

FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

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In this paper the Lie algebra analogues to groups with property E of Bechtell are investigated. Let \mathfrak{X} be the class of solvable Lie algebras with the following property: if H is a subalgebra of L , then $\phi(H) \subseteq \phi(L)$ where $\phi(L)$ denotes the Frattini subalgebra of L ; that is, $\phi(L)$ is the intersection of all maximal subalgebras of L . Groups with the analogous property are called E -groups by Bechtell. The class \mathfrak{X} is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal N of $L \in \mathfrak{X}$ be the Frattini subalgebra of L . Only solvable Lie algebras of finite dimension are considered here.

The following notation will be used. We let $N(L)$ be the nil radical of L and $S(L)$ be the socle of L ; that is, $S(L)$ is the union of all minimal ideals of L . If A and B are subalgebras of L , let $Z_B(A)$ be the centralizer of A in B . The center of A will be denoted by $Z(A)$. If $[B, A] \subseteq A$, we let $\text{Ad}_A(B) = \{\text{ad } b \text{ restricted to } A; \text{ for all } b \in B\}$. L' will be the derived algebra of L and $L'' = (L)'$.

PROPOSITION 1. *Let L be a Lie algebra such that L' is nilpotent. Then the following are equivalent:*

- (1) $\phi(L) = 0$.
- (2) $N(L) = S(L)$ and $N(L)$ is complemented by a subalgebra.
- (3) L' is abelian, is a semi-simple L -module and is complemented by a subalgebra.

Under these conditions, Cartan subalgebras of L are exactly those subalgebras complementary to L' .

Proof. Assume (1) holds. Nilpotency of L' implies $\phi(L) \supseteq L''$, so L' is abelian and may be regarded as an L/L' -module. We may assume $L' = \sum \bigoplus V_\rho$, V_ρ indecomposable L/L' -submodules. If M is a maximal subalgebra of L and if $V_\rho \not\subseteq M$, then $M \cap V_\rho$ is an ideal of L . If S is an L/L' -submodule of V_ρ properly contained between $M \cap V_\rho$ and V_ρ , then $M + S$ is a subalgebra of L properly contained between M and L , contradicting the maximality of M . Therefore M contains all maximal submodules of V_ρ for each ρ . Then $\phi(L) = 0$ implies the intersection of all maximal submodules of V_ρ is zero for each ρ . If V_1, \dots, V_s are maximal submodules of V_ρ with $V_1 \cap \dots \cap V_s = 0$ and are minimal with respect to this property, we have $V = V_2 \cap \dots \cap V_s \neq 0$

and $V \cap V_1 = 0$ so that $V \oplus V_1 = V_\rho$, contradicting indecomposability. Therefore each V_ρ is irreducible and L' is a completely reducible L/L' -module and is also a completely reducible L -module. Since L is solvable it contains Cartan subalgebras by Theorem 3 of [1]. Let H be a Cartan subalgebra of L and let L_0 and L_1 be the Fitting null and one component of L with respect to H . Then $L = L_0 + L_1 = H + L_1 \cong H + L'$ shows $L = H + L'$. We claim that $H \cap L' = 0$. If $H \cap L' \neq 0$, then, since L' is abelian, H is nilpotent and L' is a completely reducible L -module, L' is a sum of irreducible H -modules, U_1, \dots, U_r , such that for each U_i $[\dots \underbrace{[U_i, H] \dots}_k H] = 0$ for some k , hence $[U_i, H] = 0$.

Thus $[H, L' \cap H] = 0$. One sees that each U_i is a central minimal ideal of L , and since $\phi(L) = 0$, U_i is complemented by a maximal subalgebra M . Therefore U_i is a one-dimensional direct summand of L , contradicting $U_i \subseteq L'$. Hence $L' \cap H = 0$ and H is a complement to L' in L . Since $[H, H] \subseteq H \cap L' = 0$, H is abelian. Any minimal ideal not in L' satisfies $[L, A] \subseteq A \cap L' = 0$, so is central. Therefore $S(L) = L' + Z(L)$ and, since H is a Cartan subalgebra, $Z(L) \subseteq H$. Let H_0 be a complementary subspace to $Z(L)$ in H . One sees that $N(L) = L' + Z(L) + (N(L) \cap H_0) = S(L) + (N(L) \cap H_0)$. If h is a nonzero element in $N(L) \cap H_0$, $ad h$ is nilpotent but not zero which implies $[V_\rho, h] = V_\rho$ for some $V_\rho \subseteq L'$ and $[\dots \underbrace{[V_\rho, h] \dots}_k h] = 0$ for some k , a contradiction. Thus $S(L) = N(L)$ and H_0 is a complement. Consequently (1) implies (2).

