = X$. It follows that $V \cup Y\backslash\phi(X\backslash U) = Y$. Since $\phi^{-1}(y)$ is compact and $\mathscr{U}$ is closed under finite unions, $\mathscr{V}$ is a maximal open cover. The space Y is a-realcompact, so there exists a countable subcover $\{V_i\}$ of $\mathscr{V}$. Let $U_i \in \mathscr{U}$ such that $V_i = Y\backslash\phi(X\backslash U_i)$. Then $\{U_i\}$ is a countable subcover of X and X is a-realcompact.

In order for the perfect preimage of an almost realcompact space to be almost realcompact one has to assume the domain is regular. No such condition is needed for a-realcompactness. Next we show that a-realcompactness is a generalization of realcompactness.

THEOREM 1.6. *Let X be a regular space. If X is almost realcompact, then X is a-realcompact.*

Proof. Let $\mathscr{M}$ be a free closed ultrafilter. Set $\mathscr{U} = \{U\colon U$ is open and there exists a $F \in \mathscr{M}$ such that $F \subset U\}$. $\mathscr{U}$ is contained in an open ultrafilter $\mathscr{U}'$ and by regularity $\cap \overline{\mathscr{U}'} = \emptyset$. Since X is almost realcompact, there exists a countable subcollection $\mathscr{U}''$ of $\mathscr{U}'$ such that $\cap \overline{\mathscr{U}''} = \emptyset$. If $U \in \mathscr{U}'$, then $\bar{U}$ meets each element of $\mathscr{M}$ so $\bar{U} \in \mathscr{M}$. Thus $\overline{\mathscr{U}''}$ is a countable subfamily of $\mathscr{M}$ that has empty intersection. Hence X is a-realcompact.

A space is said to be a *cb-space* (weak *cb*) [8] if given a decreasing sequence $\{F_n\}$ of closed (regular closed) sets in X with empty intersection, there exists a sequence $\{Z_n\}$ of zero-sets with empty intersection such that $Z_n \supset F_n$. The referee has pointed out that the assertion that normality implies *cb* is either independent of existing set-theoretic axioms (through the axiom of choice) or else false according to M.E. Rudin in 1955 Canadian Journal and recent results of Tennenbaum and Solovay. However, normality and countably paracompactness implies *cb*. One of the nice properties of the zero sets is that a zero ultrafilter $\mathscr{M}$ has the countable intersection property if and only if $\mathscr{M}$ contains a prime filter which has the countable intersection property. Unfortunately, this is not necessarily true for closed ultrafilters. We do, however, have the following result.

THEOREM 1.7. *Let X be a cb-space. A closed ultrafilter $\mathscr{M}$ has the countable intersection property if and only if $\mathscr{M}$ contains a prime filter having the countable intersection property.*

Proof. The sufficiency is clear. Let $\mathscr{M}$ be a closed ultrafilter and $\mathscr{M}'$ be a prime filter contained in $\mathscr{M}$ such that $\mathscr{M}'$ has the countable intersection property. Suppose there exists a collection $\{F_i\} \subset \mathscr{M}$ such that $\cap F_i = \varnothing$. Set $G_n = \bigcap_{i=1}^n F_i$. Since X is *cb*, there exists a collection of zero sets $\{Z_n\}$ with empty intersection such that $Z_n \supset G_n$. Pick $f_i \in C(X)$ such that $Z_i = Z(f_i)$, $0 \leqq f_i \leqq 1$ and set $f = \sum 2^{-i} f_i$. Let $C_n = \{x: f(x) \geqq 2^{-n}\}$ and $B_n = \{x: f(x) \leqq 2^{-(n-1)}\}$. Now if $x \in \bigcap_{i=1}^{n+1} Z_i$, then $f(x) \leqq 2^{-(n+1)}$. Hence $x \notin C_n$ and $C_n \notin \mathscr{M}'$. Since $\mathscr{M}'$ is prime, $B_n \in \mathscr{M}'$. However $\cap B_n = \varnothing$, which contradicts the countable intersection property of $\mathscr{M}'$. Thus $\mathscr{M}$ has the countable intersection property.

