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ON EMBEDDINGS OF 1-DIMENSIONAL COMPACTA IN A HYPERPLANE IN E^4

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In this note a proof of the following theorem is given.

THEOREM 1. Suppose that X is a 1-dimensional compactum in a 3-dimensional hyperplane E^3 in euclidean 4-space E^4, that $\varepsilon > 0$, and that $f: X \to E^3$ is an embedding such that $d(x, f(x)) < \varepsilon$ for each $x \in X$. Then there exists an ε-push h of (E^4, X) such that $h|X = f$.

The proof of Theorem 1 is based on a technique exploited by the first author in [3]. This method requires that one be able to push X off of the 2-skeleton of an arbitrary triangulation of E^4 using a small push of E^4. This could be done very easily if it were possible to push X off of the 1-skeleton of a given triangulation of E^3 via a small push of E^3. Unfortunately, this cannot be accomplished unless X has some additional property (such as local contractibility) as demonstrated by the examples of Bothe [2] and McMillan and Row [9]. However, we are able to overcome this difficulty by using a property of twisted spun knots obtained by Zeeman [10].

In the following theorem let B^4 denote the unit ball in E^4, B^3 the intersection of B^4 with the 3-plane $x_4 = 0$, and D^2 the intersection of B^4 with the 2-plane $x_1 = x_2 = 0$.

THEOREM 2. *Let X be a 1-dimensional compactum in B^3 such that $X \cap \operatorname{Bd} D^2 = \emptyset$. Then there exists an isotopy $h_t: B^4 \to B^4$ $(t \in [0, 1])$ such that*

(i) $h_0 =$ *identity,*
(ii) $h_t | \operatorname{Bd} B^4 =$ *identity for each* $t \in [0, 1]$, *and*
(iii) $h_1(X) \cap D^2 = \emptyset$.

Proof. Let $I = D^2 \cap B^3$. Since X does not separate B^3, there exists a polygonal arc J in $B^3 - X$ joining one endpoint of I to the other. We may assume, by applying an appropriate isotopy of B^4, that J_+, the intersection of J with the half-space $x_3 \geqq 0$ is contained in I. Let F be a 3-cell in B^3 such that $F \cap J = J_+$ and $F \cap X = \emptyset$, and let J_- be the intersection of J with the half-space $x_3 \leqq 0$. Now spin the arc J_- about the plane $x_3 = x_4 = 0$, twisting once, so that at time $t = \pi$, J_- lies in F. (See Zeeman [10] for the details of this construction.) Observe that the boundary of the 2-cell C traced out by J_- is the same as $\operatorname{Bd} D^2$.

It follows from [10, Corollary 2] that the pair (B^4, C) is equivalent to the pair (B^4, D^2) by an isotopy that keeps $\operatorname{Bd} B^4$ fixed. Such an isotopy, of course, will push X off of D^2.

THEOREM 3. *Let X be a 1-dimensional compactum in a 3-plane E^3 in E^4. Then for each 2-complex K in E^4 and each $\varepsilon > 0$, there exists an ε-push h of (E^4, X) such that $h(X) \cap K = \emptyset$.*

Proof. Given a 2-complex K and $\varepsilon > 0$, we may assume first of all that none of the vertices of K lies in E^3. Also, we may move the 1-simplexes of K slightly so that they do not meet X.

Let σ be a 2-simplex of K such that $\sigma \cap X \neq \emptyset$. By moving X an arbitrarily small amount, keeping it in E^3, we can ensure that each component of $\sigma \cap X$ not only lies in $\operatorname{Int} \sigma$, but has diameter less than ε. Hence, we can get $\sigma \cap X$ into a finite number of mutually exclusive line segments $I_1, \cdots, I_n$ in $\operatorname{Int} \sigma \cap E^3$, each of which having diameter less than ε. Let $B_1, \cdots, B_n$ be a collection of mutually exclusive 4-cells in E^4, each of diameter less than ε, such that each triple $(B_j, B_j \cap E^3, B_j \cap \sigma)$ is equivalent to the triple (B^4, B^3, D^2) (as defined above) and such that $B_j \cap \sigma \cap E^3 = I_j$. Now apply Theorem 2 to each of the $B_j (j = 1, \cdots, n)$.

