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ON A CLASS OF NÖRLUND MEANS AND FOURIER SERIES

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By considering a class of Nörlund means that covers as a subclass the corresponding (C) means, we obtain in the present paper, several results concerning absolute Nörlund summability and deduce from these the corresponding $|C|$ results as special cases. What is indeed remarkable, is that a special case of our Theorem 2 improves an earlier result due to Bosanquet and Hyslop in dropping one of the two independent conditions used by them. Further, the proofs of some of our results are shorter and even more direct than the proofs given for the corresponding special cases by using equivalent Riesz means instead of (C) means.

1. **Definitions and notations.** Let $\sum_n v_n$ be a given infinite series with the sequence of partial sum $\{s_n\}$. We shall consider sequence to sequence transformations of the type

$$t_n = \sum_{k=0}^{\infty} d_{nk}s_k;\ d_{nk} = 0 \quad \text{for} \quad k > n\ ;$$

in which the elements of the matrix $D = ((d_{nk}))$ are real or complex constants. t_n is called the n-th D-mean of $\{s_n\}$.

Let $\{p_n\}$ be a sequence of constants, real or complex and let $P_n = p_0 + p_1 + \cdots + p_n \neq 0$, $P_{-1} = p_{-1} = 0$. Then the matrix D defines a Nörlund matrix (N, p), if

$$d_{nk} = p_{n-k}/P_n\ , \qquad n \geqq k \geqq 0\ .$$

In the special case in which

$$(1.1) \qquad p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)},\ \alpha \neq -1,\ -2,\ \cdots\ ,$$

the (N, p) mean reduces to the familiar (C, α) mean.

The $(N, p)(C, 1)$ matrix is defined as the product of a (N, p) matrix with the $(C, 1)$ matrix. Thus the $(N, p)(C, 1)$ mean of $\{s_n\}$ is

$$t_n = \frac{1}{P_n}\sum_{r=0}^{n} p_{n-r}\frac{1}{r+1}\sum_{k=0}^{r} s_k\ .$$

Similarly, one defines the $(C, 1)(N, p)$ mean [5].

Let $f(t)$ be integrable (L) in $(-\pi, \pi)$ and periodic with period 2π. We assume as we may without any loss of generality that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_n A_n(t) .$$

Then the conjugate series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_n B_n(t) .$$

We shall also consider the series

$$\sum_n \frac{1}{n}\left\{\sum_{k=1}^{n} A_k(t) - s\right\} = \sum_n A_n^*(t) ,$$

where s is an appropriate number, independent of n.

Throughout the present paper we write $L^\alpha(t)$ for the series $\sum_n n^\alpha A_n(t)$, $\tilde{L}^\alpha(t)$ for $\sum_n n^\alpha B_n(t)$, $(\alpha \geqq 0)$, $L^*(t)$ for $\sum_n A_n^*(t)$ and $\mathscr{A}$ for the class $\{L^\alpha(t), \tilde{L}^\alpha(t), L^*(t)\}$.

Let E_f be a point set in the interval $(-\pi, \pi)$ for each function $f(t)$ and such that at every point $x \in E_f$, $f(x)$ has a finite definite value and satisfies a prescribed condition of regularity.

DEFINITION 1. A method of summation $D = (\!(d_{nk})\!)$ is said to be $|A(x), E_f|$-*effective*, if for each $x \in E_f$

$$\sum_{n=1}^{\infty} | t_n(A(x)) - t_{n-1}(A(x)) | < \infty ,$$

symbolically, $\{t_n(A(x))\} \in BV$; where $t_n(A(x))$ denotes the nth D-mean of $A(x) \in \mathscr{A}$.

We write

$$\phi(t) = \frac{1}{2}\{f(x + t) + f(x - t)\} ; \ \phi^*(t) = \phi(t) - s ;$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - u)^{\alpha-1}\phi(u)du, \ \alpha > 0 ; \ \Phi_0(t) = \phi(t) ;$$

$$\phi_\alpha(t) = \Gamma(\alpha + 1)t^{-\alpha}\Phi_\alpha(t), \ \alpha \geqq 0 ;$$

$$\psi(t) = \frac{1}{2}\{f(x + t) - f(x - t)\} ,$$

$\Psi_\alpha(t)$ and $\psi_\alpha(t)$ have similar meanings.

