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FINITE LINEAR GROUPS OF DEGREE SEVEN II

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The determination of finite groups which can be represented as a group of 7×7 matrices irreducible over the complex numbers is finished in this paper. To simplify the cases, the matrices are assumed unimodular and the groups are primitive. The groups discussed here are essentially simple and have orders $7 \cdot 5^a \cdot 3^b \cdot 2^c$. The theory of groups with a prime to the first power in the group order and of course the representation of degree seven are used heavily in the determination.

This paper is the third in a series of papers discussing linear groups, the first two being [24, 25]. We shall prove the following result.

THEOREM I. *Suppose G has an irreducible complex representation X of degree 7 which is faithful, primitive and unimodular. Suppose, further, that G has an abelian 7-Sylow subgroup. Then, by* [5, 4A], *Z, the center of G, has order 1 or 7 and $G = G_1 \times Z$ for a subgroup G_1 of G. We prove that G_1 is one of the following groups. Let $|G_1|$ be the cardinality of G_1.*

I. $G_1 \cong PSL(2,13)$ $\quad |G_1| = 13 \cdot 7 \cdot 3 \cdot 2^2 = 1092.$

II. $G_1 \cong PSL(2,8)$ $\quad |G_1| = 7 \cdot 3^2 \cdot 2^3 = 504.$

III. $G_1 \cong A_8$ $\quad |G_1| = 8!/2 = 20160.$

IV. $G_1 \cong PSL(2,7)$ $\quad |G_1| = 7 \cdot 3 \cdot 2^3 = 168.$

V. $G_1 \cong U_3(3)$ $\quad |G_1| = 7 \cdot 3^3 \cdot 2^5 = 6048.$

VI. $G_1 \cong S_6(2)$ $\quad |G_1| = 7 \cdot 5 \cdot 3^4 \cdot 2^9 = 1451520.$

VII. *An extension of* III, IV, V *by an automorphism of order* 2 *or an extension of* II *by an automorphism of order* 3. *For* III *it is S_8; for* IV *it is induced by $PGL(2,7)$. For* V *and* II *it is induced by field automorphisms. For V it is $G_2(2)$.*

This result together with [25, Th. 4.1] determines the linear groups of degree 7. The proof is in several parts. Rather than use the notation G_1 we assume G is as stated in the theorem and assume $Z = e$. Set $|G| = g$, the order of G. By [5, 4A] we know $|G| = 7 \cdot g_0$ where $7 \nmid g_0$. Let $|G| = g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. We use the notation of [24, 25]. Thus χ is the character of X. As χ is of zero 7-defect, $\chi(\xi) = 0$ where ξ is an element of order 7 [7, Th. 1]. This means the eigenvalues are all distinct and so by [5, 3F] $C(P) = P$ where P

is a 7-Sylow group. The characters of G satisfy many properties described in [5, § 8]. These are different depending on the value $s = |N(P)/P|$. The possible values are 2, 3, or 6. The value $s = 1$ is impossible as by Burnside's theorem there would be a normal 7-complement contradicting the primitivity of X. The cases 2, 3, 6 are treated separately. The case $s = 6$ is by far the most difficult. It is treated first (§ 2–§ 5) because some of the ideas are used for the case $s = 3$ (§ 7). However much of the treatment for $s = 3$ (§ 7) and all of the treatment for $s = 2$ (§ 6) is independent of the earlier sections and can be read independently.

If there are primes higher than 7 occurring in $g = |G|$, G is case I by [17, 5, 2D]. In the remaining cases we assume $g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. We know by [5, 3E and 25, 2.6] that $a \leqq 6$, $b \leqq 8$, $c \leqq 10$. A flow chart for the order of the elimination is given at the end.

As in [24 or 25], some notation is standard. Thus if K is a subset of G, $C(K)$ and $N(K)$ are the centralizer and normalizer of N. If $\gamma \in G$ we define $N(\gamma) = N(\langle\gamma\rangle)$. Also set $C(K) \cap K = Z(K)$. Let $|K|$ denote the cardinality of K. We have set $|G| = g$. We mention Theorem 2.1 of [25] which says that a 5-Sylow group of G is abelian. We label the principal p-block $B_0(p)$ for any prime p dividing g.

2. Preliminary properties of χ when $s = 6$. We first assume $s = 6$, $G = G'$. By [5, 8A], G is simple. As in [5, § 8] the results of [2] apply to G. There are seven characters χ_i, $i = 0, 1, \cdots, 6$, in $B_0(7)$ of G. Their degrees x_i, $i = 0, 1, \cdots, 6$ are congruent to $\pm 1 \pmod 7$. We set $\chi_i(\xi) = \delta_i$, $i = 0, 1, \cdots, 6$. Here $\delta_i = \pm 1$, ξ is an element of order 7. The degree equation for $B_0(7)$ is $\sum_{i=0}^{6} \delta_i x_i = 0$ [2, Th. 11 or 5, § 8]. We assume χ_0 is the identity character.

If χ is the character of degree 7 corresponding to X we have

$$\chi\bar{\chi} = \chi_0 + \sum_{i=1}^{6} a_i \chi_i + \eta \,.$$

Here η is a sum of irreducible characters of G of zero 7-defect. There must be some i with $a_i \neq 0$ $i = 1, 2, \cdots, 6$ for which $\delta_i = -1$. This is because $\chi(\xi) = 0$ and so $\chi\bar{\chi}(\xi) = 0$. The possible values of x_i are 6, 20, 27, and 48. This means G must have a character, say χ_i, of degree 6, 20, 27, or 48. We will consider each of these cases individually in this and later sections. The case $x_i = 6$ is easily eliminated by considering the restriction of χ to $N(\xi)$. As this analysis will be needed in later sections, we include more than is necessary here.

Let $N = N(\xi)$. We are assuming $|N| = 42 = 7 \cdot 6$. Let τ be an element of order 6 in N. The character table is as follows where $\varepsilon = e^{2\pi i/3}$.

TABLE I. *Character table for $N = N(\zeta)$.*

element	1	ξ	τ	τ^2	τ^3	τ^4	τ^5
ψ_0	1	1	1	1	1	1	1
ψ_1	1	1	$-\varepsilon$	ε^2	-1	ε	$-\varepsilon^2$
ψ_2	1	1	ε^2	ε	1	ε^2	ε
ψ_3	1	1	-1	1	-1	1	-1
ψ_4	1	1	ε	ε^2	1	ε	ε^2
ψ_5	1	1	$-\varepsilon^2$	ε	-1	ε^2	$-\varepsilon$
ψ_6	6	-1	0	0	0	0	0 .

If χ_j is a character of G of degree 6 $\chi_j \mid N = \psi_6$. However the eigenvalues of the representation corresponding to $\psi_6(\tau)$ are 1, -1, $-\varepsilon$, $-\varepsilon^2$, ε, ε^2. This means the determinant is -1. The representation X_j corresponding to χ_j cannot be unimodular and so $G \neq G'$. As we are assuming in this section that $G' = G$ we can assume there are no characters χ_j of degree 6.

For later use we require some further results. The restriction of χ to N must contain ψ_6 and ψ_j $0 \leqq j \leqq 5$ as constituents. In order that $X(\tau)$ be unimodular, $j = 3$. This gives

$$\chi \mid N = \psi_3 + \psi_6 \,. \tag{2.1}$$

Let $P_2(\chi)$ be the character corresponding to the symmetric tensors of rank 2 for X and $C_2(\chi)$ be the character corresponding to the skew symmetric tensors of rank 2. Their restrictions to N are as follows

$$C_2(\chi) \mid N = 3\psi_6 + \psi_1 + \psi_5 + \psi_3 \,, \tag{2.2}$$

$$P_2(\chi) \mid N = 4\psi_6 + 2\psi_0 + \psi_2 + \psi_4 \,. \tag{2.3}$$

We obtain similar results for a character χ_j of degree 8 or 20. Here $\chi_j \mid N$ will have two linear constituents. The character ψ_6 of degree six will appear once if χ_j has degree 8 and three times if χ_j has degree 20. As the representation corresponding to χ_j is unimodular there are three possibilities for the two linear constituents. These are $\psi_0 + \psi_3$, $\psi_1 + \psi_2$, $\psi_4 + \psi_5$. If χ_j has degree 8 this gives

$$\text{(2.4)}\quad \begin{array}{ll} \text{(i)} & \chi_j \mid N = \psi_6 + \psi_0 + \psi_3 \\ \text{(ii)} & \chi_j \mid N = \psi_6 + \psi_1 + \psi_2 \\ \text{(iii)} & \chi_j \mid N = \psi_6 + \psi_4 + \psi_5 \,. \end{array}$$

If χ_j has degree 20 the corresponding result is

$$(2.5)\quad \begin{array}{ll} \text{(i)} & \chi_j \mid N = 3\psi_6 + \psi_0 + \psi_3 \\ \text{(ii)} & \chi_j \mid N = 3\psi_6 + \psi_1 + \psi_2 \\ \text{(iii)} & \chi_j \mid N = 3\psi_6 + \psi_4 + \psi_5 . \end{array}$$

If χ_j is real the only possibilities are (2.4)(i) or (2.5)(i).

The following lemma will be needed several times.

LEMMA 2.1. *Let Q be a 5-Sylow group of G. If a character μ of G is not real and $\mu \mid Q$ is not rational, there are at least four distinct nonreal conjugates of μ.*

Proof. Let K be a splitting field for G containing $\lambda_1 = e^{2\pi i/25}$. Let K_1 be the subfield of K containing any r-th roots of 1 lying in K where r runs over the primes in g other than 5. We may pick $\sigma \in G(K/K_1)$ the Galois group of K over K_1 so that $\sigma(\lambda_1) = (\lambda_1)^2$. Suppose $\mu, \mu^\sigma, \bar{\mu}, \bar{\mu}^\sigma$ are not all distinct. As there are no elements of order 5^3 in $G[5, 3B]$, $\mu \mid Q \neq \mu^\sigma \mid Q$ by hypothesis. Also $\mu \neq \bar{\mu}$ by hypothesis. This means $\bar{\mu} = \mu^\sigma$. In particular $\mu^\sigma \mid Q = \bar{\mu} \mid Q$. Let $\eta = \sigma^{10}$. We have $\mu^\eta \mid Q = \bar{\mu} \mid Q = \mu^\sigma \mid Q$. This implies $\mu^\sigma \mid Q = \mu \mid Q$ a contradiction. We see $\mu, \mu^\sigma, \bar{\mu}, \bar{\mu}^\sigma$ are all distinct. Clearly none can be real. This proves the lemma.

Several times we will need to study the case in which Q is cyclic of order 5. That is $g = 7\cdot 5\cdot 3^b\cdot 2^c$. The results of [2] can be applied for $p = 5$. Let π be an element of order 5. By Burnside's theorem $C(\pi) = \langle\pi\rangle \times V$. As there are no elements of order $5\cdot 7$, $|V| = 3^\beta 2^\gamma$. Let $|N(\pi)/C(\pi)| = w$. As G has no normal 5-complement w is 2 or 4 by Burnside's theorem. Each 5 block contains e nonexceptional characters and $4/e$ exceptional characters where e is 1, 2, or 4. Let $B_1(5)$ be the 5-block containing χ. If χ is nonexceptional there are two possibilities for $\chi \mid C(\pi)$ as can be seen from close inspection of [2, II, Th. 1]

$$\chi \mid \langle\pi\rangle \times V = \theta + \sum_{i=0}^{4} \lambda^i \cdot \varphi \tag{2.6}$$

where θ is of degree 2 with $\theta(\pi) = 2$, λ is the linear character of $\langle\pi\rangle$ such that $\lambda(\pi) = e^{2\pi i/5}$, φ is a linear character of V.

$$\chi \mid \langle\pi\rangle \times V = \varphi_1 + \varphi_2 + \sum_{i=0}^{4} \lambda^i \varphi \tag{2.7}$$

where λ, φ are as in (2.6) and φ_1, φ_2 are distinct linear characters of $c(\pi)$ conjugate in $N(\pi)$. Also $\varphi_i(\pi) = 1$, $i = 1, 2$. In the case (2.6)

V is nonabelian; in the case (2.7) V is abelian.

