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APPROXIMATE UNITS AND MAXIMAL ABELIAN C^*-SUBALGEBRAS

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In recent years it has become clear that the study of C^*-algebras without a unit element is more than just a mildly interesting extension of the "typical" case of a C^*-algebra with unit. A number of important examples of C^*-algebras rarely have a unit, for example the group C^*-algebras and algebras of the form $I \cap I^*$ where I is a closed left ideal of a C^*-algebra. J. Dixmier's book, *Les C^*-algebras et leurs representations,* carries through all the basic theory of C^*-algebras for the no-unit case, and his main tool is the approximate identity which such algebras have. Many C^*-algebra questions can be answered for a C^*-algebra without unit by embedding such an algebra in a C^*-algebra with unit. Some problems, especially those which involve approximate units, are not susceptible to this approach. This paper will study some problems of this type.

Theorem 1.1 states that if $\mathscr{U}$ is a norm separable C^*-algebra and $\{f_1, \cdots f_n\}$ is a finite set of orthogonal pure states of $\mathscr{U}$ (i.e., $||f_i - f_j|| = 2$ if $i \neq j$), then there exists a maximal abelian C^*-subalgebra A of $\mathscr{U}$ such that $f_k \mid A$ is pure $(k = 1, \cdots, n)$ and $f_k \mid A$ has unique pure state extension to $\mathscr{U}$ $(k = 1, \cdots, n)$. This extends the prototype result of Aarnes and Kadison by (a) allowing a finite number of pure states instead of just one, (b) dropping the assumption that $1 \in \mathscr{U}$, and (c) proving uniqueness of the pure state extension. In § 2 two examples are constructed which show that the uniqueness assertion of Theorem 1.1 cannot be extended to the nonseparable case, and that even in the separable case the subalgebra A must be carefully chosen to insure uniqueness of pure state extension. Theorem 1.2 and Example 2.3 show that a very desirable majorization property of approximate units does not quite carry over from the abelian case to the general case. (If it did, several important problems, including the Stone-Weierstrass problem would have been solved.) Theorem 1.3 extends the author' characterization of approximate units of C^*-algebras to approximate right units for left ideals of C^*-algebras.

1. Theorems.

THEOREM 1.1 *Let $\mathscr{U}$ be a norm separable C^*-algebra and $\{f_1, \cdots, f_n\}$ a finite set of orthogonal pure states (i.e., $||f_j - f_i| = 2$ if $i \neq j$) of $\mathscr{U}$. Then there exists a maximal abelian C^*-subalgebra*

$A \subset \mathscr{U}$ such that $f_k \mid A$ is pure $(k = 1, \cdots, n)$, and f_k is the unique pure state of $\mathscr{U}$ which extends $f_k \mid A$ $(k = 1, \cdots, n)$.

Proof. We shall consider $\mathscr{U}$ acting on H under its universal representation (the direct sum of all the cyclic representations due to positive linear functionals on $\mathscr{U}$ [5, p. 43]). Recall [5, p. 236] that the weak closure of $\mathscr{U}$ can be identified with the second dual $\mathscr{U}^{**}$ of $\mathscr{U}$. Let $x_k \in H_{f_k} \subset H$ be a unit vector such that for any $a \in \mathscr{U}$ we have $\langle ax_k, x_k \rangle = f_k(a)$. By [6, Corollary 7] we may choose a positive operator $b \in \mathscr{U}$ with $\| b \| \leqq 1$ and $bx_k = (k/n)x_k$ for $k = 1, \cdots, n$. Choose real-valued functions $\{\varphi_k\}_{k=1}^n$ of a real variable with the following properties: (1) $\varphi_k(t)\varphi_j(t) = 0$ if $k \neq j$, (2) $\varphi_k(k/n) = 1$, (3) $0 \leqq \varphi_k(t) \leqq 1$, (4) φ_k is continuous; where t is any real number and $k, j = 1, \cdots, n$.