$[x]$ denotes the greatest integer not greater than x.

By '$F(t) \in BV(a, b)$', we mean that $F(t)$ is a function of bounded variation in (a, b) and by '$\{\lambda_n\} \in B$' that $\{\lambda_n\}$ is a bounded sequence.

K denotes a positive constant, not necessarily the same at each occurrence.

DEFINITION 2. For some $\alpha \geqq 0$, the point x is said to be
(i) $|F_\alpha|$ – *regular*, if $\phi_\alpha(t) \in BV(0, \pi)$,
(ii) $|\tilde{F}_\alpha|$ – *regular*, if $\Psi_\alpha(+0) = 0$ and $\int_0^\pi t^{-\alpha} |d\Psi_\alpha(t)| \leqq K$,
(iii) $|F^\alpha|$ – *regular*, if $\int_0^\pi t^{-\alpha} |d\phi(t)| \leqq K$,
(iv) $|\tilde{F}^\alpha|$ – *regular*, if $\psi(+0) = 0$ and $\int_0^\pi t^{-\alpha} |d\psi(t)| \leqq K$,
(v) $|F^*|$ – *regular*, if $\int_0^\pi t^{-1} |d\phi^*(t)| \leqq K$,
(vi) $|\tilde{F}^*|$ – *regular*, if $\int_0^\pi t^{-1} |\psi(t)| dt \leqq K$.

Denoting the set of $|X|$ – *regular* points with respect to $f(t)$ in $(-\pi, \pi)$ by $E|X, f|$, we know the following ([14], § 13.24)

$$E|\tilde{F}^*, f| \not\subset E|\tilde{F}_0, f| \quad \text{and} \quad E|\tilde{F}_0, f| \not\subset E|\tilde{F}^*, f|^1 . \tag{1.2}$$

DEFINITION 3. A method of summation, which is $|A(x), E_f|$ – *effective* is said to be
(i) $|F_\alpha|$ – *effective*, if $A(x) = L^0(x)$ and $E_f = E|F_\alpha, f|$;
(ii) $|\tilde{F}_\alpha|$ – *effective*, if $A(x) = \tilde{L}^0(x)$ and $E_f = E|\tilde{F}_\alpha, f|$;
(iii) $|\tilde{F}^\alpha|$ – *effective*, if $A(x) = L^\alpha(x)$ and $E_f = E|F^\alpha, f|$;
(iv) $|\tilde{F}^\alpha|$ – *effective*, if $A(x) = \tilde{L}^\alpha(x)$ and $E_f = E|\tilde{F}^\alpha, f|$;
(v) $|F^*|$ – *effective*, if $A(x) = L^*(x)$ and $E_f = E|F^*, f|$;
(vi) absolute *α-effective* or $|\alpha|$-*effective*, if it is effective in the sense of (i)–(iv) simultaneously.

The following notations will be used throughout. If for $n = 0, 1, 2, \cdots$

$$p_n > 0,\ p_{n+1}/p_n \leqq p_{n+2}/p_{n+1} \leqq 1 ,$$

then we shall write $\{p_n\} \in M$. If $\{p_n\} \in M$ and for some α,

$$P_k \sum_{n=k}^{\infty} \frac{1}{n^{1-\alpha} P_n} \leqq K k^\alpha , \qquad k = 1, 2, \cdots \tag{1.3}$$

then we write $\{p_n\} \in M_\alpha$.