THEOREM 1.8. *If every prime closed nonconvergent filter $\mathscr{M}$ on X has a countable subcollection $\mathscr{M}''$ such that $\cap \overline{\mathscr{M}''} = \varnothing$, then X is almost realcompact.*

Proof. Let $\mathscr{S}$ be a maximal open filter such that $\cap \overline{\mathscr{S}} = \varnothing$. Set $\mathscr{M} = \{F: F$ is closed and there exists a $U \in \mathscr{S}$ such that $U \subseteqq F\}$. Let F and G be closed sets such that $F \cup G = X$. Then if $F \notin \mathscr{M}$, $X \backslash F \in \mathscr{S}$, so $G \in \mathscr{M}$. Since $\mathscr{M}$ is a prime closed nonconvergent filter, there exists a countable subcollection $\mathscr{M}'$ of $\mathscr{M}$ such that $\cap \overline{\mathscr{M}'} = \varnothing$. Pick $U_F \in \mathscr{S}$ such that $U_F \subset F$. Then $\cap_{F \in \mathscr{M}'} \bar{U}_F = \varnothing$ and X is almost realcompact.

Since almost realcompact and weak-*cb* implies realcompact, and for *cb* spaces, a-realcompactness implies that every prime closed nonconvergent filter $\mathscr{M}$ on X has a countable subcollection $\mathscr{M}''$ such that $\cap \overline{\mathscr{M}''} = \varnothing$, we have the following:

COROLLARY 1.10. *If X is completely regular, Hausdorff, a-real-*

compact and cb, then X is realcompact.

2. Complete systems of covers. Frolik introduces the concept of a complete system of open covers on a space X in [3, p. 128]. Let ν be a collection of coverings of a space X. A *ν-Cauchy* family is a family $\mathscr{M}$ of subsets of X with the finite intersection property such that for every $\mathscr{U} \in \nu$ there exists a $B \in \mathscr{M}$ and $U \in \mathscr{U}$ with $U \supset B$. The collection ν will be called *complete* if $\cap \overline{\mathscr{M}} \neq \varnothing$ for every ν-Cauchy family $\mathscr{M}$ of open sets. Then the following theorem is proved.

THEOREM 2.0. *A space X is almost realcompact if and only if the collection ν of all countable open coverings of X is complete.*

Also, if X is almost realcompact, then the collection of all countable closed covers is complete and if the collection of all countable closed covers is complete, then X is a-realcompact.

However, Frolik gives another definition of a complete family of open coverings in [2]. These two definitions are not equivalent so we shall introduce the term a-complete. A family of coverings ν is said to be *a-complete* if the following condition is satisfied:

If $\{F\}$ is a ν-Cauchy family of closed sets then $\cap\{F\} \neq \varnothing$.

THEOREM 2.1. *A space X is a-realcompact if and only if the collection ν of all countable open coverings of X is a-complete.*

Proof. Suppose X is a-realcompact, and let $\mathscr{F}$ be a ν-Cauchy closed ultrafilter. If $\mathscr{F}$ is free, there exists $\{F_i\} \subset \mathscr{F}$ such that $\cap F_i = \varnothing$. Since $\{X \backslash F_i\}$ is a countable open cover of X there exists a $F \in \mathscr{F}$ and an integer i such that $F \subset X \backslash F_i$, which is impossible. Thus $\cap \mathscr{F} \neq \varnothing$ and ν is a-complete.

Conversely, suppose the collection ν of all countable open coverings of X is a-complete. Let $\mathscr{M}$ be a free closed ultrafilter. There exists a $\mathscr{U} \in \nu$ such that if $U \in \mathscr{U}$ then $F \nsubseteq U$ for any $F \in \mathscr{M}$. Hence $X \backslash U \in \mathscr{M}$ and $\bigcap_{U \in \mathscr{U}} X \backslash U = \varnothing$, so $\mathscr{M}$ does not have the countable intersection property. Thus X is a-realcompact.