Define $b_k = \varphi_k(b)$ by the spectral theorem. Let p_k be the range projection of b_k, and let p be the orthogonal projection on the subspace spanned by $\{x_1, \cdots, x_n\}$. Using the terminology of [3] we see that $(1 - p)$ and $\{p_k\}_{k=1}^n$ are open projections for $\mathscr{U}$, and hence for $\tilde{\mathscr{U}}$, the C^*-algebra formed from $\mathscr{U}$ by adjoining the unit 1. Note that p commutes with b and hence with each p_k. Set

$$I_k = \{a \in \mathscr{U}: p_k(1 - p)a(1 - p)p_k = a\}$$

for each $k = 1, \cdots, n$. Since $\mathscr{U}$ is norm separable, we may choose a strictly positive element $a_k \in I_k$ with $\| a_k \| = 1$ by [1]. (I.e., for any positive linear functional h on the C^*-algebra I_k, $h(a_k) = 0$ implies $h = 0$.) By [3, II. 7] $p_k(1 - p)$ is open, hence $1 - p_k(1 - p)$ is the null projection of a_k for each $k = 1, \cdots, n$.

Now we define $c_k = b_k - b_k a_k b_k$ for $k = 1, \cdots, n$. Each c_k is self-adjoint and $c_k x_j = \delta_{kj} x_j$ (Kronecker Delta). Also $c_k c_j = 0$ if $j \neq k$. Define $I_0 = \{a \in \mathscr{U}:(1 - p)a(1 - p) = a\}$. Then let $\Gamma_0 = \{a \in I_0: c_k a = ac_k$ for all $k = 1, \cdots, n\}$. Let Γ_1 be the C^*-algebra generated by $\{c_k\}_{k=1}^n$ and Γ_0, and let A be a maximal abelian C^*-subalgebra of Γ_1 containing $\{c_k\}_{k=1}^n$. It is clear that $f_k \mid A$ is multiplicative, hence pure, on A, since $f_k \mid I_0 = 0$ for each $k = 1, \cdots, n$. We shall show that A is a maximal abelian C^*-subalgebra of $\mathscr{U}$.

Suppose $c \in \mathscr{U}$ with $ca = ac$ for all $a \in A$. We may suppose $c \geqq 0$ since $\{d \in \mathscr{U}: da = ad$ for all $a \in A\}$ is a C^*-algebra, hence generated by its positive elements. For each $k = 1, \cdots, n$, we define a scalar λ_k as follows. If $cx_k = 0$, let $\lambda_k = 0$. If $cx_k \neq 0$, let $y = cx_k/\| cx_k \|$. Since $cc_k = c_k c$, we get $c_k(cx_k) = c(c_k x_k) = cx_k$, so $c_k y = y$. Thus $1 = \langle c_k y, y \rangle = \langle b_k y, y \rangle - \langle a_k b_k y, b_k y \rangle$. Since $b_k \geqq 0$, $a_k \geqq 0$, $\| b_k \| \leqq 1$, this means $b_k y = y$ and $a_k b_k y = 0 = a_k y$. By the above, this means $(1 - p_k(1 - p))y = y$, since $(1 - p_k(1 - p))$ is the null projection of a_k.

But $b_k y = y$ implies $y = p_k y$. Hence $y = p_k p y$. This means that $y = \lambda_k x_k$, since $p_k p$ is the projection on the one dimensional subspace spanned by x_k. Since the foregoing was valid for each $k = 1, \cdots, n$, we can define $d = c - \sum_{k=1}^{n} \lambda_k c_k$, and note that $(1 - p)d(1 - p) = d$, so that $d \in I_0$. But $da = ad$ for all $a \in A$, so $d \in \Gamma_0$, hence $d \in A$ by maximality of A. This proves that A is a maximal abelian C^*-subalgebra of $\mathscr{U}$.

Lastly, we shall prove the uniqueness assertion. Suppose g is a pure state of $\mathscr{U}$ with $g \mid A = f_k \mid A$. Then there exists a unit vector $y \in H_g \subset H$ such that $\langle ay, y\rangle = g(y)$ for all $a \in \mathscr{U}$. Thus $\langle c_k y, y\rangle = \langle c_k x_k, x_k\rangle = 1 = \langle b_k y, y\rangle - \langle a_k b_k y, y\rangle$. Using the same argument as above, we get $b_k y = y$ and $a_k y = 0$. As before, this implies $y = \lambda x_k$, so $g = f_k$.

One is tempted to ask if the hypothesis of separability can be dropped. Example 2.1 shows that we cannot hope for unique state extensions in the (very) nonseparable case. However, the question of the existence of the maximal abelian C^*-subalgebra is still open. For the separable case one might ask if a pure state of a maximal abelian C^*-subalgebra always has unique pure state extension to the whole algebra. Example 2.2 shows that this is not the case.