For a given series $v = \sum_n v_n$,

$$\sigma_n(v) = \sum_{k=0}^{n} p_{n-k} k v_k .$$

We also write

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[1] I.e., $x \in E|\tilde{F}^*, f|$ does not imply that $x \in E|\tilde{F}_0, f|$ and conversely.

$$h(n, t) = \frac{2}{\pi} \sum_{k=0}^{n} p_{n-k} \exp(ikt) ,$$

$$H(n, u) = \frac{1}{\Gamma(1-\alpha)} \int_u^{\pi} (t-u)^{-\alpha} \frac{d}{dt} h(n, t) dt ,$$

$g(n, t)$ and $\tilde{g}(n, t)$ for the imaginary and real parts respectively of $h(n, t)$. $J(n, u)$ (or $\tilde{J}(n, u)$) is $H(n, u)$ with $h(n, t)$ replaced by $g(n, t)$ (or $\tilde{g}(n, t)$). Further

$$V(n, u) = \frac{1}{\Gamma(1+\alpha)} \int_0^u v^{\alpha} \frac{d}{dv} J(n, v) dv .$$

2. Introduction. Concerning $|\alpha| -$ *effectiveness* of the (C)-method we have the following result which is known to be the best possible in the sense that it breaks down if $\delta = 0$.

THEOREM A. *If* $0 < \alpha < 1$, *then* $(C, \alpha + \delta)$ *for any* $\delta > 0$ *is* $|\alpha| -$ *effective.*

Starting with the proof of $|F_\alpha| -$ *effectiveness* of $(C, \alpha + \delta)$ given by Bosanquet [1] in 1936, Theorem A has been completed in stages by different authors. Thus $|\tilde{F}_\alpha| -$ *effective* part of Theorem A is due to Bosanquet and Hyslop ([2], Th. 2) and $|F^\alpha|$ and $|\tilde{F}^\alpha| -$ *effective* parts are due to Mohanty ([11], Th. 1 and Th. 2). It is somewhat peculiar to observe that Theorem A may be extended to cover the case $\alpha = 0$, as far as $|F_\alpha|$ or $|F^\alpha| -$ *effective* parts are concerned but for $|\tilde{F}_0|$ or equivalently $|\tilde{F}^0| -$ *effectiveness* of the (C) method, Bosanquet and Hyslop ([2], Th. K, with $\alpha = 0$) require an additional condition that x is $|\tilde{F}^*| -$ *regular* also, in the following.

THEOREM B. *If* x *is* $|\tilde{F}^*| -$ *regular, then the* (C, δ) *method is* $|\tilde{F}_0| -$ *effective for each* $\delta > 0$.

The condition that x is $|\tilde{F}^*| -$ *regular* is independent of the condition that x is $|\tilde{F}_0| -$ *regular* in view of (1.2).

In Theorem 1 of the present paper we prove $|\alpha| -$ *effectiveness* $(0 < \alpha < 1)$ of a (N, p) method which covers the corresponding (C) method as a special subclass and deduce the various results of Theorem A as particular cases of our Theorem 1. What is indeed remarkable is that in Theorem 2, we have succeeded in extending Theorem 1 to the case $\alpha = 0$ by proving $|\tilde{F}_0| -$ *effectiveness* of the (N, p) method, even without using the hypothesis that x is a $|\tilde{F}^*| -$ *regular* point. Thus the following special case of Theorem 2 improves Theorem B in dropping the condition that x is a $|\tilde{F}^*| -$

regular point.

THEOREM C. *The* (C, δ) *method is* $|\tilde{F}_0| -$ *effective for every* $\delta > 0$.

Covering as a special case, an earlier result due to Mohanty and Mohapatra ([12], Th. 3) we prove in Theorem 3, $|F^*| -$ *effectiveness* of the (N, p) method and thus demonstrate that the $|N, p|$ of $L^*(x)$, is a local property of its generating function [6].

It may be observed that the proofs of some of our theorems are shorter and even more direct than the proofs given in support of the corresponding special cases by using equivalent Riesz methods instead of the (C) – methods.

3. We prove the following.

THEOREM 1. *If* $0 < \alpha < 1$ *and* $\{p_n\} \in M_\alpha$, *then* (N, p) *is* $|\alpha| -$ *effective.*

THEOREM 2. *If* $\{p_n\} \in M_0$, *then* (N, p) *is* $|F_0|$ *and* $|\tilde{F}_0| -$ *effective.*

THEOREM 3. *If* $\{p_n\} \in M_0$, *then* (N, p) *is* $|F^*| -$ *effective.*

4. **Some preliminary results.** We need the following lemmas, of which Lemma 1 is the same as Theorem 6 of Das [3].