Frolik used the following definition for only completely regular spaces, but it clearly makes sense for any space. The collection ν will be called *b-complete* if $\cap \overline{\mathscr{M}} \neq \varnothing$ for every ν-Cauchy family $\mathscr{M}$. Note that if X is regular, then completeness of a family of open covers is equivalent to b-completeness. It is known [3] that if X is realcompact, then the collection of all countable zero covers is b-complete. Hence, the collection of all countable closed covers μ will be b-complete. Frolik also shows that if X is normal and countably paracompact and μ is b-complete, then X is realcompact.

THEOREM 2.2. *If the collection μ of all countable closed covers is a-complete, then X is a-realcompact.*

Proof. Let $\mathscr{M}$ be a free closed ultrafilter. There exists an $\mathscr{U} \in \mu$ such that if $F \in \mathscr{M}$ and $G \in \mathscr{U}$, then $F \not\subset G$. Since $G \notin \mathscr{M}$, there exists a $F_G \in \mathscr{M}$ such that $G \cap F_G = \emptyset$. It follows that $\bigcap_{G \in \mathscr{U}} F_G = \emptyset$ and X is a-realcompact.

Similarly we have the following theorem.

THEOREM 2.3. *If X is a-realcompact and cb, then the collection μ of all countable closed covers is b-complete.*

Thus for completely regular, Hausdorff, cb-spaces, realcompactness is equivalent to the condition that the collection μ of all countable closed covers is a-complete, b-complete, or complete.

Note that every space is a-complete with respect to the collection ν of all maximal open covers of X. However, the property of being b-complete with respect to the collection ν of all maximal open covers of X appears to be a stronger property. Clearly it is closed-hereditary.

Next we show that it is preserved under perfect maps.

THEOREM 2.4. *Let ϕ be a perfect map of X onto Y. If X is b-complete with respect to the collection ν of all maximal open covers on X, then Y is b-complete with respect to the collection σ of all maximal open covers on Y.*

Proof. Assume X is complete with respect to ν and let $\mathscr{U}$ be a maximal open cover of X. Then $\mathscr{V} = \{Y \backslash \phi(X \backslash U)\colon U \in \mathscr{U}\}$ is a maximal open cover of Y, since ϕ is perfect (proof of 1.5). Let $\mathscr{M}$ be a σ-Cauchy family. If $\mathscr{U} \in \nu$, there exists a $U \in \mathscr{U}$ and $M \in \mathscr{M}$ such that $M \subset Y \backslash \phi(X \backslash U)$. Thus $\phi^{-1}(\mathscr{M})$ is a ν-Cauchy family. Since X is complete with respect to ν, we have that $\overline{\cap \phi^{-1}(\mathscr{M})} \neq \emptyset$. But ϕ is continuous, so $\cap \overline{\mathscr{M}} \neq \emptyset$. Whence, Y is complete with respect to the collection σ.

3. c-realcompact spaces.

Let us call X *c-realcompact* if X is completely regular (Hausdorff) and for every $p \in \beta X \backslash X$ there exists a normal lower semicontinuous function [8] on βX such that $f(p) = 0$ and f is positive on X.

THEOREM 3.1. *Let X be a completely regular, weak cb, c-realcompact space. Then X is realcompact.*

Proof. Let $p \in \beta X \backslash X$ and let g be a n.l.s.c. function g on βX such that $g(p) = 0$ and g is positive on X. Since X is weak cb, there exists a $f \in C(X)$ such that $0 < f(x) \leqq g(x)$ for each $x \in X$. Let f^* be the Stone-Cech extension of $f \wedge 1$ to βX. Since n.l.s.c. functions are determined on dense subsets, $0 \leqq f^*(p) \leqq g(p) = 0$. Thus X is realcompact.

Next we investigate the relation between c-realcompact and almost realcompact.