LEMMA. *Suppose that $X \subset E^3 \subset E^4$ and $f: X \to E^3$ are as in the statement of Theorem 1 with $d(x, f(x)) < \varepsilon$ for each $x \in X$. Then for each $\delta > 0$ there exists an ε-push h of (E^4, X) such that $d(h(x)), f(x)) < \delta$ for each $x \in X$.*

Proof. Apply the proof of Lemma 2 of [3] with $p = 2$ and $q = 1$.

The proof of Theorem 1 is now obtained by applying the technique employed in the proof of Theorem 4.4 of [7]. The only additional observation that should be made is that if X is a compactum in E^4 satisfying the conclusion of Theorem 3 and if g is a homeomorphism of E^4, then $g(X)$ also satisfies the conclusion of Theorem 3 with respect to 2-complexes in the piecewise linear structure on E^4 induced by g.

COROLLARY. *Let X be a 1-dimensional compactum in a 3-hyperplane in E^4. Then for each $\varepsilon > 0$ there exists a neighborhood of X in E^4 that ε-collapses to a 1-dimensional polyhedron.*

This follows from the fact that every 1-dimensional compactum can be embedded in E^3 so as to have this property in E^3.

Bothe [2] and McMillan and Row [9] have examples which show that not every embedding of the Menger universal curve in E^3 has small neighborhoods with 1-spines.

REMARK 1. Notice that Theorem 1 is a consequence of a special case of a theorem of Bing and Kister [1] if X is either a 1-dimensional polyhedron or a 0-dimensional compactum. If X is a 2-dimensional polyhedron, then Theorem 1 is false in general as pointed out by Gillman [6]. It would be interesting to known for what 2-dimensional compacta Theorem 1 holds. For example, this theorem is true if X is a compact 2-manifold [5].

REMARK 2. One of the important properties of a compactum X in a hyperplane in E^n is that $E^n - X$ is 1-ALG (see [8]). If $n - \dim X \geqq 3$, this is equivalent to saying that $E^n - X$ is 1-ULC. In [3] and [4] it is shown that any two such embeddings of X into E^n (regardless of whether they lie in a hyperplane) are equivalent, provided $n \geqq 5$ and $2 \dim X + 2 \leqq n$. Although there is no hope of improving this theorem by lowering the codimension of the embedding (at least for arbitrary compacta), Theorem 1 lends credence to the conjecture that this result holds when $n = 4$.

REFERENCES

1. R. H. Bing and J. M. Kister, *Taming complexes in hyperplanes*, Duke Math. J. **31** (1964), 491-512.
2. H. G. Bothe, *Ein eindimensionales Kompactum im E^3, das sich nicht lagetreu im die Mengersche Universalkurve einbetten lässt*, Fund. Math. **54** (1964), 251-258.
3. J. L. Bryant, *On embeddings of compacta in euclidean space*, Proc. Amer. Math. Soc. **23** (1969), 46-51.
4. ———, *On embeddings of 1-dimensional compacta in E^5*, Duke Math. J. (to appear)
5. D. Gillman, *Unknotting 2-manifolds in 3-hyperplanes in E^4*, Duke Math. J. **33** (1966), 229-245.
6. ———, *The spinning and twisting of a complex in a hyperplane*, Ann. of Math. (2) **85** (1967), 32-41.
7. H. Gluck, *Embeddings in the trivial range*, Ann. of Math. (2) **81** (1965), 195-210.
8. O. G. Harrold, Jr., *Euclidean domains with uniformly abelian local fundamental groups*, Trans. Amer. Math. Soc. **67** (1949), 120-129.
9. D. R. McMillan, Jr. and H. Row, *Tangled embeddings of one-dimensional continua*, Proc. Amer. Math. Soc. **22** (1969), 378-385.
10. E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471-495.

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