Assume (2) holds and proceed by induction on the dimension of L . Since $L' \subseteq N(L) = S(L)$ and minimal ideals are abelian, L' is abelian. If every minimal ideal of L is contained in L' , then $S(L) = L'$ and (3) follows. Therefore let A be a minimal ideal of L such that $A \not\subseteq L'$. Hence $A \not\subseteq \phi(L)$ and there exists a maximal subalgebra M of L such that $L = M + A$. Since $[L, A] \subseteq A \cap L' = 0$, A is central, hence one-dimensional. It follows that L is the Lie algebra direct sum of M and A . Since M inherits the condition (2), M satisfies (3) by induction. It now follows that L also satisfies (3).

Assume (3) holds. Then L' is a sum of minimal ideals of L , which we denote by A_1, \dots, A_k , and $L = L' + H$, H a subalgebra of L . Since $H' \subseteq H \cap L' = 0$, H is abelian. One sees that $L' = [L', H]$ and, consequently, $A_i = [A_i, H]$ for all i . Since $Z_{A_i}(H)$ is central in L , $Z_{A_i}(H)$ is an ideal in L contained in A_i . Since $Z_{A_i}(H) \neq A_i$, $Z_{A_i}(H) = 0$. It follows that H is its own normalizer, hence is a Cartan subalgebra of L . Now $H + A_1 + \dots + \hat{A}_i + \dots + A_k$ is a maximal subalgebra of L since any containing algebra has a nonzero projection on A_i which is $ad H$ stable, hence equal to A_i . Therefore $\phi(L) \subseteq H$ and $\phi(L) \subseteq H \cap L' = 0$. Hence (1) holds.

That complements to L' are Cartan subalgebras is shown in (3) implies (1). That Cartan subalgebras are complements to L' is shown in (1) implies (2). This completes the proof of Proposition 1.

THEOREM 1. *Let L be a Lie algebra such that L' is nilpotent and $\phi(L) = 0$. Then, for any subalgebra M of L , $\phi(M) = 0$.*

Proof. Suppose $L' \subseteq M$. Let H be a complement to L' in L , so $H \cap M$ is a complement to L' in M . Since L acts completely reducibly on L' and L' is abelian, H acts completely reducibly on L' . Then, since H is abelian, $H \cap M$ acts completely reducibly on L' , hence so does M . Therefore $L' = M' \oplus A$ for some ideal A in M where M acts completely reducibly on M' and $A + (H \cap M)$ is a complementary subalgebra of M' in M . Thus by Proposition 1, $\phi(M) = 0$.

Suppose $L' \not\subseteq M$. Since $M + L'$ falls in the preceding case, we may assume $M + L' = L$. Since L' is abelian, $L' \cap M$ is an ideal in L , $M/(L' \cap M)$ complements $L'/(L' \cap M) = (L/L' \cap M)'$ in $L/(L' \cap M)$ and $M/(L' \cap M)$ acts completely reducibly in $L'/(L' \cap M)$, $M/(L' \cap M)$ is a Cartan subalgebra of $L/(L' \cap M)$. Let C be a Cartan subalgebra of M . By Lemma 4 of [1], C is a Cartan subalgebra of L . Thus C is a complement to L' and $C + (L' \cap M) = M$ since $C \subseteq M$. Hence C is a complement to $L' \cap M$ in M . Since M acts completely reducibly on $L' \cap M$ and $M' \subseteq L' \cap M$, M acts completely reducibly on M' , $L' \cap M = M' \oplus (L' \cap Z(M))$ and, since $Z(M) \subseteq C$, $Z(M) \cap L' \subseteq C \cap L' = 0$. Therefore $C = M' = M$ and $C \cap M' = 0$. Now M satisfies part (3) of Proposition 1, hence $\phi(M) = 0$.

If L is a solvable Lie algebra it has been shown in [2] that $\phi(L)$ is an ideal of L . We look for a condition on the subalgebras of $L/\phi(L)$ which are necessary and sufficient that $L \in \mathfrak{X}$. In order to do this the following concept is introduced.

We shall say that a Lie algebra L is the reduced partial sum of an ideal A and a subalgebra B if $L = A + B$ and for any subalgebra C of L such that $L = A + C$ and $C \subseteq B$ then $C = B$. It is noted that if $A \not\subseteq \phi(L)$, then there exists a $B \neq L$ such that L is the reduced partial sum of A and B . On the other hand, if $A \subseteq \phi(L)$ and L is the reduced partial sum of A and B , then $B = L$.