Suppose χ is exceptional. If $s = 4$ and $e = 1$ or 2 an examination of [2, II, Th. 1] shows χ cannot have degree 7. If $e = 4$ any of the characters in $B_1(5)$ can be chosen exceptional and so χ can be chosen nonexceptional. In any case χ satisfies (2.6) or (2.7). If $s = 2$, there are at least two characters χ and χ' of degree 7 in $B_1(5)$. By [7, p. 579] $\chi\chi'$ has a constituent in $B_0(5)$. As $\chi \neq \chi'$ this constituent is not x_0 and so $B_0(5)$ has a nontrivial character whose degree is at most 49. By [2] there are three characters in $B_0(5)$ besides x_0. Their degrees must be congruent to ± 1 or 0 mod 7. By examining the possibilities one sees the smallest such degree equation for $B_0(5)$ is $1 + 63 = 64$. This means χ cannot be exceptional and so χ must satisfy (2.6) or (2.7).

3. The case $\chi\bar{\chi} = \chi_0 + \chi_1$, $\deg \chi_1 = 48$, $g = 7\cdot 5^a \cdot 3^b \cdot 2^c$. In this section we consider the case IB where $\chi\bar{\chi} = \chi_0 + \chi_1$, $\deg \chi_1 = 48$. We still assume $s = 6$. This case is eliminated by first showing χ is rational when restricted to 5-Sylow group and so $a \leqq 1$. The case $a = 1$ is eliminated by finding $C(\pi)$ where π is an element of order 5. The case $a = 0$ is then eliminated using some results in [6].

Suppose χ_1 is not rational. In particular let σ be an element of the Galois group of a splitting field K for G over the rationals for which $(\chi_1)^\sigma \neq \chi_1$. Clearly $\chi^\sigma\bar{\chi}^\sigma = \chi_0 + \chi_1^\sigma$. This implies $\chi\chi^\sigma\bar{\chi}\bar{\chi}^\sigma$ has χ_0 as a constituent with multiplicity 1. This means $\chi\chi^\sigma$ of degree 49 is irreducible giving a contradiction. We see χ_1 is rational. Also $\chi\bar{\chi} = \chi_0 + \chi_1$ must be rational

Let π be an element of order 5. Suppose $\chi \mid \langle\pi\rangle = \sum_{i=1}^{t} b_i\lambda_i$ where the λ_i, $i = 1, 2, \cdots, t$ are distinct linear characters of $\langle\pi\rangle$, $b_i \neq 0$. Certainly $t \leqq 5$ and $\sum_{i=1}^{t} b_i = 7$. If $c = \sum_{i=1}^{t}(b_i)^2$ we have $\chi\bar{\chi}(\pi) = c - b$ where $c + 4b = 49$. This means $c \equiv 1 \pmod 4$. If the numbers $\{b_1, \cdots, b_t\}$ are arranged in decreasing order the following possibilities occur:

$$\{3, 1, 1, 1, 1\}; \{2, 2, 2, 1\}; \{4, 2, 1\}; \{3, 2, 2\}; \{6, 1\}; \{5, 2\}; \{4, 3\} .$$

Now let λ be the linear character of $\langle\pi\rangle$ with $\lambda(\pi) = e^{2\pi i/5}$. Suppose $\chi \mid \langle\pi\rangle = \sum_{i=0}^{4} a_i\lambda^i$. This means

$$\chi\bar{\chi} \mid \langle\pi\rangle = \{\sum_{i=0}^{4} a_i\lambda^i\}\{\sum_{i=0}^{4} a_i\lambda^{-i}\} = \sum_{i=0}^{4}\{\sum_{j=0}^{4} a_j a_{j-i}\}\lambda^i .$$

As $\chi\bar{\chi}$ is rational we obtain

$$\sum_{j=0}^{4} a_j a_{j-i} = b , \qquad i = 1, 2, 3, 4 . \tag{3.1}$$

The nonzero entries among $\{a_0, a_1, \cdots, a_4\}$ are the values $\{b_1, \cdots, b_t\}$ given in the above paragraph possibly rearranged. A routine check of these possibilities shows $\{3, 1, 1, 1, 1\}$ to be the only set for which (3.1) can be satisfied. The checking is facilitated by noting that the first two integers a_0, a_1 can be picked arbitrarily from $\{b_1, \cdots, b_t, 0, \cdots, 0\}$ without changing the form of (3.1). Here the bracketed integers are completed with zeroes to give 5 terms. The case $[4, 2, 1\}$ or $\{3, 2, 2\}$ can be chosen as $\{0, 0, \delta_1, \delta_2, \delta_3\}$ where $\delta_1, \delta_2, \delta_3$ is 4, 2, 1 or 322 in some order. Equation (3.1) gives $\delta_1\delta_2 + \delta_2\delta_3 = \delta_1\delta_3$. This is impossible for any of the choices of $\delta_1, \delta_2, \delta_3$. The case $\{2, 2, 2, 1\}$ can be taken as $\{0, 1, 2, 2, 2\}$. Equation (3.1) is not satisfied. In the cases $\{b_1, b_2\}$ we consider $\{b_1, b_2, 0, 0, 0\}$. Equation (3.1) cannot be satisfied. In the case $\{3, 1, 1, 1, 1\}$ the unimodularity of X gives $\chi \mid \langle\pi\rangle = 3\lambda^0 + \lambda + \lambda^2 + \lambda^3 + \lambda^4$. This means $\chi(\pi) = 2$.

Suppose π has order 5^2. As π^5 has order 5 the constituents of $\chi \mid \langle\pi^5\rangle$ are $\{3\lambda^0, \lambda, \lambda^2, \lambda^3, \lambda^4\}$ where $\lambda(\pi^5) = e^{2\pi i/5}$. Let the linear constituents of $\chi \mid \langle\pi\rangle$ be $\{\lambda_1, \lambda_2, \lambda_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ where $(\varepsilon_i)^5 \mid \langle\pi^5\rangle = \lambda^i$, $(\lambda_i)^5 \mid \langle\pi^5\rangle = \lambda^0$. Suppose $\lambda_i = \lambda_j$, $i \neq j$, $i, j = 1, 2,$ or 3. This means $\varepsilon_1\bar{\lambda}_i$ and $\varepsilon_1\bar{\lambda}_j$ are equal and so all twenty conjugates appear with equal multiplicity at least 2 in $\chi\bar{\chi} \mid \langle\pi\rangle$. The following thirteen linear characters also appear in $\chi\bar{\chi} \mid \langle\pi\rangle$: $\varepsilon_i\bar{\varepsilon}_i$, $i = 1, 2, 3, 4$; $\lambda_j\lambda_k$, $j, k = 1, 2, 3$. None of these are conjugate to $\varepsilon_1\bar{\lambda}_i$ and so we have too many constituents. This means $\lambda_1, \lambda_2, \lambda_3$ are all distinct. Also $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are all distinct as their fifth powers are distinct. This means the trivial character of $\langle\pi\rangle$ occurs seven times. The number of conjugates of any nontrivial character is 4 or 20 and hence divisible by 4. However $4 \nmid 42$. This means there are no elements of order 5^2 in G.

We have shown χ is rational on a 5-Sylow group. In particular, by [21] $a \leqq 1$. We now consider this case, case IB(ii) of the flow chart. There are two possibilities for $\chi \mid C(\pi)$, (2.6) and (2.7). In case (2.7), $C(\pi)$ is abelian as $\chi \mid C(\pi)$ has seven linear constituents. In case (2.6), V is nonabelian as $\chi \mid V$ has a constituent of degree 2.

We may also consider the restriction $\chi_1 \mid V \times \pi$. Here $\chi_1(\pi) = 3$. The only possibility [2, II, Th. 1] is $\chi_1 \mid V \times \pi = \theta\cdot\lambda^0$, $\deg\theta = 3$. This means V is nonabelian as θ is irreducible. In particular, case (2.7) above does not occur. The character θ is rational as χ_1 is rational. As $\chi\bar{\chi} = \chi_0 + \chi_1$ we have $\theta_1\bar{\theta}_1 = \theta_0 + \theta$ where θ_0 is the trivial character of V, $\chi \mid V \times \langle\pi\rangle = \theta_1 + \sum_{i=0}^{4}\lambda^i\cdot\varphi$.

We know $\chi \mid V = \theta_1 + 5\varphi$. If R is in the kernel of θ_1, $(\varphi(R))^5 = 1$ by the unimodularity of X. This means $\varphi(R) = 1$ and so θ_1 is faithful. Let V_3 be a 3-Sylow group of V. As θ_1 is faithful of degree 2, V_3 must be abelian. We see $X \mid V_3$ has at most 3 distinct linear characters and so $|V_3| \leqq 3^2$ [5, 3D]. Let V_2 be a 2-Sylow group of V. As

V_2 has a faithful representation of degree 2 there is an abelian subgroup A of index 2. As $X \mid A$ has at most 3-distinct linear characters, $|A| \leqq 2^2$ [5, 3D]. Here V_2 is nonabelian if and only if $|V_2| = 2^3$. In this case an involution in $Z(V_2)$ must satisfy $\theta_1(J) = -2$. Clearly $\varphi(J) = 1$. In particular $J \in Z(V)$.

We now consider θ. Let K be the kernel of θ. As $\theta_1\bar{\theta}_1 = \theta_0 + \theta$, if $R \in K$, $|\theta_1(R)| = 2$. In particular $R \in Z(V)$. Suppose there is an element R of order 3 in K. As θ is faithful and rational on V/K $|V| = 3^2 \cdot 2^\delta$. Also $\theta_1(R)$ is $2u$ or $2u^2$ where $u = e^{2\pi i/3}$. We can assume $\theta_1(R) = 2u$ by taking R^2 if necessary. By the unimodularity of X, $\varphi(R) = u^2$. If there is an element J in V for which $\theta_1(J) = -2$, $\varphi(J) = 1$ and Blichfeldt's theorem is violated for X and JR. This means $\delta \leqq 2$. As V/K has a representation of degree 3, $\delta = 2$. If V_2 is elementary abelian there is an element J for which $\theta_2(J) = -2$. We see V_2 is cyclic. This means there is a normal 2-complement. However, in this case there can be no character of degree 3 by Ito's theorem [12, 53.18]. This shows there are no elements of order 3 in K.

As V/K has a rational character of degree 3 $|V| = 3 \cdot 2^\delta$. Here V has characters of degree 3 and 2. As $3^2 + 2^2 > 12$ we see $|V| = 3 \cdot 2^3$. This means V_2 is nonabelian and so there is an involution in $Z(V)$ for which $\theta_1(J) = -2$. Let T be an element of order 3 in V. If $\theta_1(T) = u + \bar{u}$ where $u = e^{2\pi i/3}$ the element TJ would contradict Blichfeldt's theorem. This means $\theta_1(T) = 1 + u$ or $1 + u^2$. We see $\bar{\theta}_1 \neq \theta_1$. By the unimodularity of X, $\chi(T) = 1 + 6u$. We see T is not conjugate to T^{-1} in G. There must be a normal three complement to $\langle T \rangle$ in V and so the number of linear characters of V is a multiple of three.