We now turn to a quite different problem, that of majorizing an element of a C^*-algebra by an element of a subalgebra. If $\mathscr{U}$ is an abelian C^*-algebra and $B \subset \mathscr{U}$ a C^*-subalgebra which contains a positive increasing approximate unit for all of $\mathscr{U}$, then for every $a \geqq 0$ in $\mathscr{U}$ there is $b \in B$ with $b \geqq a$. In fact, with a slightly more refined argument one can choose b so that $\| b \| = \| a \|$. Example 2.3 shows that this last assertion is (in general) false if the hypothesis of commutativity is dropped. The following result shows how close we can come to the abelian case.

THEOREM 1.2 *Let A be a C^*-algebra and A_0 a C^*-subalgebra of A which contains a positive, increasing approximate unit $\{a_\alpha\}_{\alpha \in I}$ for A. Then given $a_0 \geqq 0$ in A and $\varepsilon > 0$, there exists $b \geqq 0$ in A_0 with $b \geqq a_0$ and $\| b \| \leqq \| a_0 \| + \varepsilon$.*

Proof. First let us note that if the theorem is true for all $a_0 \in A$ with $\| a_0 \| = 1$, then it is true for all $a_0 \in A$. Thus we may assume $\| a_0 \| = 1$. Given $\varepsilon > 0$ we may choose $\alpha_1 \in I$ such that $\| a_0 - a_{\alpha_1} a_0 a_{\alpha_1} \| < \varepsilon/2$. Since $\{a_\alpha\}$ is an increasing positive approximate identity and $\| a_0 \| = 1$, $a_{\alpha_1} \geqq a_{\alpha_1} a_0 a_{\alpha_1}$. Set $a_1 = a_0 - a_{\alpha_1} a_0 a_{\alpha_1}$. Then find $\alpha_2 \in I$ such that $\| a_1 - a_{\alpha_2} a_1 a_{\alpha_2} \| < \varepsilon/4$. Now $\| (2/\varepsilon) a_1 \| \leqq 1$, so as above

$$a_{\alpha_2} \geqq a_{\alpha_2}^2 \geqq a_{\alpha_2}\left(\frac{2}{\varepsilon}a_1\right)a_{\alpha_2} .$$

Thus $(\varepsilon/2)a_{\alpha_2} \geqq a_{\alpha_2}a_1a_{\alpha_2}$. Set $a_2 = a_1 - a_{\alpha_2}a_1a_{\alpha_2}$, and continue by induction to get sequences $\{\alpha_n\}_{n=1}^{\infty} \subset I$ and $\{a_n\}_{n=0}^{\infty} \subset A$ so that $a_n = a_{n-1} - a_{\alpha_n}a_{n-1}a_{\alpha_n}$ for $n > 0$ (a_0 as given), $\| a_n \| \leqq \varepsilon/2^n$ for $n > 0$, and

$$\left(\frac{\varepsilon}{2^{n-1}}\right)a_{\alpha_n} \geqq a_{\alpha_n}a_{n-1}a_{\alpha_n} .$$

Thus the series $\sum_{n=1}^{\infty} a_{\alpha_n}a_{n-1}a_{\alpha_n}$ is absolutely convergent to a_0. Also

$$a_0 = \sum_{n=1}^{\infty} a_{\alpha_n}a_{n-1}a_{\alpha_n} \leqq a_{\alpha_1} + \sum_{n=2}^{\infty}\left(\frac{\varepsilon}{2^{n-1}}\right)a_{\alpha_n} ,$$

the right hand side also converging absolutely to an element $b \in A_0$. Clearly

$$\| b \| \leqq \| a_{\alpha_1} \| + \sum_{n=2}^{\infty}\left\|\left(\frac{\varepsilon}{2^{n-1}}\right)a_{\alpha_n}\right\| \leqq 1 + \varepsilon ,$$

so we have the theorem.

In order to state the last result of this section we introduce a definition.

DEFINITION. We say a pure state f of $\mathscr{U}$ is *pure on a closed left ideal* I if f is pure on the C^*-algebra $I \cap I^*$.