LEMMA 1. *If* $\{p_n\} \in M$, *then a necessary and sufficient condition that* $\{t_n(v)\} \in BV$, *for a given series* $\sum_n v_n$ *is that*

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} |\sigma_n(v)| \leqq K ,$$

where $t_n(v)$ *is the n-th* (N, p) *mean of* $\sum_n v_n$.

LEMMA 2. *If* $\{p_n\}$ *is a nonnegative monotonic nonincreasing sequence, then for any* n *and* $0 \leqq a \leqq b$

$$\left| \sum_{k=a}^{b} p_k \exp i(n-k)t \right| \leqq KP_{[1/t]} ,$$

uniformly in $0 < t \leqq \pi$.

Lemma 2 is given in McFadden [10].

LEMMA 3. *If* $\{p_n\}$ *is a positive nonincreasing sequence, then*

$$|H(n, u)| = \begin{cases} O(n^\alpha P_n), & for\ all\quad u, \\ O(n^\alpha P_{[1/u]}), & for\quad u \geqq \dfrac{1}{n}. \end{cases}$$

Proof. We write

$$\begin{aligned} \Gamma(1-\alpha)H(n, u) &= \left\{\int_u^{u+(1/n)} + \int_{u+(1/n)}^{\pi}\right\}(t-u)^{-\alpha}\frac{d}{dt}h(n, t)dt \\ &= H_1 + H_2, \end{aligned}$$

say. If $u \geqq 1/n$, then by Abel's Lemma and Lemma 2,

$$|H_1| \leqq K\int_u^{u+(1/n)}(t-u)^{-\alpha}nP_{[1/t]}dt \leqq Kn^\alpha P_{[1/u]},$$

since $\{P_n\}$ is nondecreasing. Next, we have

$$\begin{aligned} |H_2| &= \left|n^\alpha\int_{u+(1/n)}^{\eta}\frac{d}{dt}h(n, t)dt\right|, \qquad u + \frac{1}{n} < \eta < \pi, \\ &\leqq Kn^\alpha P_{[1/u]}, \end{aligned}$$

by virtue of Lemma 2.

This proves the second part of the lemma. The other part follows by a similar reasoning when one observes that

$$\left|\sum_{k=0}^{n} p_{n-k}k \exp(ikt)\right| \leqq KnP_n.$$

LEMMA 4. *If $\{p_n\}$ is a positive monotonic nonincreasing sequence, then*

$$|V(n, u)| = \begin{cases} O(n^\alpha u^\alpha P_n) & for\ all\quad u; \\ O(n^\alpha) + O(n^\alpha u^\alpha P_{[1/u]}) & for\quad u \geqq \dfrac{1}{n}. \end{cases}$$

For the proof of Lemma 4, reference may be made to ([9], p. 265).

5. **Proof of Theorem 1.** (I) $|F_\alpha|$ − *effectiveness*: We have

$$nA_n(x) = \frac{2}{\pi}\int_0^\pi \phi(t)\frac{d}{dt}\sin ntdt$$

and

$$\sigma_n(L^0(x)) = \int_0^\pi \phi(t)\frac{d}{dt}g(n, t)dt.$$

As in ([1], proof of Theorem 1), on integration by parts, we get

$$\sigma_n(L^0(x)) = \Phi_\alpha(\pi)J(n, \pi) - \phi_\alpha(\pi)V(n, \pi) + \int_0^\pi V(n, u)d\phi_\alpha(u) .$$

Thus, by Lemma 3

$$\sigma_n(L^0(x)) = O(n^\alpha) - \phi_\alpha(\pi)V(n, \pi) + \int_0^\pi V(n, u)d\phi_\alpha(u) .$$