The characters obtained so far have degrees 1, 1, 1, 3, 2, 2. There must be one further character of degree 2. As $\theta_1\bar{\theta}_1 = \theta_0 + \theta$ has two irreducible constituents, $\theta_1\theta_1 = P_2(\theta_1) + C_2(\theta_1)$ must have two irreducible constituents. These are the characters corresponding to the symmetric tensors of rank 2, $P_2(\theta_1)$, and the skew symmetric tensors of rank 2, $C_2(\theta_1)$. We see $P_2(\theta_1) = \theta$. Similarly $\chi\chi = P_2(\chi) + C_2(\chi)$. As $\chi\bar{\chi} = 1 + \chi_1$, $P_2(\chi)$ and $C_2(\chi)$ are irreducible. Clearly $\chi\chi(\pi v) = \theta_1\theta_1(v) = (\theta + C_2(\theta_1))(v)$, where $v \in V$. Let $\psi = P_2(\chi)$. Clearly $\psi(\pi v) = \theta(v)$. This means ψ is in the same 5-block as χ_1. Denote this 5-block by $B_1(5)$. Evaluating $\psi(T)$ we find $\psi(T) = \{(1 + 6u)^2 + (1 + 6u^2)\}/2 = -5 + 15u^2$. This means $\psi \neq \bar{\psi}$. However $\psi(\pi v) = \bar{\psi}(\pi v)$ for $v \in V$ and so $\bar{\psi} \in B_1(5)$. We show case I B(ii) is impossible by showing the block $B_1(5)$ cannot be completed without giving a contradiction.

There cannot be two exceptional characters in $B_1(5)$ or there would be too many characters. Here ψ, $\bar{\psi}$ cannot be the exceptional characters as $28 \equiv 48 \pmod 5$. This means there are two missing characters with degrees R and S. There are two cases for the degree equation $104 + R = S$ and $104 = R + S$. There must be one more character from $B_0(7)$ and one whose degree is not divisible by 3. If the degree of the character from $B_0(7)$ is not divisible by 3 it must be 8, 64, or 512. The only solution is $48 + 28 + 28 + 8 = 112$. If the character of degree 8 is denoted by χ_6, $\chi_6 \mid V = \theta + 5\zeta$ where ζ is linear. As $\theta(J) = 3$ we see $\zeta(J) = 1$. This means J is in the kernel of χ_6 contradicting the simplicity of G. This means R or S is of the form $7 \cdot 2^B$ where $7 \cdot 2^B \equiv \pm 2 \pmod 5$. The degree of the character in $B_0(7)$ is divisible by 3. We see $B = 2, 4, 6, 8, 10$, $7 \cdot 2^B \equiv 1 \pmod 3$. We need only consider the cases $7 \cdot 2^B \equiv 3 \pmod 5$. The values are 28, 448, 7168. There are no solutions. This case is therefore impossible and we can assume $g = 7 \cdot 3^b \cdot 2^c$.

We now begin Case IB (iii). We assume $s = 6$, $g = 7 \cdot 3^b \cdot 2^c$. We know $b \leqq 8$. By Sylow's theorem $b = 1, 3, 5, 7$. There must be a character of degree 27 or 729 and one of degree 8,64, or 512.

If a character of degree 729 occurs it must be in a 3-block containing 3 characters of degree 729, for if not there would be a character of degree at least $6 \cdot 729$. The degree equation would then be $1 + 729 + 729 + 729 + \begin{Bmatrix} 8 \\ 64 \\ 512 \end{Bmatrix} = 48 + \chi_6$. There is no solution.

There must be a character χ_3 of degree 27. Let $g = 7 \cdot 3^b \cdot 2^c$. Sylow's theorem gives $b = 3, 5, 7$. Suppose first $b = 3$. Then $c = 3, 6, 9$. If $c = 3$ or 6, $g < 20{,}000$. All simple groups of order at most 20,000 are listed in [19] and none have this order. If $c = 9$ the result [6, 1H] is contradicted as $2^9 \geqq 12 \cdot 3^3 = 324$. When $b = 5$ or 7 [6, 1L] can be applied to show there is no character of degree 512. As there must be a character of degree 2^B it must be 8 or 64. Each of these cases can be eliminated with a routine elimination of degree equations using block separation and Schur's theorem [21]. We do not include the details.

4. The case $\chi\bar{\chi} = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 20$, $g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. In this section we consider the case IC where $\chi\chi = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 20$. The case is eliminated by first showing $a = 1$. This case is eliminated by considering $\chi \mid C(\pi)$ where π is an element of order 5. The relations (2.1)–(2.7) are used.

We begin with a preliminary discussion regarding the tree for the prime 7 [2]. The character χ is a principal 7-indecomposable and so $\chi\bar{\chi}$ is a sum of principal 7-indecomposables [9]. There is exactly one principal indecomposable containing χ_0 as a constituent. This is $\chi_0 + \chi_j$ where χ_j is adjacent to χ_0 on the tree. This means $\chi_0 + \chi_j$ is a constituent of $\chi\bar{\chi}$. We have already eliminated the case

$x_j = 6$ or $x_j = 48$. This means x_j is 20 or 27. We are assuming $x_1 = 20$. If $x_j = 27$, $\chi\bar{\chi}$ would have two linear constituents contradicting the simplicity of G or the irreducibility of χ. This means $x_j = 20$.

Suppose $j \neq 1$. This would mean $\chi\bar{\chi} = \chi_0 + \chi_1 + \chi_j + \mu$. Here μ would be of degree 8. Clearly χ_j is real as it is adjacent to χ_0 on the stem. Also μ is real as it is the only constituent of $\chi\bar{\chi}$ of degree 8 and $\chi\bar{\chi}$ is real. Using (2.4) and (2.5) we see $\chi_0 + \mu + \chi_j \mid N$ has ψ_0 as constituent with multiplicity 3. However (2.2) and (2.3) imply $\chi\bar{\chi} \mid N$ has ψ_0 as constituent with multiplicity 2. This means $j = 1$ and so χ_1 is adjacent to χ_0 on the stem.

Let $\chi\bar{\chi} = \chi_0 + \chi_1 + \mu$. As above χ_1 is real. From (2.2)-(2.5) we see

$$\mu \mid N = 4\psi_6 + \psi_1 + \psi_5 + \psi_2 + \psi_4 \,. \tag{4.1}$$

In particular ψ_0 and ψ_3 are not constituents of $\mu \mid N$. This means by (2.1) that μ has no irreducible constituent of degree 7. By (2.5), μ has no real constituent of degree 20. As μ itself is real it can have no constituent of degree 20. By (2.4), μ has no real constituent of degree 8. If there is a nonreal constituent its conjugate also appears as μ is real. This leaves a remaining constituent of degree at most 12 which is impossible as no χ_i has degree 6. This means μ is irreducible or has two constituents of degree 14.

We can now show χ_1 is rational. Suppose there is some element σ of the Galois group of K such that $\chi_1^\sigma \neq \chi_1$. Then $(\chi\bar{\chi})^\sigma = (\chi^\sigma)(\bar{\chi}^\sigma) = \chi_0 + \chi_1^\sigma + \mu^\sigma$. As μ^σ has no constituent of degree 20, χ_1^σ must be adjacent to χ_0 on the stem. However this means $\chi_1 = \chi_1^\sigma$.

We now show that χ_1 is not in the principal 5-block $B_0(5)$. In fact we show that a character η of degree 20 in $B_0(5)$ must be irrational when restricted to a 5-Sylow group Q. Suppose not. If π has order 5, $\eta(\pi) = -5$. If there is an elementary abelian subgroup of Q of order 5^2 summing the character η over the subgroup gives $(-5)(24) + 20 < 0$ giving a contradiction. If Q has order 5^2, η has 5-defect 1 and so $\eta \notin B_0(5)$. This gives the result as there are no elements of order 5^3 in G. Let $B_2(5)$ be the 5-block containing χ_1.

We will now assume $a \geqq 2$. This is Case IC(i) in the flow chart. We have

$$\chi\bar{\chi} = \chi_0 + \chi_1 + \mu \,. \tag{4.2}$$

Again, let Q be a 5-Sylow group. As $a \geqq 2$, $\chi \mid Q$ cannot be rational by Schur [21]. Let σ be an element of the Galois group of a splitting field K which fixes all p-th roots of unity for primes p other than 5. Set

$$\chi^{\sigma}\bar{\chi} = \sum a_i \zeta_i \ . \tag{4.3}$$

Here the ζ_i are irreducible characters of G. Let 4.2 be written in terms of ζ_i as

$$\chi\bar{\chi} = \sum b_i \zeta_i \ . \tag{4.4}$$

We see $\chi\bar{\chi}$ and $\chi^{\sigma}\bar{\chi}$ are equal on 5-regular element. This means for B any 5-block

$$\textstyle\sum_{\zeta_i \in B} a_i \zeta_i(\rho) = \sum_{\zeta_i \in B} b_i \zeta_i(\rho) \tag{4.5}$$

for any 5-regular ρ.

We apply (4.5) with $B = B_2(5)$, the 5-block containing χ_1. The character χ_1 appears with multiplicity one on the right hand side. There is possibly one second character of degree 14 appearing with no zero coefficient on the right hand side. The degree is therefore 20 or 34. In particular it is congruent to $-1 \pmod 7$. This means the left hand side must contain a character χ_j of degree 20 or 27. As (4.3) is a sum of principal 7-indecomposables there is a character χ_k whose degree is congruent to $1 \pmod 7$. Its degree must be 8 or 15. Also χ_j and χ_k are adjacent on the tree.

Suppose $x_k = 8$. It follows from the discussion in the above paragraph that $\chi_k \notin B_2(5)$. As N is 5-regular we may use (4.5), (4.1), and (2.4) to see χ_k cannot be real. If $\chi_k \mid Q$ is not rational where Q is a 5-Sylow group of G, then Lemma 2.1 gives four nonreal conjugates of χ_j. The degree equation for $B_0(7)$ must now be

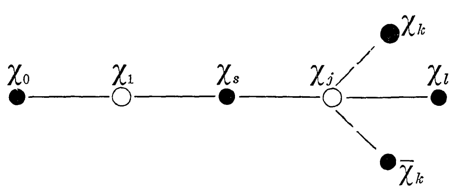
$$1 + 8 + 8 + 8 + 8 = 20 + 13 \ .$$

This is impossible and so $\chi_k \mid Q$ is rational. By Schur's theorem [21], $a \leqq 2$. As we are assuming $a \geqq 2$ we have $a = 2$. Also $B_2(5)$ is of defect 1 and so the right hand side of (4.5) is χ_1. This means χ_j has degree 20 and $\chi_1 = \chi_j$ for 5-regular elements.

Suppose $j \neq 1$ and so $\chi_j \neq \chi_1$. If χ_j is not real the degree equation becomes

$$1 + 8 + 8 + 43 = 20 + 20 + 20$$

which is impossible. If χ_j is real it is on the stem. The stem has 5 characters giving



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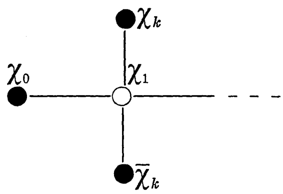
The character χ_l has degree 8 or 15. This is impossible as $8+8+8>20$ and so $2x_k + x_l > x_j$. This means $j = 1$ and so $\chi_j = \chi_1$.