THEOREM 1.3 *Let I be a closed left ideal of a C^*-algebra $\mathscr{U}$. An increasing directed net of positive operators $\{a_\alpha\} \subset I$ is an approximate right unit for I if $f(a_\alpha) \to 1$ for every pure state f of $\mathscr{U}$ which is pure on I.*

Proof. Let $\{a_\alpha\} \subset I$ be a positive increasing directed net in I with $f(a_\alpha) \to 1$ every pure state f of $\mathscr{U}$ which is pure on I. Now set $I_0 = I \cap I^*$, and note that $\{a_\alpha\} \subset I_0$. By [4, 5.1] $\{a_\alpha\}$ is an approximate unit for I_0, since every pure state of I_0 has (by [7]) a pure state extension to $\mathscr{U}$. Let $b \in I$. Then

$$\begin{aligned} \| b - ba_\alpha \|^2 &= \| (b^* - a_\alpha b^*)(b - ba_\alpha) \| \\ &= \| b^*b - b^*ba_\alpha - a_\alpha b^*b + a_\alpha b^*ba_\alpha \|_\alpha \to 0 \end{aligned}$$

since $b^*b \in I_0$ and $\{a_\alpha\}$ is an approximate unit for I_0. Thus $\| \mathrm{b} - ba_\alpha \|_\alpha \to 0$.

2. Examples. This first example will be a nonseparable C^*-algebra A which has no positive increasing approximate unit $\{a_\alpha\}$ consisting of pairwise commuting elements. That is, no maximal abelian C^*-subalgebra of A contains an approximate unit for A. By considering $\tilde{A}$ the C^*-algebra obtained from A by adjoining a unit, we get a pure state f on $\tilde{A}$ with $f \mid A = 0$ (considering $A \subset \tilde{A}$) which cannot be the unique pure state extension of any pure state of any maximal abelian C^*-subalgebra of $\tilde{A}$.

EXAMPLE 2.1 Let Γ be an index set of cardinality 2^c, where c is the cardinality of the set of real numbers. Let H be a Hilbert space with an orthonormal basis of cardinality 2^{2^c}. Choose a family $\{H_\gamma\}_{\gamma \in \Gamma}$ of orthogonal subspaces of H with $\dim(H_\gamma) = \dim(H)$ for all $\gamma \ni \Gamma$. For each fixed $\gamma \in \Gamma$, choose a family $\{H_\gamma^\alpha\}_{\alpha \in \Gamma}$ of subspace of H_γ which are orthogonal and such that $\dim(H_\gamma^\alpha) = \dim(H)$ for all $\gamma, \alpha \in \Gamma$. Let $H_0 = \Sigma \oplus \{H_\gamma : \gamma \in \Gamma\}$, and we shall work in $B(H_0)$, the algebra of all bounded operators on H_0. For each pair $(\alpha, \gamma) \in \Gamma \times \Gamma$ with $\alpha \neq \gamma$, choose projections $p_{\alpha\gamma}$ and $q_{\alpha\gamma}$ on H_γ^α such that $p_{\alpha\gamma} q_{\alpha\gamma} \neq q_{\alpha\gamma} p_{\alpha\gamma}$. Define for each $\beta \in \Gamma$ a projection $p_\beta \in B(H_0)$ by defining p_β on each subspace H_γ^α as follows: $p_\beta \mid H_\gamma^\alpha = 0$ if $\alpha \neq \beta$ and $\gamma \neq \beta$, $p_\beta \mid H_\gamma^\alpha = 1_{\beta\beta}$ if $\alpha = \beta = \gamma$, $p_\beta \mid H_\beta^\alpha = p_{\alpha\beta}$ if $\alpha \neq \beta$, and $p_\beta \mid H_\alpha^\beta = q_{\beta\alpha}$ if $\beta \neq \alpha$.

Now we let A be the C^*-algebra generated by $\{p_\gamma\}_{\gamma \in \Gamma}$. We note the following facts.

(1) $p_\gamma p_\alpha \neq p_\alpha p_\gamma$ unless $\gamma = \alpha$.

(2) If $1_{\alpha\gamma}$ denotes the projection on H_γ^α, then $1_\gamma^\alpha \in A'$, the commutant of A in $B(H_0)$, and $1_\gamma^\alpha \cdot 1_\rho^\beta = 0$ unless $\alpha = \beta$ and $\gamma = \rho$.