THEOREM 3.2. *Let X be a completely regular space. The following statements are equivalent*:

(a) *X is almost realcompact.*

(b) *Let BX be any Hausdorff compactification of X. If $\mathscr{A}$ is any open ultrafilter on BX such that there exists a $p \in BX \backslash X$ such that $\cap \overline{\mathscr{A}} = \{p\}$, then there exists a* n.l.s.c. *function f on BX such that f is positive on $X, f(p) = 0$ and there exists a countable subfamily $\{U_i\}$ of $\mathscr{A}$ such that $x \in \bar{U}_n$ implies that $f(x) < 1/n$.*

Proof. Assume that X is almost realcompact. Let $\mathscr{A}$ be an open ultrafilter on BX such that $\cap \overline{\mathscr{A}} = \{p\} \in BX \backslash X$. Then $\mathscr{A}' = \{U \cap X\colon U \in \mathscr{A}\}$ is an open ultrafilter on X and $\bigcap_{U \in \mathscr{A}'} \mathrm{cl}_X U = \emptyset$. Since X is almost realcompact, there exists a decreasing subcollection $\{U_i\}$ of $\mathscr{A}$ such that $\bigcap \mathrm{cl}_X(U_i \cap X) = \emptyset$. Define $f_i(x) = 0$ if $x \in \mathrm{cl}_{BX} U_i$ and $f_i(x) = 1$ otherwise. Set $f = \sum 2^{-i} f_i$. Then f is n.l.s.c., $f(p) = 0$, f is positive on X and if $x \in \bar{U}_n$, then $f(x) < 1/n$.

Conversely, assume the conditions of (b) are satisfied. Let $\mathscr{A}$ be an open ultrafilter on X such that $\cap \overline{\mathscr{A}} = \emptyset$ and set $\mathscr{A}' = \{U\colon U$ is open in BX and $U \cap X \in \mathscr{A}\}$. There exists a $p \in BX \backslash X$ such that $\cap \overline{\mathscr{A}'} = \{p\}$ and since $\mathscr{A}'$ is an open ultrafilter on BX there exists a n.l.s.c. function f on BX such that f is positive on $X, f(p) = 0$, and there exists a countable subfamily $\{U_n\}$ of $\mathscr{A}'$ such that $x \in \bar{U}_n$ implies that $f(x) < 1/n$. Then $\{U_n \cap X\}$ is a countable subfamily of $\mathscr{A}$ and $\cap \mathrm{cl}_X(U_n \cap X) = \emptyset$, since f is positive on X. Thus X is almost realcompact.

COROLLARY 3.3. *Let X be almost realcompact and BX a compactification of X. Then if $p \in BX \backslash X$, there exists a* n.l.s.c. *function f such that f is positive on X and $f(p) = 0$. In particular, if X is almost realcompact, then X is c-realcompact.*

There exists an almost realcompact space and hence c-realcompact space that is not realcompact [1]. The problem is that the n.l.s.c. function may not be continuous at the point p.

THEOREM 3.4. *Let X be a completely regular (Hausdorff) space. If for any $p \in \beta X \backslash X$ there exists a nonnegative function f on βX such that f is positive on X, $f(p) = 0$ and f is continuous at p, then X is realcompact.*

Proof. Consider $f^*(x) = \inf\{\sup_{y \in N} f(y)\colon N$ is a neighborhood of $x\}$. Now f^* is upper semicontinuous, $f^* \geqq f$ and $f^*(p) = 0$. By a result of John Mack, if for every $p \in \beta X \backslash X$ there exists a nonnegative upper semicontinuous function f on βX such that f is positive on X and $f(p) = 0$, then X is realcompact.

Recall that X is realcompact if and only if $X = \cap\{\text{cozero } f\colon f \in C(\beta X)$ and cozero $f \supset X\}$. If f is a nonnegative n.l.s.c. function then cozero f is a countable union of regular open sets. Now X is c-realcompact if and only if $X = \cap\{\text{cozero } f\colon f$ is nonnegative n.l.s.c. on βX and cozero $f \supset X\}$. Thus for any completely regular space X there exists a smallest c-realcompact space uX between X and βX, and this space is the largest subspace of βX to which every nonnegative upper semicontinuous function on X can be extended. If X is weak cb, then $uX = \upsilon X$. Further $\upsilon \tilde{X}$ is the absolute of uX. [5].

4. Applications to realcompact and topologically complete spaces.

Mrowka [9] has shown that if a normal space X is the union of a countable collection of closed realcompact spaces, then X is realcompact.