As $\chi_j = \chi_1$ and there are no characters with degree smaller than 8, the tree must be

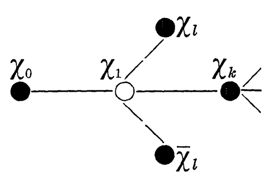


The 7-modular constituents of χ_1 are therefore of degree 1, 8, 8, and 3. By [23] the constituent of degree 3 is realizable in GF(7). However $5^2 \nmid |GL(3,7)|$ giving a contradiction. This means $x_k = 15$.

Suppose now χ_k has degree 15. Let χ_k be in the 5-block $B_*(5)$. We assume first $B_*(5) \neq B_0(5)$. Apply equation (4.5) with $B = B_*(5)$. A character of degree 15 cannot be fitted into the sum (4.5) over $B_2(5)$ as that sum is of degree 20 or 34. This means $B_*(5) \neq B_2(5)$. The possible sums of degree over $B_*(5)$ on the right of (4.5) are 14 and 28. However $14 < 15$. If the sum is 28, (4.3) must have a linear constituent giving a contradiction. This means $\chi_k \in B_0(5)$.

Because $\chi_k \in B_0(5)$, $\chi_k(\pi)$ must be irrational for any element π of order 5. This means $\chi_k \mid Q$ is irrational where Q is a 5-Sylow group of G. If χ_i is not real Lemma 2.1 gives four nonreal conjugates of χ_k. The degree equation is impossible. Therefore χ_k is real. If χ_k^σ is a conjugate of χ_k a similar argument with χ^σ shows χ_k^σ is real. There is at least one such $\chi_k^\sigma \neq \chi_k$.

Assume first $\chi_j = \chi_1$. As χ_k and χ_k^σ are real they are on the stem. The stem contains at least five characters. Also χ_k is adjacent to χ_1 on the stem. This forces a branch at χ_1. The tree must be



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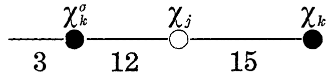
Clearly $x_l = 8$ and G has a 7-modular representation of degree 3. This contradicts $a \geqq 2$ and implies $\chi_j \neq \chi_1$.

We now eliminate this case using the tree and the degree equation. The tree has at least 5 real characters as $\chi_0, \chi_1, \chi_j, \chi_k, \chi_k^\sigma$ are all real. A branch at χ_1 implies a 7-modular character of degree 3 which is a contradiction. This means the character χ_l adjacent to χ_1 other than χ_0 must have degree at least 19 and so cannot be χ_k or χ_k^σ. In turn the stem must have 7 characters. If χ_j has degree 27 the configuration



implies χ_k^σ has a 7-modular constituent of degree 3 again giving a contradiction as $a \geqq 2$ [23]. It follows that $x_j = 20$. The degree equation is

$$1 + 15 + 15 + x_l = 20 + 20 + x_m$$

where χ_m is the character adjacent to χ_k^σ above. As χ_l is a constituent of $\chi_1\bar{\chi}_1 = \chi_1\chi_1$ its degree is at most $20 \cdot 21/2$. The only such degrees are 36, 50, 64, 120, and 162. There is only one solution

$$1 + 15 + 15 + 36 = 20 + 20 + 27 .$$

This is eliminated by 5-block separation using $B_2(5)$. We have completed all cases where $a \geqq 2$ which is part IC(i) of the flow chart.

We now consider the case in which $a = 1$. This is IC(ii) of the flow chart. We use the results (2.6) and (2.7). The case is eliminated by a careful examination of the decompositions of $\chi\bar{\chi}$ and $\chi\chi$ and their restrictions to $C(\pi)$. The results of [2] provide contradictions for each of the possibilities for decompositions of $\chi\bar{\chi}$ and $\chi\chi$.

We know from (2.6) and (2.7) that χ is 5-rational, that is χ lies in the field of $g/5$th roots of unity. This can also be shown using (4.5). We know that $\chi\bar{\chi} = \chi_0 + \chi_1 + \mu$. From (4.1) we know that μ is either irreducible or has two constituents μ_1 and μ_2 of degree 14. Also (4.1) shows $\mu_1 \mid N \neq \mu_2 \mid N$. As χ is 5-rational so is μ. As N consists of 5-regular elements, μ_1 and μ_2 are 5-rational also when μ is reducible. This means that the constituents of $\chi\bar{\chi}$ of full 5-defect are all 5-rational and consequently nonexceptional for $p = 5$. [2].

Let S_0 be the character of $\langle\pi\rangle$ defined by $S_0(e) = 5$, $S_0(\pi) = 0$. In case (2.6) let $\gamma = \theta$, in case (2.7) let $\gamma = \varphi_1 + \varphi_2$. Then by (2.6) or (2.7), $\chi \mid C(\pi) = \gamma + \varphi S_0$. Let $\gamma\bar{\gamma} = \xi_0 + \xi_1$ where ξ_0 is the trivial character of $C(\pi)$. This means

$$\chi\bar{\chi} \mid C(\pi) = \xi_0 + \xi_1 + \gamma\bar{\varphi}S_0 + \bar{\gamma}\varphi S_0 + 5\varphi\bar{\varphi}S_0 . \tag{4.6}$$

Assume now μ is reducible. By [2, II, Th. 1] set $\mu_i \mid C(\pi) = \pm \bar{\psi}_i + S_i$ where ψ_i is a sum of τ_i irreducible characters θ_i^j, $j = 1, 2, \cdots, \tau_i$ containing π in their kernel. The S_i are $\gamma_i S_0$ where γ_i is a character of V. The θ_i^j, $j = 1, 2, \cdots, \tau_i$ are conjugate in $N(\pi)$ and so $\tau_i = 1, 2, 4$. Using 4.6 and 4.2 we see $\pm\psi_1 \pm \psi_2 = \xi_1$. If $\psi_1 \neq \psi_2$ both signs must be plus. Interchanging ψ_1 with ψ_2 if necessary we may assume ψ_1 has degree 1, $\tau_1 = 1$. But then degree $\mu_1 \equiv 1 \pmod 5$ giving a contradiction. If $\psi_1 = \psi_2$ then both signs must be equal.

As ξ_1 has odd degree this is impossible.

This contradiction implies that μ must be irreducible. We may set $\mu \mid C(\pi) = \pm \psi + S$ as before. This time $\psi = \xi_1$. As ψ is a sum of τ irreducible characters θ^j of $C(\pi)$ with the same degree and $\tau = 1, 2, 4$, we see $\tau = 1$ and θ^1 has degree 3. In particular V is nonabelian and so case (2.6) holds. This gives $\theta\bar{\theta} = \xi_0 + \theta^1$.

The same technique can be applied to $\chi\chi$. We have first that $\chi\chi = P_2(\chi) + C_2(\chi)$. Using (2.2), (2.3), (2.4) and (2.5) it can be seen that the constituents of $\chi\chi$ are all of zero 7-defect. As $\chi\bar{\chi}$ has three distinct irreducible constituents so does $\chi\chi$. Using (2.2) and (2.3) we see that either $P_2(\chi)$ has two distinct constituents of degree 14 and $C_2(\chi)$ is irreducible or $P_2(\chi)$ is irreducible and $C_2(\chi)$ has constituents of degrees 14 and 7. Let these be η_1, η_2, η_3. We may again check using (2.2), (2.3), (2.4) that they are 5-rational.

We also have $\theta^2 = P_2(\theta) + C_2(\theta)$. Also $P_2(\theta)$ and $C_2(\theta)$ are irreducible as $\theta\bar{\theta} = \xi_0 + \theta^1$. As before we may set $\eta_i \mid C(\pi) = \pm \psi_i + S_i$ where the ψ_i are sums of characters conjugate in $N(\pi)$ each with π in the kernel. We have

$$\pm\psi_1 \pm \psi_2 \pm \psi_3 = P_2(\theta) + C_2(\theta) .$$

If the ψ_i, $i = 1, 2, 3$ are all distinct there would be three constituents on the right. It is easy to see that we may assume $\psi_1 = \psi_2$ and the signs are opposite. Therefore $\psi_3 = P_2(\theta) + C_2(\theta)$ which is impossible as $P_2(\theta)$ is irreducible of degree 3 and ψ_3 is a sum of irreducible characters of the same degree. This contradiction finishes this section.

5. **The case $\chi\bar{\chi} = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 27$, $g = 7\cdot 5^a\cdot 3^b\cdot 2^c$.** In this section we consider the case ID where $\chi\bar{\chi} = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 27$. Here $s = 6$. There is exactly one group of this form $S_6(2)$ of order $7\cdot 5\cdot 3^4\cdot 2^9$. Cases are eliminated by first showing χ is rational when restricted to a 3-Sylow group. This is done much as in § 4 where it was shown that χ restricted to a 5-Sylow group was rational. Here the character χ_1 of degree 27 cannot be in the principal 3-block by [8]. We can therefore use relations like (4.5) for the 3-block containing χ_1. Once it is known that χ restricted to a 3-Sylow group is rational, the value of b is at most 4 by Schur [21]. As χ_1 is of degree 27, $b \geqq 3$. The two cases $b = 3$ and $b = 4$ are treated separately. For $b = 3$, the generalized decomposition numbers for χ on $C(\tau^2)$ are examined where τ^2 is of order 3 and normalizes a 7-Sylow group. These lead to a contradiction. For $b = 4$, the 3-Sylow group is determined explicitly. The character χ_1 is of 3-defect 1. The various possibilities for the tree are eliminated except of course the one

leading to $S_6(2)$. In this case it is shown there is an involution J for which $\chi(J) = -5$. Adjoining $-I$ to the matrices $X(G)$ gives a group generated by reflections. These groups are all known and we obtain $S_6(2)$.

The analysis here is much longer than in preceding sections and we do not give all the details. Where arguments are similar to earlier arguments they are not repeated. In eliminating cases, the various known techniques involving block separation, cyclic defect groups, etc., are used implicitly. Consequently, only the most troublesome cases are treated.

Let $\chi\bar{\chi} = \chi_0 + \chi_1 + \mu$. If μ contained a constituent of 7-defect 1 its degree would by 6, 8, or 20. We have ruled out a degree 6 in § 2. As μ has degree 21, μ has no constituents of 7-defect 1. As in § 4, $\chi_0 + \chi_1$ is a principal 7-indecomposable and so χ_1 adjoins χ_0 on the stem. In particular χ_1 is real. As in § 4, χ_1 is rational.