If in particular, we suppose $\phi(t) = 1$ for all t, in which case $\phi_\alpha(t) = 1$ for all t and $\sigma_n(L^0(x)) = 0$ for every n, we obtain

$$0 = O(n^\alpha) - \phi_\alpha(\pi)V(n, \pi)$$

and therefore

$$\sigma_n(L^0(x)) = O(n^\alpha) + \int_0^\pi V(n, u)d\phi_\alpha(u) .$$

Thus,

$$\sum_{n=1}^\infty \frac{1}{nP_n}|\sigma_n(L^0(x))|$$

$$\leqq K\sum_{n=1}^\infty \frac{1}{n^{1-\alpha}P_n} + K\int_0^\pi \sum_{n=1}^\infty \frac{1}{nP_n}|V(n, u)||d\phi_\alpha(u)| \leqq K ,$$

since by hypothesis $\int_0^\pi |d\phi_\alpha(u)| \leqq K$ and by Lemma 4,

$$\sum_{n=1}^\infty \frac{1}{nP_n}|V(n, u)| \leqq Ku^\alpha \sum_{n\leqq 1/u} n^{\alpha-1} + K\{1 + u^\alpha P_{[1/u]}\}\sum_{n>1/u}\frac{1}{n^{1-\alpha}P_n} \leqq K ,$$

by virtue of the hypothesis that $\{p_n\} \in M_\alpha$.

This completes the proof of $|F_\alpha| -$ *effective* part of Theorem 1, when one appeals to Lemma 1.

(II) $|\tilde{F}_\alpha| -$ *effectiveness*: We have

$$nB_n(x) = -\frac{2}{\pi}\int_0^\pi \psi(t)\frac{d}{dt}\cos ntdt$$

and therefore

$$\sigma_n(\tilde{L}^0(x)) = -\int_0^\pi \psi(t)\frac{d}{dt}\tilde{g}(n, t)dt .$$

As in ([2], proof of Theorem 2), we have

$$\begin{aligned}\sigma_n(\tilde{L}^0(x)) &= -\int_0^\pi \frac{d}{dt}\tilde{g}(n, t)\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}d\Psi_\alpha(u)\right\}dt \\ &= -\frac{1}{\Gamma(1-\alpha)}\int_0^\pi d\Psi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}\tilde{g}(n, t)dt \\ &= -\int_0^\pi \tilde{J}(n, u)d\Psi_\alpha(u) .\end{aligned}$$

$|\widetilde{F}_\alpha|$ — *effectiveness* of the (N, p) mean now follows from Lemma 1 and the hypothesis that $\int_0^\pi u^{-\alpha}|d\Psi_\alpha(u)| \leqq K$, when we observe that uniformly in $0 < u \leqq \pi$

$$u^\alpha \sum_{n=1}^\infty \frac{1}{nP_n} |\widetilde{J}(n, u)| \leqq Ku^\alpha \sum_{n \leqq 1/u} n^{\alpha-1} + Ku^\alpha P_{[1/u]} \sum_{n>1/u} \frac{1}{n^{1-\alpha}P_n} \leqq K ,$$

by virtue of Lemma 3 and the hypothesis that $\{p_n\} \in M_\alpha$.

(III) $|F^\alpha|$ — *effectiveness*: Integrating by parts, we have

$$n^{\alpha+1}A_n(x) = \frac{2}{\pi}\int_0^\pi \phi(t)n^{\alpha+1}\cos ntdt = -\frac{2}{\pi}\int_0^\pi n^\alpha \sin ntd\phi(t) .$$

Thus

$$\sum_{n=1}^\infty \frac{1}{nP_n}|\sigma_n(L^\alpha(x))| \leqq \int_0^\pi \left\{\sum_{n=1}^\infty \frac{1}{nP_n}\left|\sum_{k=0}^n p_{n-k}k^\alpha \sin kt\right|\right\}|d\phi(t)| .$$

$|F^\alpha|$ — *effectiveness*, of the (N, p) mean now follows from Lemma 1 and the hypothesis that $\int_0^\pi t^{-\alpha}|d\phi(t)| \leqq K$, when one observes that uniformly in $0 < t \leqq \pi$

$$\begin{aligned} & t^\alpha \sum_{n=1}^\infty \frac{1}{nP_n}\left|\sum_{k=0}^n p_{n-k}k^\alpha \sin kt\right| \\ & \leqq Kt^{\alpha+1}\sum_{n\leqq 1/t} n^\alpha + Kt^\alpha \sum_{n>1/t}\frac{1}{n^{1-\alpha}P_n}\max_{0\leqq\nu\leqq n}\left|\sum_{k=0}^{\nu} p_k \sin(n-k)t\right| \\ & \leqq K + Kt^\alpha P_{[1/t]}\sum_{n>1/t}\frac{1}{n^{1-\alpha}P_n} \leqq K , \end{aligned}$$

by Abel's lemma, Lemma 2 and the hypothesis that $\{p_n\} \in M_\alpha$.