THEOREM 4.1. *If X is the union of two closed, almost realcompact spaces, then X is almost realcompact.*

Proof. Let $X = F \cup G$ where F and G are closed, almost realcompact spaces and set $U = X \backslash F$ and $V = X \backslash \overline{X \backslash F}$. Then $X = \bar{U} \cup \bar{V}$ where U and V are open sets and $\bar{U}$ and $\bar{V}$ are almost realcompact, since they are regular closed subsets of an almost realcompact space. Let $\mathscr{A}$ be an open ultrafilter on X such that $\cap \overline{\mathscr{A}} = \varnothing$. Since $U \cup V$ is dense in X, it follows that $U \cup V \in \mathscr{A}$. But $\mathscr{A}$ is prime, so $U \in \mathscr{A}$ or $V \in \mathscr{A}$. Assume that $V \in \mathscr{A}$ and set $\mathscr{A}' = \{U \cap \bar{V}\colon U \in \mathscr{A}\}$. Since $\mathscr{A}'$ is an open ultrafilter on $\bar{V}$, and $\bar{V}$ is almost realcompact, there exists a countable subcollection $\{A_i'\}$ of $\mathscr{A}'$ such that $\cap \bar{A}_i' = \varnothing$. Now let $A_i \in \mathscr{A}$ such that $A_i' = A_i \cap \bar{V}$. Then $\cap \overline{(A_i \cap V)} = \varnothing$ and hence X is almost realcompact.

THEOREM 4.2. *If X is the union of a countable collection of open sets such that the closure of each open set is almost realcompact, then X is almost realcompact.*

Proof. Let $X = \cup U_i$ where each U_i is open and $\bar{U}_i$ is almost realcompact. Let $\mathscr{A}$ be an open ultrafilter on X such that $\cap \overline{\mathscr{A}} = \varnothing$. Now suppose that $U_i \notin \mathscr{A}$ for each positive integer i. Pick $V_i \in \mathscr{A}$ such that $V_i \cap U_i = \varnothing$. Then $\cap \bar{V}_i = \varnothing$, and we are finished. If there exists a $U_i \in \mathscr{A}$, then proceed as in 4.1 and obtain a countable subcollection $\{A_i\}$ of $\mathscr{A}$ such that $\cap \bar{A}_i = \varnothing$. Thus X is almost realcompact.

COROLLARY 4.3. *Let X be a completely regular (Hausdorff) weak cb-space. If X is the union of two closed realcompact spaces, then X is realcompact. If X is the union of a countable collection of open sets such that the closure of each open set is realcompact, then X is realcompact.*

THEOREM 4.4. *If X is the union of a countable collection of a-realcompact spaces, then X is a-realcompact.*

Proof. Let $X = \cup F_i$ where each F_i is a-realcompact. Let $\mathscr{M}$ be a free closed ultrafilter. Suppose that for each positive integer i there exists $G_i \in \mathscr{M}$ such that $F_i \cap G_i = \varnothing$. Then $\cap G_i = \varnothing$, and we are finished. Now assume that there exists an i such that $\mathscr{M}' = \{G \cap F_i : G \in \mathscr{M}\}$ is a filter. Then $\mathscr{M}'$ is a free closed ultrafilter in F_i and since F_i is a-realcompact, there exists a countable subcollection $\{G_i\}$ of $\mathscr{M}'$ such that $\cap G_i = \varnothing$. If $\cap \operatorname{cl}_X G_i \neq \varnothing$, then there is a $G \in F$ such that $(\cap \operatorname{cl}_X G_i) \cap G = \varnothing$. Hence X is a-realcompact.

COROLLARY 4.5. *Let X be a completely regular (Hausdorff) cb-space. If X is the union of a countable collection of realcompact spaces, then X is realcompact.*

It is not known if normality implies weak *cb*. The following theorem replaces the normal and weak *cb* hypothesis of [1, 2.4] with *cb* and provides a further generalization of [6, 7.5, p. 477].