We now consider the case in which $\chi \mid S$ is not rational. Here S is a 3-Sylow group of G. This is part ID(i) of the flow chart. Let $B_1(3)$ be the 3-block containing χ_1. By [8] it is not of full defect. This means that $B_1(3) \neq B_0(3)$ where $B_0(3)$ is the principal 3-block. Also the degrees of the characters in $B_1(3)$ are all divisible by 3. Let σ be an element of the Galois group of K which fixes all roots of unity except 3rd roots of unity and for which $\chi^\sigma \mid S \neq \chi \mid S$. This will mean $\chi^\sigma\bar{\chi}$ and $\chi\bar{\chi}$ are equal on 3-regular elements. Let

$$\chi^\sigma\bar{\chi} = \sum b_i\zeta_i \; . \tag{5.2}$$

Let $\mu = \sum c_i\zeta_i$. We have for 3-regular elements ρ

$$\chi_1(\rho) + {\sum}^1 c_i\zeta_i(\rho) = {\sum}^1 b_i\zeta_i(\rho) \tag{5.3}$$

where the sum $\sum^1$ is taken only over the characters ζ_i in $B_1(3)$. If some $c_i \neq 0$ appearing in (5.3) the degree ζ_i must be 21 as such degrees must be divisible by 3. This means some constituent of (5.2) is linear giving a contradiction. Therefore the degree of $\sum^1 b_i\zeta_i$ is 27. There must be a $b_i \neq 0$ appearing in (5.3) for which the degree is congruent to $-1 \pmod 7$. It can only be 27. This gives for 3-regular elements ρ

$$\chi_1(\rho) = \zeta_1(\rho) \tag{5.3$'$}$$

where ζ_1 is an irreducible character of G in $B_1(3)$. As in § 4 there are two cases (i) $\zeta_1 = \chi_1$ and (ii) $\zeta_1 \neq \chi_1$.

In either case there must be exactly one further character ζ_2 in $B_0(7)$ appearing in (5.2). Its degree must be 8 or 15. As $\chi^\sigma\bar{\chi}$ must be a sum of principal indecomposables $\zeta_1 + \zeta_2$ must be a sum of principal indecomposables and so ζ_2 is adjacent to ζ_1 on the tree.

It is now possible to eliminate each of these cases by careful analysis of the tree. The method is routine using block separation, properties of the tree, and the degree equation. Lemma 2.1 and Schur's theorem [21] are used when ζ_2 is of degree 8 and nonreal. We do not give any further details of this enumeration.

We now treat the cases in which $\chi \mid S$ is rational. Again S is a 3-Sylow group. By Schur's theorem [21] the order of S, 3^b, is at most 3^4. As χ_1 has degree 27 we have either $b = 3$ or 4. We will first treat the case $b = 3$, ID (ii) of the flow chart. We then treat the case $b = 4$, ID (iii) of the flow chart.

Assume then that $|S| = 3^3$. We will show first that S is nonabelian. Suppose first S is abelian. Let $\chi \mid S = \sum_{i=1}^{7} \lambda_i$ where the λ_i are linear characters of S. Suppose there is an element T of order 9 in S. There must be an i for which $\lambda_i(T)$ is a primitive 9-th root of unity. As χ is rational all six conjugates must appear amongst the λ_i, $i = 1, 2, \cdots, 7$. For the remaining character λ_j we have $\lambda_j(T) = 1$ by the unimodularity of X. There can be no element of S independent from T. As there are no elements of order 27 in G [5, 3B] we see $|S| \leqq 9$. Therefore S must be elementary abelian. We can write $\chi \mid S = \lambda_1 + \bar{\lambda}_1 + \lambda_2 + \bar{\lambda}_2 + \lambda_3 + \bar{\lambda}_3 + \lambda_0$ where λ_0 is the trivial character. We know χ_1 of degree 27 appears as a constituent in $\chi\bar{\chi}$. As $\chi_1 \mid S$ is the character of the regular representation of S, we see all linear characters of S appear as constituents of $\chi\bar{\chi} \mid S$. Checking with the characters $\lambda_1, \lambda_2, \lambda_3$ shows that $\chi\bar{\chi} \mid S$ cannot contain 27 distinct linear characters as constituents. This means S is nonabelian.

As $\chi \mid S$ is faithful it must contain a nonlinear constituent. As S is a 3-group its degree must be 3. Let μ be this nonlinear character and let U be the representation corresponding to it. Here U must be faithful as any proper quotient group of a group of order 27 is abelian. There is an element R in $Z(S)$ for which $\mu(R) = 3u$ where $u = e^{2\pi i/3}$. Here $U(R) = uI_3$ where I_3 is the 3×3 identity matrix. This means μ is nonreal. As $\chi \mid S$ is rational $\bar{\mu}$ must also appear as a constituent. We see $\chi \mid S = \mu + \bar{\mu} + \lambda_1$ where λ_1 is linear.

The constituents of $\chi \mid S$ are all distinct and so $|C(S)|$ is divisible by the primes 3 and 7 only by [5, 3F]. As $7 \nmid |C(S)|$, we see $C(S) = Z(Q)$. This means that the principal 3-block $B_0(3)$ is the only 3-block of full defect [3, I6D]. We know $Z(S)$ has order 3. Let $\langle R \rangle = Z(S)$. Then $\chi(R) = 3u + 3\bar{u} + \lambda_1(R) = -3 + \lambda_1(R)$. Clearly $\lambda_1(R) = 1$ and so $\chi(R) = -2$. In particular, if S is a 3-element and $\chi(S) \neq -2$, then S is not in the center of any 3-Sylow group and so $3^3 \nmid |C(S)|$.

We will apply this to the element τ^2 of order 3 given in Table

I, § 2. Here $\tau^2 \in N(\xi)$ where ξ is an element of order 7. We may assume $\tau^2 \in S$. By (2.1), $\chi(\tau^2) = 1$ and so $3^3 \nmid |C(\tau^2)|$. Let $|C(\tau^2)| = 3^2 \cdot c_0$ where $3 \nmid c_0$. Here $|C(\tau^2)/\langle\tau^2\rangle| = 3c_0$. For the sake of simplicity we will replace τ^2 with T. This case will be eliminated by considering the generalized decomposition numbers for T.

Let $C = C(T)$, $\bar{C} = C/\langle T\rangle$. We know $|C| = 3^2 \cdot c_0$, $|\bar{C}| = 3c_0$. If b is a 3-block of full defect of C there is a corresponding block $\bar{b}$ of full defect for $\bar{C}$. The modular characters of b all have T in their kernel and can be considered as modular characters of the block $\bar{b}$ of $\bar{C}$. If $\bar{C}_1$ is the Cartan matrix for $\bar{b}$, the Cartan matrix, C_1, for b is $3\bar{C}_1$, [4, p. 154].

Any 3-block of full defect of $\bar{C}$ has a cyclic defect group of order 3. The theory of such defect blocks can readily be applied [2]. There are two cases, (a) and (b). In case (a) there is one modular character and three ordinary characters of the same degree. The Cartan matrix is (3). In case (b) there are two modular characters and three ordinary characters. If f_1 and f_2 are the degrees of the modular characters, the degrees of the ordinary characters are $f_1, f_1 + f_2, f_2$. Also $f_1 \equiv f_2 \pmod 3$. The Cartan matrix is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

We apply these results to the principal 3-block $\overline{b_0(3)}$ of $\bar{C}$. We first show case (a) is impossible. Suppose, then, $\bar{C}$ had one modular character in $\bar{b}_0(3)$. Then the principal 3-block $b_0(3)$ of C has one modular character which is of course the trivial character φ_0. We know that $\chi \in B_0(3)$ as there is only one 3-block of full defect. Results on generalized decomposition numbers in [3, 4] show $\chi(TS) = d\varphi_0(S) = d$ where $S \in C(T)$ and S is 3-regular. We may pick $J = \tau^3$ of order 2. From (2.1) we see $\chi(T) = d = 1$. Therefore $d = 1$. However from (2.1) we see also $\chi(TJ) = -1$. This would mean $d = -1$ giving a contradiction. This shows case (a) does not occur.

This means case (b) occurs. There are two modular constituents of $C(T)$ in $b_0(3)$. One is φ_0. Let φ_1 be the second. Let $J = \tau^3$. If d_0 and d_1 are the decomposition numbers for χ we obtain

$$\begin{aligned} \chi(T) &= d_0 + d_1\varphi_1(e) = 1 \\ \chi(TJ) &= d_0 + d_1\varphi_1(J) = -1 \,. \end{aligned} \tag{5.4}$$

Subtracting we find $d_1(\varphi_1(e) - \varphi_1(J)) = 2$. As J is an involution $\varphi_1(e) - \varphi_1(J)$ is an even integer. Therefore $d_1 = 1$, $\varphi_1(e) - \varphi_1(J) = 2$. Equations (5.4) become

$$\begin{aligned} d_0 + \varphi_1(e) &= 1 \\ d_0 + \varphi_1(J) &= -1 \,. \end{aligned} \tag{5.4$'$}$$

This means d_0 is a rational integer. As $d_0\bar{d}_0 \leqq 6$ and $\varphi_1(e) \equiv 1 \pmod 3$

we see $\varphi_1(e) = 1$, $d_0 = 0$. This shows that for any character χ of degree 7 the decomposition numbers are $d_0 = 0$, $d_1 = 1$.

We are now in a position to analyze the decomposition matrix if there are at least four characters of degree 7. Suppose then that there are four characters of degree 7. Let D be the nonzero rows of the decomposition matrix of $B_0(3)$ with respect to T. We know ${}^t\overline{D}D = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$. The entries of D are in $Z[\rho]$ where $\rho = e^{2\pi i/3}$. We let the first column correspond to φ_0, the second to φ_1. By a small amount of trial and error, we find there is one possibility to within permutations of the rows and changes in sign. This is

$$D = \begin{bmatrix} 1 & 0 \\ \pm 1 & \pm 1 \\ \pm 2 & \pm 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

This shows there is exactly one character whose degree is not congruent to zero (mod 3) other than χ_0 and the four characters of degree 7. This is the character corresponding to the second row. As $B_0(3)$ is the only 3-block of full defect, these six characters are the only ones whose degrees are not divisible by 3. This new character must therefore be in $B_0(7)$ or the degree equation could not be satisfied.

We are now in a position to obtain a contradiction if $a \geqq 2$ where $g = 7\cdot 5^a\cdot 3^b\cdot 2^c$. Certainly if $a \geqq 2$, $\chi \mid Q$ is irrational by Schur's theorem. Here Q is a 5-Sylow group of G. Suppose χ is not real. By Lemma 2.1 there are at least four conjugates of χ. The conclusions of the above paragraph apply. Let $P_2(\chi) = \sum a_i\eta_i$ where η_i are irreducible characters of G, $P_2(\chi)$ is the character corresponding to the symmetric tensors of rank two for X. As $\chi \neq \overline{\chi}$ none of the η_i are linear. By (2.3) no character of degree 7 can be a constituent of $P_2(\chi)$. As 28 is not divisible by 3, $P_2(\chi)$ is reducible by the above paragraph. In fact it is impossible to write $P_2(\chi)$ as a sum of characters satisfying the above paragraph. This means $\chi \mid Q$ is real.

Suppose $a \geqq 3$ where $g = 7\cdot 5^a\cdot 3^b\cdot 2^c$. By [25, Th. 3.1], $\chi \mid Q$ is not real. The above paragraph applies and eliminates this case. Therefore $a \leqq 2$. Sylow's theorem gives $a = 2$ or 0.

We first treat the case $a = 2$. By the paragraphs above we know χ is real. Therefore $\chi\overline{\chi} = \chi\chi = P_2(\chi) + C_2(\chi)$ where $P_2(\chi)$ is the character corresponding to the symmetric tensors of rank two. As

χ_1 is a constituent of $\chi\chi$ we see by considering degrees that $P_2(\chi) = \chi_0 + \chi_1$. As χ_0 and χ_1 are rational this means $P_2(\chi)$ is rational. Using the formula for $P_2(\chi)$ we have that $\chi^2(R) + \chi(R^2)$ is rational for any $R \in G$. We prove a lemma regarding this situation. We assume $\chi \mid Q$ is real.