(IV) $|\widetilde{F}^\alpha|$ — *effectiveness*: Integrating by parts and observing that $\psi(+0) = 0$, we have

$$\begin{aligned} n^{\alpha+1}B_n(x) &= \frac{2}{\pi}\int_0^\pi \psi(t)n^{\alpha+1}\sin ntdt \\ &= -\frac{2}{\pi}\psi(\pi)n^\alpha \cos n\pi + \frac{2}{\pi}\int_0^\pi n^\alpha \cos ntd\psi(t) . \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{nP_n}|\sigma_n(\widetilde{L}^\alpha(x))| \leqq |\psi(\pi)| \sum_{n=1}^\infty \frac{1}{nP_n}\left|\sum_{k=0}^n p_{n-k}k^\alpha \cos k\pi\right| \\ + \int_0^\pi \left\{\sum_{n=1}^\infty \frac{1}{nP_n}\left|\sum_{k=0}^n p_{n-k}k^\alpha \cos kt\right|\right\}|d\psi(t)| . \end{aligned}$$

$|\widetilde{F}^\alpha|$ — *effectiveness* now follows from Lemma 1 and the hypothesis that $\int_0^\pi t^{-\alpha}|d\psi(t)| \leqq K$, when one observes that uniformly in $0 < t \leqq \pi$

$$\begin{aligned} & t^\alpha \sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{k=0}^{n} p_{n-k} k^\alpha \cos kt \right| \\ & \leqq Kt^\alpha \sum_{n \leqq 1/t} n^{\alpha-1} + Kt^\alpha \sum_{n>1/t} \frac{1}{n^{1-\alpha}P_n} \max_{0 \leqq \nu \leqq n} \left| \sum_{k=0}^{\nu} p_k \cos (n-k)t \right| \\ & \leqq K + Kt^\alpha P_{[1/t]} \sum_{n>1/t} \frac{1}{n^{1-\alpha}P_n} \leqq K , \end{aligned}$$

by Lemma 2 and the hypothesis that $\{p_n\} \in M_\alpha$.

(V) $|F^*|$ — *effectiveness*: This follows from the result of Theorem 3, when one observes that $\{p_n\} \in M_\alpha$, $\alpha < 0$, implies $\{p_n\} \in M_0$.

This completes the proof of Theorem 1.

6. Proof of Theorem 2. It may be observed that the proof of $|F^\alpha|$ — *effectiveness*, given in the preceeding section remains valid even for the case $\alpha = 0$ and therefore the (N, p) method is $|F^0|$ or equivalently $|F_0|$ — *effective*.

In order to prove $|\tilde{F}_0|$ — *effectiveness*, we observe that on integration by parts we get

$$\begin{aligned} B_n(x) &= \frac{2}{\pi}\int_0^\pi \psi(t) \sin ntdt = -\frac{2}{\pi}\int_0^\pi \frac{1-\cos nt}{n} d\psi(t) \\ &= \frac{2}{\pi}\int_0^\pi \frac{\cos nt}{n} d\psi(t) , \end{aligned}$$

since $\psi(\pi) = \psi(0) = 0$.