THEOREM 4.6. *Let ϕ be a closed map of X onto a completely regular (Hausdorff), cb, k-space Y. Then X realcompact implies that Y is realcompact.*

Proof. We shall show that Y is a-realcompact. Then realcompactness will follow from 1.10. Let $\mathscr{M}$ be a free closed ultrafilter on Y. Then $\phi^{-1}(M)$ is contained in a free closed ultrafilter $\mathscr{A}$ and if $F \in \mathscr{A}$, then $\phi(F) \in \mathscr{M}$. Since X is a-realcompact, there exists a countable decreasing subfamily $\{A_n\}$ of $\mathscr{A}$ such that $\cap A_n = \varnothing$. Define $f_n(x) = 0$ if $x \in \operatorname{cl}_{\beta X} A_n$ and $f_n(x) = 1$ otherwise. Then $f =$

$\sum 2^{-n} f_n$ is a lower semicontinuous function on βX and f is positive on X. Let Φ denote the Stone extension of ϕ from βX onto βY. Since f is a nonnegative lower semicontinuous function, $f^i(y) = \inf\{f(x)\colon x \in \Phi^{-1}(y)\}$ is a function on βY and $Z(f^i) \cap Y \subset \Phi(Z(f)) \cap Y$, which is discrete by [1, 2.3]. Set $\Phi(Z(f)) \cap Y = Y_0$. Now $\phi(A_n) \in \mathscr{M}$ and $\cap \phi(A_n) \subset Z(f^i) \cap Y \subset Y_0$. If $Y_0 \notin \mathscr{M}$, then there exists a $G \in \mathscr{M}$ such that $Y_0 \cap G = \emptyset$. Thus $\{\phi(A_n)\} \cup \{G\}$ is a countable subfamily of $\mathscr{M}$ with empty intersection. If $Y_0 \in \mathscr{M}$ then since Y_0 is realcompact and closed, there exists a countable subcollection of $\mathscr{M}$ with empty intersection. Thus Y is a-realcompact.

Topological completeness can also be characterized in terms of maximal open and maximal cozero covers.

THEOREM 4.7. *Let X be a completely regular (Hausdorff) space. The following statements are equivalent.*

(1) *X is topologically complete.*

(2) *Every maximal cozero cover is normal.*

(3) *Every maximal open cover is normal.*

Proof. $1 \Leftrightarrow 2$. Assume that X is topologically complete. Let μ be the uniformity generated by all continuous pseudometrics on X, and let $\mathscr{U}$ be a maximal cozero cover. Then $\mathscr{M} = \{X \backslash U\colon U \in \mathscr{U}\}$ is a free zero-ultrafilter. Since X is topologically complete, $\mathscr{M}$ is not μ-Cauchy. So there exists a $p \in \mu$ and $\varepsilon > 0$ such that $\mathscr{M}$ contains no set of p-diameter $< 3\varepsilon$. Let $U_\varepsilon = \{U_x\colon x \in X\}$ where $U_x = \{y \in X\colon p(x, y) < \varepsilon\}$. Then U_ε is a normal cozero cover. If $Z \in \mathscr{M}$ and $x \in X$, then $Z \not\subset U_x$ so $X \backslash U_x \cap Z \neq \emptyset$. Hence $X \backslash U_x \in \mathscr{M}$ and $U_x \in \mathscr{U}$. Thus $\mathscr{U}$ is normal.

Conversely, assume every maximal cozero cover is normal. Let μ be the collection of all continuous pseudometrics on X and let $\mathscr{M}$ be a μ-Cauchy z-ultrafilter. If $\mathscr{M}$ is free, then $\mathscr{U} = \{X \backslash F\colon F \in \mathscr{M}\}$ is a maximal cozero cover and so by the hypothesis, $\mathscr{U}$ is normal. Thus there exists a pseudometric d and $\varepsilon > 0$ such that U_ε refines $\mathscr{U}$. Since $\mathscr{M}$ is μ-Cauchy, there exists a $U \in U_\varepsilon$ and a $F_0 \in \mathscr{M}$ such that $F_0 \subset U$. Since U_ε refines $\mathscr{U}$ there exists a $F \in \mathscr{M}$ such that $U \subset X \backslash F$; but $F_0 \subset X \backslash F$ contradicts $\mathscr{M}$ being a filter. Thus $\mathscr{M}$ is fixed and X is topologically complete.

$1 \Leftrightarrow 3$. X is topologically complete if and only if every Cauchy closed ultrafilter is fixed, so the proof is identical to the preceding.

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