LEMMA 5.1. *If $P_2(\chi) \mid Q$ is rational, Q is cyclic. Here Q is the 5-Sylow group of G, $P_2(\chi)$ is the character associated with the symmetric tensors.*

Proof. Let π be an element of order 5. Let σ be an automorphism of $R[\lambda]$ mapping λ to λ^2 where $\lambda = e^{2\pi i/5}$, R = rationals. We have $\chi^\sigma(\pi) = \chi(\pi^2)$, $\chi^{\sigma^2}(\pi) = \overline{\chi(\pi)} = \chi(\pi)$. Also

$$(\chi^2(\pi) + \chi(\pi^2))^\sigma = \chi^2(\pi) + \chi(\pi^2) ,$$
$$(\chi^\sigma)^2(\pi) + \chi(\pi) = \chi^2(\pi) + \chi^\sigma(\pi) ,$$
$$(\chi^\sigma - \chi)(\chi^\sigma + \chi - 1)(\pi) = 0 .$$

This implies $\chi^\sigma(\pi) = \chi(\pi)$ or $(\chi^\sigma + \chi - 1)(\pi) = 0$. Assume first $(\chi^\sigma + \chi)(\pi) = 1$. This will be true also for π, π^2, π^3, π^4. Therefore

$$(\chi^\sigma + \chi)(\pi + \pi^2 + \pi^3 + \pi^4 + e)$$

is 18 giving a contradiction. Therefore $\chi^\sigma(\pi) = \chi(\pi)$. In particular χ is rational when restricted to elements or order 5 in Q. As Q is abelian it must be cyclic by Schur's theorem [21]. This completes the proof of the lemma.

As there are no elements of order 125, $|Q| \leqq 25$. Let π_1 be a generator such that $(\pi_1)^5 = \pi$. We know $\chi(\pi) = 2$. Suppose

$$\chi \mid Q = \lambda_1 + \overline{\lambda}_1 + \lambda_2 + \overline{\lambda}_2 + \lambda_3 + \overline{\lambda}_3 + \lambda_0 ,$$

where the λ_i are linear characters of Q, λ_0 is the trivial character. Only two of $\lambda_1, \lambda_2, \lambda_3$ can represent π_1 by a primitive 25-th root of unity. This means there are at least five conjugates of χ contradicting the above paragraphs.

We have shown then that $g = 7 \cdot 3^3 \cdot 2^c$. Sylow's theorem gives $c = 3, 6, 9$. The cases 3,6 are well within the known range under 20,000 [19] and there are no simple groups with these orders. If $c = 9$ we again mention [6, 1H] which shows $2^c \leqq 12 \cdot 3^3 = 324$.

This eliminates all cases for which $b = 3$, Case ID(ii) of the flow chart. We begin now Case ID(iii) of the flow chart in which $b = 4$. We show there is exactly one group of this form $S_6(2)$.

We are considering groups whose orders are $7 \cdot 5^a \cdot 3^4 \cdot 2^c$. Sylow's theorem gives $a \equiv 1 \pmod 2$. As $a \leqq 6$ we have $a = 1, 3,$ or 5. We will first show that $a \neq 5$.

Suppose then that $a = 5$. Let Q be a 5-Sylow group. We know from [25, Th. 2.1] that Q is abelian. By [25, Corollary 2.8] an elementary abelian subgroup of Q has order at most 5^4. In particular Q is not elementary abelian. Let π_1 be an element of order 25.

We know χ_1 is a rational character of degree 27. Let

$$\chi_1 \mid Q = \sum_{i=1}^{27} \lambda_i$$

where the λ_i, $i = 1, 2, \cdots, 27$ are linear characters of Q. As χ_1 is faithful there is an i such that $\lambda_i(\pi_1)$ is a primitive 25th root of 1. Each of the 20 conjugates of λ_i must appear as constituents in $\chi_1 \mid Q$ as χ_1 is rational. Let K be the kernel of λ_i. As there are no elements of order 125 in Q, K has order 125 and $Q \cong K \times \langle \pi_1 \rangle$. Clearly K is in the kernel of each of the conjugates of λ_i. If $\lambda_1, \cdots, \lambda_7$ are the characters λ_j not conjugate to λ_i we have $y = \sum_{i=1}^{7} \lambda_i$ a faithful rational representation of K. As K has order 5^3 this is impossible by Schur's theorem [21]. We have proved that $a \neq 5$ and so a must be 1 or 3.

The character χ_1 of degree 27 is in a 3-block of defect 1. Let $B_1(3)$ be the 3-block containing χ_1. As the defect group is cyclic of order 3 we may apply results in [13]. There are exactly three characters in $B_1(3)$. The three characters in $B_1(3)$ may all be of degree 27, in which case two are nonreal, or there will be two degrees y_1, y_2 besides 27 such that $27 + y_1 = y_2$. The degrees y_1 and y_2 will be divisible by 3^3 but not 3^4. By checking the various possibilities for the degrees of representations of the groups we are considering we find exactly one possibility. This is $y_1 = 189$, $y_2 = 216$. Block separation has been used in this elimination. This means that $B_0(7)$ contains characters with degrees 1, 27, 216 or characters with degrees 1, 27, 27, 27. In the latter case the tree has a branch.

It is now possible to eliminate all but eleven possible degree equations by straightforward techniques as described earlier. We do not include the details but list the degree equations not eliminated.

1. $1 + 15 + 120 + 512 = 27 + 216 + 405$
2. $1 + 64 + 36 + 162 = 27 + 216 + 20$
3. $1 + 120 + 120 + 162 = 27 + 216 + 160$
4. $1 + 960 + 120 + 162 = 27 + 216 + 1000$
5. $1 + 64 + 64 + 162 = 48 + 27 + 216$
6. $1 + 8 + 120 + 162 = 27 + 216 + 48$
7. $1 + 512 + 288 + 162 = 27 + 216 + 720$
8. $1 + 120 + 162 = 27 + 216 + 20 + 20$
9. $1 + 120 = 27 + 27 + 27 + 20 + 20$
10. $1 + 120 + 120 = 27 + 27 + 27 + 160$
11. $1 + 960 + 120 = 27 + 27 + 27 + 1000.$

We now separate the cases $g = 7\cdot5\cdot3^4\cdot2^c$ and $g = 7\cdot5^3\cdot3^4\cdot2^c$. We begin with $g = 7\cdot5\cdot3^4\cdot2^c$ and show that all degree equations are impossible except (1) and that this leads to $S_6(2)$.

By Sylow's theorem $c = 3, 6, 9$. The only possible degree equations for $c = 3$ are (8) and (9) as the remaining equations contain degrees divisible by 16. Equations (8) and (9) can be eliminated using the two characters of degree 20 which must be in a 2-block of defect 1. However on the tree they are off the stem and so they must be complex conjugates. This is impossible. This leaves the two cases $g = 7\cdot5\cdot3^4\cdot2^6$ and $g = 7\cdot5\cdot3^4\cdot2^9$. We will do the latter case completely to obtain $S_6(2)$. The case $g = 7\cdot5\cdot3^4\cdot2^6$ can be done similarly.

We assume then that $g = 7\cdot5\cdot3^4\cdot2^9$ or $7\cdot5\cdot3^4\cdot2^6$. The results and notation of § 2 for groups of order $7\cdot5\cdot3^b\cdot2^c$ will be used, in particular equation (2.6) and (2.7). Let π be of order 5, $C(\pi) = \langle\pi\rangle \times V$. We know $|V| = 3^r\cdot2^s$. Let V_2 be a 2-Sylow group of V and V_3 a 3-Sylow group of V. The restriction $\chi \mid V_3$ has linear constituents by (2.6), (2.7) and so V_3 is abelian. By (2.6), (2.7), $X \mid V_3$ has at most 3-distinct linear characters. Using [5, 3D] we see $|V_3| \leqq 3^2$. In particular $r \leqq 2$. If V_2 is abelian, the variety[1] of $\chi \mid V_2$ is at most 3 and so $s \leqq 2$. In case 2.7 this is always the case. In the case 2.7, V_2 at any rate has a subgroup of index two which is abelian with variety at most 3. In this case $s \leqq 3$. If V_2 is in fact nonabelian $|V_3| = 2^3$, and $\theta(J) = -2$ where $J \in Z(V_2)$.

We also know that $\chi \mid V_3$ is rational. Therefore there can be no element of order 9 in V_3 as the variety is at most 3. Furthermore if T is of order 3, the eigenvalues of $X(T)$ must be $\lambda, \bar{\lambda}, 1, 1, 1, 1, 1$ where $\lambda = e^{2\pi i/3}$. In case (2.6) then $\varphi(T) = 1$, $\theta(T) = -1$; in case (2.7), $\varphi(T) = 1$, $(\varphi_1 + \varphi_2)(T) = -1$. In particular $3^2 \nmid |V|$. If there is an involution J which commutes with T, then $X(J)$ will have either six or two eigenvalues -1. In particular, the eigenvalues of $X(TJ)$ will be $\{1, 1, 1, 1, 1, -\lambda, -\bar{\lambda}\}$ or $\{-1, -1, -1, -1, -1, \lambda, -\bar{\lambda}\}$ for T or T^2. In each case Blichfeldt's theorem [1, p. 96] is contradicted. This means that V cannot contain an element of order 6. In particular if $|V|$ is divisible by 24 this is the case. In case (2.7), V is abelian and so $|V|$ is not divisible by six.

We now consider the case $g = 7\cdot5\cdot3^4\cdot2^9$. Let $w = |N(\pi)/C(\pi)|$. If $w = 2$ by Sylow's theorem $|V| \equiv 2 \pmod 5$. This means $|V| = 2$ or 12. If $|V| = 2$, V is abelian and so case (6.7) applies. However here φ_1 must equal φ_2 giving a contradiction. If $|V| = 12$, V must be nonabelian and so case (2.6) applies. Therefore V has an irreducible representation of degree 2 and so there is an element of order 6 in

[1] The variety is defined in [1] or in [24, Introduction].

V. This shows $w \neq 2$.

Suppose $g = 7 \cdot 5 \cdot 3^4 \cdot 2^9$ and $w = 4$. In this case Sylow's theorem gives $|V| \equiv 1 \pmod 5$. This shows $|V| = 1$ or 6. If $|V| = 1$ there would be one 5-block of full defect and so no character of degree 7. Therefore $|V| = 6$. As there can be no element of order 6 in V, V must be isomorphic to S_3 as there are only two nonisomorphic groups of order six. Case (2.6) applies.

The group V is generated by an element T of order 3 and an involution J such that $JTJ = T^{-1}$. There are three irreducible characters of V; η_0, η_1, θ. Here η_0 is the trivial character; $\eta_1(T) = 1$, $\eta_1(J) = -1$, $\eta_1(e) = 1$; $\theta(T) = -1$, $\theta(J) = 0$, $\theta(e) = 2$. There are therefore three 5-blocks of full defect: $B_0(5)$ corresponding to η_0, $B_1(5)$ corresponding to η_1, and $B_2(5)$ corresponding to θ. In each case $e = 4$ and so each block consists of 5 ordinary characters. The degrees of characters in $B_0(5)$ and $B_1(5)$ are congruent to $\pm 1 \pmod 5$. In $B_2(5)$ they are congruent to $\pm 2 \pmod 5$.

Suppose there is a character, say χ_2, of degree 162. Then $\chi_2 \in B_2(5)$ and so $\chi_2(\pi T) = \pm 1$. As χ_2 is of full 3-defect this is a contradiction [7, p. 579]. This shows that cases 2, 3, $\cdots$, 8 are impossible.