Thus, we have (cf. [13])

$$\frac{\pi}{2}\sigma_n(\tilde{L}^0(x)) = \begin{cases} -\int_0^\pi \left\{ \sum_{k=0}^{n} p_{n-k}(1-\cos kt) \right\} d\psi(t) , \\ \text{or} \\ \int_0^\pi \left\{ \sum_{k=0}^{n} p_{n-k} \cos kt \right\} d\psi(t) , \end{cases}$$

and

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{nP_n} |\sigma_n(\tilde{L}^0(x))| \\ & \leqq \int_0^\pi |d\psi(t)| \left\{ \sum_{n \leqq 1/t} \frac{1}{nP_n} \left| \sum_{k=0}^{n} p_{n-k}(1-\cos kt) \right| + \sum_{n>1/t} \frac{1}{nP_n} \left| \sum_{k=0}^{n} p_{n-k} \cos kt \right| \right\} \\ & = \int_0^\pi |d\psi(t)| \{\Sigma_1 + \Sigma_2\} , \end{aligned}$$

say. $|\tilde{F}_0|$ — *effectiveness* of the (N, p) method now follows from Lemma 1 and the hypothesis that $\int_0^\pi |d\psi(t)| \leqq K$, when one observes that uniformly in $0 < t \leqq \pi$

$$\Sigma_1 \leqq Kt^2 \sum_{n \leqq 1/t} n \leqq K ,$$

since $|1 - \cos kt| \leqq k^2 t^2$ and

$$\Sigma_2 \leqq KP_{[1/t]} \sum_{n > 1/t} \frac{1}{nP_n} \leqq K ,$$

by virtue of Lemma 2 and the hypothesis that $\{p_n\} \in M_0$.
This completes the proof of Theorem 2 (cf. [7]).

7. **Proof of Theorem 3.** We have ([14], § 13.2)

$$nA_n^*(x) = \frac{1}{\pi}\int_0^\pi \phi^*(t)\left\{\sin\left(n + \frac{1}{2}\right)t/\sin\frac{1}{2}t\right\}dt .$$

Thus

$$\sum_{n=1}^\infty \frac{1}{nP_n} | \sigma_n(L^*(x)) |$$
$$\leqq \int_0^\pi \left\{ \sum_{n=1}^\infty \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \sin\left(k + \frac{1}{2}\right)t \right| \right\} \frac{|\phi^*(t)|}{t} \frac{t}{\sin\frac{1}{2}t} dt .$$

$|F^*|$ – *effectiveness* of the (N, p) method now follows from Lemma 1 and the hypothesis that $\int_0^\pi t^{-1} |\phi^*(t)| dt \leqq K$, when we observe that uniformly in $0 < t \leqq \pi$

$$\sum_{n=1}^\infty \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \sin\left(k + \frac{1}{2}\right)t \right|$$
$$\leqq Kt \sum_{n \leqq 1/t} 1 + KP_{[1/t]} \sum_{n > 1/t} \frac{1}{nP_n} \leqq K ,$$

by virtue of Lemma 2 and the hypothesis that $\{p_n\} \in M_0$.
This completes the proof of Theorem 3.

8. **Remarks.** Corresponding to our Theorem 2, we have an earlier result of Hille and Tamarkin ([8], Th. II) for ordinary (N, p) summability of $L^0(x)$ which states that under certain condition on $\phi(t)$ the hypothesis

$$\left\{ \frac{1}{P_n} \sum_{k=0}^n \frac{P_k}{k+1} \right\} \in B \tag{8.1}$$

is both necessary and sufficient for (N, p) summability of $L^0(x)$, if $\{p_n\}$ is a positive monotonic nonincreasing sequence. The intrinsic character of the hypothesis $\{p_n\} \in M_0$ of Theorem 2, emerges from the above result when one observes that the condition (8.1) implies that

$$P_n \sum_{k=n}^{\infty} \frac{1}{kP_k} \leqq K , \qquad n = 1, 2, \cdots .$$

This follows from a recent paper of the author ([4], p. 168).

The claim that the corresponding (C) method results, reduce to special cases of our theorems, follows when we observe that

$$\left\{ \binom{n+\beta-1}{\beta-1} \right\} \in M_\alpha, \; 1 \geqq \beta > \alpha \geqq 0 \, ,$$

and appeal to a well known inclusion relation for the absolute (C)-method.

Recently $|F_1|$ – *effectiveness* of $(N, p)(C, 1)$ and $(C, 1)(N, p)$ methods have been proved by the present author.

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