(3) $p_\gamma p_\alpha p_\beta = 0$ if $\gamma \neq \alpha \neq \beta \neq \gamma$.

These are immediate from the definition of the $\{p_\gamma\}$.

Now define the *homomorphism $\varphi_{\alpha\gamma}: A \to B(H_0)$ by $\varphi_{\alpha\gamma}(a) = 1_{\alpha\gamma} a$, and let $A_{\alpha\gamma}$ be the kernel of this homomorphism for each pair $(\alpha, \gamma) \in \Gamma \times \Gamma$. We note that for each $\gamma \in \Gamma$, $A = A_{\gamma\gamma} + \{\lambda p_\gamma : \lambda$ a scalar$\}$. Now suppose $A_1 \subset A$ is a maximal abelian C^*-subalgebra of A and that A_0 contains an approximate identity for all of A. Then for each $\gamma \in \Gamma$, surely $\varphi_{\gamma\gamma}$ is nonzero on A_0, so that A_0 contains a positive element of the form $p_\gamma + a_\gamma$ where $a_\gamma \in A_{\gamma\gamma}$. We shall prove that this implies that A_0 is not abelian.

For each $\gamma \in \Gamma$, we are assuming A_0 contains a positive (fixed) element of the form $p_\gamma + a_\gamma$ with $a_\gamma \in A_{\gamma\gamma}$. Now for each fixed $\gamma \in \Gamma$, a_γ can be written as the sum of a series of products of the $\{p_\alpha\}_{\alpha \in \Gamma}$ with suitable scalar coefficients. This series can be chosen so that every term of the series lies in $A_{\gamma\gamma}$. We shall fix such a series for each γ, and define $F_\gamma = \{\alpha \in \Gamma : p_\alpha$ appears as a factor in one of the terms of the series for $a_\gamma\}$. Since each F_γ is countable and card $(\Gamma) = 2^c$, we may choose a collection $\{F_\gamma : \gamma \in K\}$, where K is an uncountable subset of Γ (any subset K with card $(K) = c$ will work), and $\bigcup \{F_\gamma : \gamma \in K\} \neq \Gamma$. Then for fixed $\beta \in \Gamma \sim \bigcup \{F_\gamma : \gamma \in K)$, p_β does not appear as a factor in any term of the series for a_γ for any $\gamma \in K$. Since $(p_\beta + a_\beta) \in A_0$ for some $a_\beta \in A_{\beta\beta}$ and K is uncountable (and the series for a_β is countable), there exists some $\gamma \in K$ (which we now fix) with $p_\gamma a_\beta 1_{\gamma\beta} = 0 = 1_{\gamma\beta} a_\beta p_\gamma$.

We now show that $(p_\gamma + a_\gamma)$ and $(p_\beta + a_\beta)$ do not commute. We need only check that $1_{\gamma\beta}(p_\alpha + a_\gamma)(p_\beta + a_\beta) \neq 1_{\gamma\beta}(p_\beta + a_\beta)(p_\gamma + a_\gamma)$. Now

$$\begin{aligned} 1_{\gamma\beta}(p_\gamma + a_\gamma)(p_\beta + a_\beta) &= 1_{\gamma\beta}(p_\gamma p_\beta + p_\gamma a_\beta + a_\gamma p_\beta + a_\gamma a_\beta) = 1_{\gamma\beta}(p_\gamma p_\beta) \\ &= q_{\gamma\beta} p_{\gamma\beta} \neq p_{\gamma\beta} q_{\gamma\beta} = 1_{\gamma\beta}(p_\beta p_\gamma) \\ &= 1_{\gamma\beta}(p_\beta + a_\beta)(p_\gamma + a_\gamma) . \end{aligned}$$

This is seen as follows. $1_{\gamma\beta} p_\gamma a_\beta = 0 = 1_{\gamma\beta} a_\beta p_\gamma$ by the choice of γ above. Also $1_{\gamma\beta} a_\gamma a_\beta = 1_{\gamma\beta} a_\beta a_\gamma = 0$ because no term of the series for a_γ contains p_β are a factor, and each term lies in $A_{\gamma\gamma}$. Finally $1_{\gamma\beta} p_\beta a_\gamma = 1_{\gamma\beta} a_\gamma p_\beta = 0$ by the choice of $\beta \in \Gamma \sim \bigcup \{F_\alpha : \alpha \in K\}$. We have proved that A_0 is not abelian, so this contradiction establishes the result that A has no maximal abelian C^*-subalgebra which contains an approximate unit for all of A.