The cases 9, 10, 11 can be eliminated by block separation as χ_0 is the only possible character in $B_0(7) \cap B_0(5)$.

This leaves (1) as the only possible degree equation remaining. The degrees in $B_2(5)$ are so far 7, 27, 512. If χ is not rational the degree equation would be $7 + 7 + 27 + 512 = 553 = 7.79$ which is absurd. Therefore χ is rational.

It is now possible to show $G \cong S_6(2)$. Let J be an involution in V. As $\theta(J) = 0$, $\varphi(J) = -1$ by the unimodularity of $X(J)$. Therefore $X(J)$ has six eigenvalues -1 and one eigenvalue 1. As G is simple the conjugates of J generate G.

We consider the group $\tilde{G} = G \times Z_2$ and a representation $\tilde{X}$ of $\tilde{G}$ given by $\tilde{X}(a, b) = X(a)\eta(b)$ where $a \in G$, $b \in Z_2$. Here η is the nontrivial character of Z_2, Z_2 is the group of order 2. As $\tilde{X}$ is rational and of odd degree it can be written in the real field [22]. We may assume the matrices are orthogonal. If $Z_2 = \langle J_1 \rangle$, $\tilde{X}(JJ_1)$ is a reflection in R^7. The same is true of any conjugate. The group generated by these conjugates is $\tilde{G}$ as can be quickly checked. This group is then a group generated by reflections in R^7. These groups have all been classified as Weyl groups of certain Dynkin diagrams containing seven elements [10, 11, 26]. These are A_7, B_7, D_2, E_7. The only group with the correct order is the Weyl group of E_7. It is known that the Weyl group of E_7 has a subgroup of index 2 isomorphic to $S_6(2)$ and that it has a complex irreducible representation of degree 7 [14].

The case $g = 7\cdot5\cdot3^4\cdot2^6$ can be handled similarly. All degree equations can be eliminated.

We now proceed to the case $g = 7\cdot5^3\cdot3^4\cdot2^c$. Sylow's theorem gives $c = 4, 7, 10$. The cases 1, 2, 6, 7, 8, 9 can be eliminated by a routine use of the established techniques. For example in cases 1 and 7 the character of degree 512 implies $c = 10$. However the character of degree 512 is then of 2-defect 1 and so must occur with multiplicity two. In cases 2, 8, and 9 the characters of degree 20 cannot be in $B_0(5)$ and the cases are eliminated by 5-block separation. In case 6 there is a rational representation of degree 8 contradicting [21].

The remaining cases 3, 4, 5, 10, 11 will be eliminated by examining a 3-Sylow group of G. It will be shown there is a self centralizing element π_9 of order 9. As $5^3 \mid g$ we know that $\chi \mid P_5$ is not real by [25, Th. 3.1]. By Lemma 2.1 there are at least four conjugates of χ. The cases can be eliminated by showing it is impossible to complete the π_9 column of the character table. The decompositions of $\chi\bar{\chi}$ and $\chi\chi$ together with equations (2.1)–(2.3) will be used. As no groups arise we sometimes only sketch the arguments.

We first show there is a self centralizing element of order 9. Let T generate the defect group of $B_1(3)$. Then, as $B_1(3)$ is not of defect 0, $\chi_1(T) \neq 0$ by [3]. As $g\cdot\chi_2(T)/|C(T)|27$ is an algebraic integer $3^4 \nmid |C(T)|$. Therefore Q is nonabelian where Q is a 3-Sylow group of G. As earlier using the fact that $\chi \mid Q$ is rational we obtain $\chi \mid Q = \mu + \bar{\mu} + \lambda_0$ where μ has degree three. The representation U corresponding to μ can be written in monomial form.

There must be an abelian subgroup of order 27. Let M be any abelian subgroup of order 27. Let $\chi \mid M = \sum_{i=1}^{7} \zeta_i$. Suppose there is an element of order 9 in M. If ζ_1 represents it faithfully all six conjugates must appear in $\chi \mid M$. Therefore ζ_1 is faithful and M is cyclic. This is a contradiction and shows M is elementary abelian.

The matrices $U(M)$ may be picked as all diagonal matrices of the form

$$\begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix}, \quad (\rho_i)^3 = 1 .$$

If $R \notin M$ we may choose a basis so that

$$U(R) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ C & 0 & 0 \end{bmatrix}.$$

As $U(R^3) = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix}$ we have $C^3 = 1$. We may choose R so that $C = 1$ or ρ where $\rho = e^{2\pi i/3}$. Picking R with $C = \rho$ gives an element π_9 of order 9. The eigenvalues of $\chi(\pi_9)$ are distinct and so $|C(\pi_9)|$ is a 3-group by [5, 3F] and the fact $7 \nmid |C(\pi_9)|$. If $3^3 \,|\, |C(\pi_9)|$ there would be an abelian subgroup of order 27 which was not elementary. This means π_9 is self centralizing. We have also shown $\chi(\pi_9) = 1$.

The value of χ_0 and the four characters of degree 7 on π_9 is 1. Let $\eta_1, \cdots, \eta_t$ be the remaining characters of G for which $\eta_i(\pi_9) \neq 0$. Let $\eta_i(\pi_9) = b_i$. The orthogonality relations give

$$\sum_{i=1}^{t} b_i \bar{b}_i = 4 \qquad (5.5)$$
$$\sum_{i=1}^{t} b_i \eta_i(e) = -29 \,.$$

There must be at least one character from $B_0(7)$ amongst the η_i and in the 5 remaining cases such degrees can be given explicitly. In case 5 the characters of degree 64 must be amongst the η_i. In cases 3 and 10 the character of degree 160 and in cases 4 and 11 the character of degree 1000 must be one of the η_i. From the tree in each case it can be seen the value is real and except for case 5 is rational and so the value b_i can be obtained by its congruence mod 3.

We now show there are at least two η_i occurring as constituents of $\chi\chi$. As $\chi\bar{\chi}$ has at least three constituents the same is true of $\chi\chi$. This implies $P_2(\chi)$ or $C_2(\chi)$ must be reducible. From (2.2) and (2.3) we see that $P_2(\chi)$ has no constituents of degree 7 and $C_2(\chi)$ has at most one. If $P_2(\chi)$ is reducible there are two constituents of degree 14. As $P_2(\chi)(\pi_9)$ is 1 they are not both equal. If $C_2(\chi)$ is reducible there is one constituent of degree 7 and one of degree 14. In any case there are at least two new constituents. Their values on π_9 can be readily evaluated. In each case it is now impossible to satisfy (5.5). This completes the final case in Section ID of the flow chart.

6. The case $s = 2$. In this section we consider the case $s = 2$. There is one group PSL (2,8), of order 504, of this kind, case II in Theorem I. This is section II of the flow chart. There are five characters in $B_0(7)$, χ_0 the trivial character, χ_1 of degree congruent to ± 1 (mod 7) and three exceptional characters $\chi_0^1, \chi_0^2, \chi_0^3$ whose degrees are congruent to ± 2 (mod 7). There are two possible degree equations

$$\text{(a)} \quad 1 + x_0^1 = x_1 \quad \text{or} \quad \text{(b)} \quad 1 + x_1 = x_0^1$$

where $x_1 = $ degree χ_1, $x_0^1 = $ degree χ_0^1.

If χ is the character of degree 7 set

$$\chi\bar{\chi} = \chi_0 + a\chi_1 + b(\chi_0^1 + \chi_0^2 + \chi_0^3) + \eta \qquad (6.1)$$

where η has constituents of zero 7-defect. Here a and b are nonnegative integers. As the degree of $\chi\bar{\chi}$ is 49 and the degree of η is divisible by 7 it is clear that either $a \neq 0$ and $x_1 \equiv -1 \pmod 7$ or $b \neq 0$ and $x_0^1 \equiv 2 \pmod 7$. This means that $x_1 = 6, 20, 27, 48$ or $x_0^1 = 2, 9, 16$. The only degree equations possible for G are then

(a) $1 + 5 = 6$, (b) $1 + 1 = 2$, (c) $1 + 8 = 9$, (d) $1 + 15 = 16$.

The first case (a) is impossible by [5] or [17]. In case (b) G' is of index 2. But then G' has a normal 7-complement and is not simple [5]. It can also be eliminated by [1] or [2]. In case (c) 2-block and 3-block separation imply $g = 7 \cdot 5^a \cdot 3^2 \cdot 2^3$. As there is a rational character of degree 8 $a \leqq 2$ by Schur [21]. Sylow's theorem gives $a = 0$. There is one simple group PSL (2,8) of order 504 [19]. It has a representation of degree 7 by [15]. One can also work out the character table quite easily.

In case (d) the character χ_1 of degree 15 is rational. A 5-Sylow group is abelian. Therefore χ_1 cannot be in $B_0(5)$ as $\chi_1(S)/15 \equiv 1 \pmod 5$ for any 5-element S. This would imply $\chi_1(S) = 15$ or $\chi_1(S) = -10$. Neither are possible in G. This argument is similar to one in § 4 where a character of degree 20 was involved. If $\chi_1 \notin B_0(5)$, 5-block separation implies $g = 7 \cdot 5 \cdot 3^b \cdot 2^c$. However the four characters χ_0, χ_0^1, χ_0^2, χ_0^3 are all in $B_0(5)$. By [2, Th. 11] there must be one further character in $B_0(5)$ of degree 49. This is a contradiction and completes this section.

7. The case $s = 3$. We now consider the case $s = 3$, $G = G'$. Thus G is simple by [5, 8A]. This is section III of the flow chart. In this case $B_0(7)$ contains χ_0, two characters χ_1, χ_2 whose degrees are congruent to $\pm 1 \pmod 7$, and two exceptional characters χ_0^1, χ_0^2 whose degrees are congruent to $\pm 3 \pmod 7$. If $x_i =$ degree χ_i, $x_0^i =$ degree χ_0^i for $i = 1, 2$ the degree equation becomes one of

$$\begin{array}{llll} & \text{(a)} & 1 + x_2 = x_1 + x_0^1 & x_1 \equiv -1 \pmod 7,\ x_2 \equiv 1 \pmod 7\,, \\ (7.1) & \text{(b)} & 1 + x_0^1 = x_1 + x_2 & x_1, x_2 \equiv -1 \pmod 7\,, \\ & \text{(c)} & 1 + x_1 + x_2 = x_0^1 & x_1 = x_2 \equiv 1 \pmod 7\,. \end{array}$$

As in §§ 2 and 6 we have

$$\chi\bar{\chi} = \chi_0 + a_1\chi_1 + a_2\chi_2 + b(\chi_0^1 + \chi_0^2) + \eta \tag{7.2}$$

where again a_1, a_2, b are nonnegative integers and the constituents of η are of zero 7-defect. We may assume that either (i) $a_1 \neq 0$, $x_1 \equiv -1 \pmod 7$ or (ii) $b \neq 0$ $x_0^1 \equiv 3 \pmod 7$. We may also assume that χ_1 or χ_0^1 adjoins χ_0 on the stem as $\chi\bar{\chi}$ is a sum of principal indecomposables [9]. The possibilities for x_1 in (i) are 6, 20, 27, 48. The possibilities for x_0^1 in (ii) are 3, 10, 24. It is clear that case c in (7.1) is impossible. This means the tree contains four real characters all on the stem.

If there is a character of degree 3, then $G \cong PSL\ (2,7)$ by [1, 2, or 17]. Also, $PSL\ (2,7)$ has a character of degree 7 by [15]. Alternatively, the character table for $PSL\ (2,7)$ can be quickly worked out. This is Case IV of Theorem I.

We will show that when $x_1 = 6$ $G \cong PSL\ (2,7)$; when $x_1 = 20$, $G \cong A_8$, when $x_1 = 27$, $G \cong U_3(3)$. The remaining cases can all be eliminated. The methods are similar to those of earlier chapters but in general much simpler because there are only two missing degrees in the degree equation. The details will not all be given.

Suppose that $x_1 = 6$ and χ_1 adjoins χ_0 on the stem. This means that $\chi_1\chi_1$ contains χ_2 as a constituent in case (a) or χ_0^1 as a constituent in case (b) by an argument involving the tree. In particular χ_2 or χ_0^1 has degree at most 21 as $(\chi_1)^2$ has constituents of degrees 15 and 21 corresponding to the symmetric and skew symmetric tensors. The possible degree equations are

$$\text{(i)}\quad 1 + 8 = 6 + 3 \quad \text{and} \quad \text{(ii)}\quad 1 + 15 = 6 + 10\,.$$

Case (i) is again by [1 or 2], $PSL\ (2,7)$.

In case (ii), χ_1 is in $B_0(5)$ by 5-block separation. [As χ_1 is rational, $5^2 \nmid g$ and $3^5 \nmid g$ by [21]. This means $g = 7\cdot 5\cdot 3^b\cdot 2^c$; $b \leqq 4$. As in § 2 we apply the results in [2, II, Th. 1] for the prime 5 this time with χ replaced by χ_1. Let $C(\pi) = \langle\pi\rangle \times V$ where π is a 5-element. Then $\chi_1 \mid V = \varphi_0 + 5\varphi_1$ where φ_0 is the trivial character of V, φ_1 is a linear character of V. It is immediate from the unimodularity of the representation corresponding to χ_1 that $\varphi_1 = \varphi_0$. This means $|V| = 1$. In particular there is only one 5-block of full defect by [3] and so $\chi \in B_0(5)$. This means $B_0(5)$ contains χ_0, χ_1, χ and a fourth character χ^* conjugate to χ.

As there are no elements of order $5\cdot 2$ or $5\cdot 3$ block separation applies to $B_0(5)$. If $b = 1$, $B_0(5) \cap B_0(3)$ contains three characters χ_0, χ, χ^*. This contradicts the theory of cyclic 3-defect groups. Therefore $b \geqq 2$. By block separation $B_0(5) \cap B_0(3) = B_0(5)$. In particular $\chi_1 \in B_0(3)$ and so $b \geqq 3$ [12, 90.19]. Let π_3 be an element of order 3 in the center of a 3-Sylow group. As χ_1 is rational, $\chi_1(\pi_3) = 3, 0, -3$. As $\chi_1 \in B_0(3)$, $\chi_1(\pi_3) = -3$. This implies $\chi(\pi_3) = \chi^*(\pi_3) = -2$. Computing $a(\pi, \pi, \pi_3)$ [6–3.1, 3.2] now gives a negative value and so a contradiction. This case is therefore impossible.

Suppose $x_0^1 = 10$ and χ_0^1 adjoins χ_0 on the stem. Using arguments involving the tree we see $x_2 \leqq 100$. The only degree equations possible are

$$(\text{i})\quad 1 + 15 = 6 + 10 \qquad (\text{ii})\quad 1 + 36 = 10 + 27\,.$$

Case (i) is eliminated as in the above paragraph. In case ii, χ_0^1 is rational when restricted to a 5-Sylow group and so $5^3 \nmid g$. The case $g = 7 \cdot 5^2 \cdot 3^b \cdot 2^c$ is eliminated by 5-block separation on χ_0^1 and χ_0^2. The case $7 \cdot 5 \cdot 3^b \cdot 2^c$ is eliminated by 5-block separation with $B_0(5)$.

If $\chi_1 = 48$ or $x_0^1 = 24$ an argument similar to that of § 3 applies to give $g = 7 \cdot 3^b \cdot 2^c$. This case can be eliminated as there must be a degree congruent to ± 1 of the form 2^β or 3^γ. These few cases can all be easily eliminated.

There are two cases remaining, $x_1 = 20$ and $x_1 = 27$ with χ_1 adjacent to χ_0 on the stem. We can apply the same tree arguments to see that χ_0^1 or χ_2 has degree at most 210 if $x_1 = 20$ and at most 378 if $x_1 = 27$. There are only a few possibilities and all but the following three can be easily eliminated using techniques already discussed.

(i) $1 + 64 = 20 + 45$
(ii) $1 + 32 = 27 + 6$
(iii) $1 + 50 = 27 + 24$.

In case (ii) the order is $7 \cdot 5^a \cdot 3^3 \cdot 2^6$ by block separation. As χ_1 has degree 6 and is rational $a \leqq 1$ by [21]. By Sylow's theorem $a = 0$. There is exactly one simple group with this order, $U_3(3)$, [19]. It is known to have a representation of degree 7. For example, a character table is given in [16].

In case (i), χ_1 of degree 20 is rational. As in § 4, χ_1 cannot be in $B_0(5)$. Block separation now gives $g = 7 \cdot 5 \cdot 3^b \cdot 2^6$. This means $b = 2$ or 8. For $b = 2$ the order is 20160. There are two known simple groups of this order A_8 and PSL (3,4). Only A_8 has a character of degree 7. The character table can be completed in a routine way to the character table of A_8. By [20], G must be A_8. The case $b = 8$ can be eliminated by closely examining $C(\pi)$ and noting $|V| = 2$, 12, or 72.

The third case is, curiously enough, quite troublesome. We do not give full details but just sketch the argument. Block separation on the characters of degrees 50 and 27 gives $g = 7 \cdot 5^2 \cdot 3^3 \cdot 2^c$. Sylow's theorem gives $c = 3, 6, 9$.

If χ is real $P_2(\chi) = \chi_0 + \chi_1$ and so $P_2(\chi)$ is rational. By Lemma 5.1 a 5-Sylow group is cyclic. This case can now be eliminated using [13]. This shows χ is not real. By Lemma 2.1 there are at least four conjugates of χ.

As in § 5 it can be shown that the 3-Sylow group is nonabelian. Let T be an element of order 3 not in the center. The decomposition numbers for $C(T)$ can be analyzed. The analysis is not as easy as in § 5 as there is no involution inverting a 7-element. As in § 5 there are two possible Cartan matrices (9) and $\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$. Using χ_0, χ_2, and the four characters of degree 7 and carefully analyzing the possible decomposition matrices this case can be eliminated.

8. The case $G \neq G'$. We now discuss the case in which $G \neq G'$. This is case IV of the flow chart. It is shown in [5] that G' is a simple group and so $X(G')$ is one of the representations already obtained. The candidates for G' are $PSL\ (2, 7)$, $PSL\ (2,8)$, A_8 and $U_3(3)$. G' could not be $S_6(2)$ as $|N(P)/C(P)| = 6$ in this case and so any extension would have an element in $C(\xi)$ where ξ is an element of order 7. For the same reason $|G: G'| = 2$ if G' is $PSL\ (2,7)$, A_8 or $U_3(3)$; $|G: G'| = 3$ if G' is $PSL\ (2,8)$.

If α is an element of G not in G' the map $\theta_\alpha: \xi \longrightarrow \xi^\alpha$, $\xi \in G'$ is an automorphism of G'. As α cannot commute with an element of order 7, θ_α cannot be the identity automorphism. It cannot be an inner automorphism as no element $\alpha\eta$, $\eta \in G'$, can commute with an element of order 7. Therefore θ_α is an outer automorphism. The automorphisms of our groups are all known. A very readable account without proofs using the fact they are all Chevalley groups can be found in [9].

In the case of $PSL\ (2,8)$, $(\theta_\alpha)^3$ is an inner automorphism as $\alpha^3 \in G'$. By taking $\theta_\alpha\eta$ where η is an inner automorphism we can assume $(\theta_\alpha)^3 = \theta_e$ as in G a 7-element is self centralizing. We see then that G is the semidirect product of G by $\langle\theta_\alpha\rangle$. If A is the automorphism group of $PSL\ (2,8)$, $I(A)$ the inner automorphism group, then $A/I(A)$ has an element of order 3 generated by a field automorphism. There are seven such extensions all isomorphic. We may assume then the extension is induced by a field automorphism. From the character table [15], $PSL\ (2,8)$ has four characters of degree 7 and so one must lift.

In the remaining case $(\theta_\alpha)^2$ is an inner automorphism as $\alpha^2 \in G'$. In each of the remaining cases there is exactly one element of order 2 in $A/I(A)$. It must be θ_α. By taking an element $\alpha\eta$ instead of α we can assume $(\theta_\alpha)^2 = \theta_e$ the identity automorphism. In this case α^2 must commute with all elements of G' and so must be e. This shows that G is uniquely determined as the semidirect product of G' and $\langle\alpha\rangle$ with the automorphism θ_α.

In each of the groups $PSL\ (2,7)$, A_8, and $U_3(3)$, there is exactly one rational character of degree 7. This means θ_α must leave it fixed and so the character of degree 7 can be lifted to G.

In the case of $PSL\ (2,7)$ there is certainly only one representation of degree 7 as the sum of the squares of the degrees in $B_0(7)$ is $168-7^2$. The element α can be taken $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Big/ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in the usual matrix form of PSL (2,7).

In the case of A_8 one may check the character table [18] to see there is only one representation of degree 7. However, this is not necessary because S_8 is known to have an irreducible unimodular representation of degree 7 and so G is S_8.

For the final case $U_3(3)$ one must check a character table [16]. Here there are three representations of degree 7, two of them are conjugate and the third rational. The automorphism is a field automorphism. The group G must be the group $G_2(2)$ as $G_2(2)$ cannot be $Z_2 \times G'$ or it would have an automorphism of order 2 [9].

FLOW CHART. $|G| = g = 7\cdot5^a\cdot3^b\cdot2^c$.

I. $s = 6$. $\chi\bar{\chi} = \chi_0 + \sum_{i=1}^6 a_i\chi_i + \eta$. $G = G'$.
- A(§ 2). One of χ_i, $i = 1, 2, \cdots, 6$ has degree 6.
- B(§ 3). Some $a_i = 1$, $x_i = 48$. We assume $a_1 = 1$, $x_1 = 48$.
  - (i) χ irrational on a 5-Sylow group.
  - (ii) $g = 7\cdot5\cdot3^b\cdot2^c$.
  - (iii) $g = 7\cdot3^b\cdot2^c$.
- C(§ 4). Some $a_i = 1$, $x_i = 20$. We assume $a_1 = 1$, $x_1 = 20$.
  - (i) $a \geqq 2$.
  - (ii) $a = 1$.
- D(§ 5). Some $a_i = 1$, $x_i = 27$. We assume $a_1 = 1$, $x_1 = 27$.
  - (i) χ restricted to a 3-Sylow group is irrational.
  - (ii) $b = 3$.
  - (iii) $b = 4$. (This case gives $S_6(2)$).

II. (§ 6). $s = 2$ $G = G'$. (This case gives $PSL_2(8)$.)

III. (§ 7). $s = 3$ $G = G'$. (This case gives $PSL_2(7)$, A_8, and $U_3(3)$.)

IV. (§ 8). $G \neq G'$. (This is VII of Theorem I).

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