Now if $\tilde{A}$ is the C^*-algebra consisting of A with the identity adjoined, define the pure state f on $\tilde{A}$ by $f(a + \lambda 1) = \lambda$, where $a \in A$ (every element of $\tilde{A}$ can be written uniquely in the form $a + \lambda 1$ with $a \in A$). If $\tilde{A}_0$ is a maximal abelian C^*-subalgebra of $\tilde{A}$, then $\tilde{A}_0 = A_0 + \{\lambda 1\}$, where A_0 is a maximal abelian C^*-subalgebra of A, and A_0 is the kernel of $f \mid \tilde{A}_0$. If $\{a_\alpha\}_{\alpha \in I} \subset A_0$ is a positive increasing approximate unit for A_0, the only pure state of $\tilde{A}_0$ vanishing on all the $a_\alpha (\alpha \in I)$ is $f \mid \tilde{A}_0$. However, it follows immediately from [4, p. 531] and [3, Th. II. 17] that there are at least two pure states of A vanishing on $\{a_\alpha\}$ or else $\{a_\alpha\}$ would be an approximate unit for all of A—contradicting the conclusion above.

In the next example we construct a separable C^*-algebra A with unit and a maximal abelian C^*-subalgebra A_0 of A and two pure states f, g of A with $f = g$ and $f \mid A_0 = g \mid A_0$.

EXAMPLE 2.2 For each $n = 1, 2, \cdots$ let H_n be a two-dimensional Hilbert space with fixed orthonormal basis $\{e_n, e'_n\}_{n=1}^{\infty}$. Let $H = \sum_{n=1}^{\infty} \oplus H_n$. Let C be the C^*-algebra of all operators b on H such that $b(H_n) \subset H_n$ and $\lim_{n\to\infty} \| b \mid H_n \| = 0$. ($C$ is the $C^*(\infty)$ direct sum of the algebras $B(H_n)$.) Let p be a projection on H defined on each H_n by the matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ with respect to the given basis. Finally let A be the C^*-algebra generated by C, p, and 1 (the identity operator on H). Let A_0 be the maximal abelian C^*-subalgebra of A consisting of all operators b in A with $be_n = \lambda_n e_n$, $be'_n = \lambda' e'_n$ for all $n = 1, 2, \cdots$. (A_0 is the algebra of operators in A diagonalized by the given basis.) Now $A/A \cap C$ is two-dimensional so there are two pure states f and g on A with $f \neq g$ and $\ker(f) \supset A \cap C$, $\ker(g) \supset A \cap C$. Since $A_0/A_0 \cap C$ is one dimensional, $f \mid A_0 = g \mid A_0$.

Our last result is an example which proves that one cannot take $\varepsilon = 0$ in Theorem 1.2.

EXAMPLE 2.3 Let H be a separable Hilbert space and $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis for H. Let A be the algebra of all compact operators on H and $A_0 = \{a \in A$: each e_i is an eigenvector of $a\}$. That means A_0 is the algebra of compact diagonal operators for the basis $\{e_n\}$. For each $n = 1, 2, \cdots$ define q_n to be the orthogonal projection on the subspace spanned by $\{e_1, \cdots, e_n\}$. It is known that $\{q_n\}$ is a positive, increasing approximate identity for A. Also A_0 is the C^*-algebra generated by $\{q_n\}$. Thus the theorem applies.

Now let p be the orthogonal projection on the subspace spanned by any vector $x = \sum_{i=1}^{\infty} x_i e_i$, where we assume $x_i \neq 0$ for all i and $\| x \| = 1$. To see that the ε condition of the theorem is necessary, suppose there was some $b \in A_0$, $\| b \| = 1$, $b \geqq p$. Thus

$$\langle bx, x \rangle = \sum_{n=1}^{\infty} \langle be_n, e_n \rangle |x_n|^2 \geqq \langle px, x \rangle = \langle x, x \rangle = \sum_{n=1}^{\infty} |x_n|^2 .$$

This would mean $\langle be_n, e_n \rangle = 1$ for all n, contradicting the compactness of the operator b.

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