LEMMA 1. *Let L be the reduced partial sum of A and B . Then $A \cap B \subseteq \phi(B)$.*

Proof. Suppose $C = A \cap B \not\subseteq \phi(B)$. Then B contains a subalgebra D such that $C + D = B$. Then $L = A + B = A + C + D = A + D$. This contradicts the minimality of B .

LEMMA 2. *Let L be the reduced partial sum of A and B . Then $\phi(L/A) \simeq A + \phi(B)/A$.*

Proof. Since $A \cap B \subseteq \phi(B)$, $A \cap \phi(B) = A \cap B$. Since $L/A \simeq A + B/A \simeq B/A \cap B$, $\phi(L/A) \simeq \phi(B/A \cap B) \simeq \phi(B)/A \cap B = \phi(B)/A \cap \phi(B) \simeq A + \phi(B)/A$.

PROPOSITION 2. *The following are equivalent for the Lie algebra L :*

- (1) $L \in \mathfrak{X}$.
- (2) For any subalgebra H of $L/\phi(L)$, $\phi(H) = 0$.

Proof. Let L satisfy (1) and let $\pi: L \rightarrow L/\phi(L)$ be the natural homomorphism. Then $\phi(\pi(L)) = \pi(\phi(L)) = 0$. Let \bar{W} be a subalgebra of $L/\phi(L)$ and let W be the subalgebra of L which contains $\phi(L)$ and corresponds to \bar{W} . Since L satisfies (1), $\phi(W) \subseteq \phi(L)$. If $\phi(W) = \phi(L)$, then $\phi(\pi(W)) = \pi(\phi(W)) = \pi(\phi(L)) = 0$. Suppose then that $\phi(W) \subset \phi(L)$. Then W can be represented as a reduced partial sum $W = \phi(L) + K$. Let T be a subalgebra of W such that $T/\phi(L) \simeq \phi(W/\phi(L))$. If $T/\phi(L) \neq 0$, then $T = T \cap (\phi(L) + K) = (T \cap \phi(L)) + (T \cap K) = \phi(L) + (T \cap K)$. Consequently there exists an $x \in T \cap K$, $x \notin \phi(L)$. Since $\phi(K) \subseteq \phi(L)$, $x \notin \phi(K)$ and there exists a maximum subalgebra S of K such that $x \in S$. We claim that either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in W . Suppose $\phi(L) + S \neq W$ and let J be a subalgebra of W which contains $\phi(L) + S$. Then $S \subseteq J \cap K$, so, by the maximality of S , either $J \cap K = S$ or $J \cap K = K$. If $J \cap K = S$, then $\phi(L) + S = \phi(L) + (J \cap K) = J \cap (\phi(L) + K) = J \cap W = J$. If $J \cap K = K$, then $J \supseteq K$ and, since $J \supseteq \phi(L)$, $J \supseteq \phi(L) + K = W$, hence $J = W$. Consequently there exist no subalgebras of W properly contained between $\phi(L) + S$ and W , hence either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in W . If $\phi(L) + S = W$, then $\phi(L) + K$ is not a reduced partial sum which is a contradiction. If $\phi(L) + S$ is maximal in W , then $\phi(L) + S/\phi(L) \supseteq \phi(W/\phi(L)) \simeq T/\phi(L)$. Hence $T \subseteq \phi(L) + S$. Since $S \subseteq \phi(L) + S$ and $x \in T \cap K \subset T \subseteq \phi(L) + S$, $K = \{S, x\} \subseteq \phi(L) + S$. Then $W = \phi(L) + K \subseteq \phi(L) + S \subseteq W$ implies $\phi(L) + K$ is not a reduced partial sum, a contradiction. Hence $\phi(\bar{W}) = T/\phi(L) = 0$ and (2) is satisfied.

If $L/\phi(L)$ satisfies (2), then $\pi(\phi(H)) \subseteq \phi(\pi(H)) = 0$ for every subalgebra H of L . Then $\phi(H) \subseteq \phi(L)$ for every subalgebra H of L .

Combining Proposition 2 and Theorem 1 we have

THEOREM 2. *Let L be a Lie algebra such that L' is nilpotent. Then $L \in \mathfrak{X}$.*

THEOREM 3. *Let $L \in \mathfrak{X}$ and let T be a Lie homomorphism of L .*

Then $T(\phi(L)) = \phi(T(L))$.

Proof. $T(\phi(L))$ is always contained in $\phi(T(L))$ by Proposition 1 in [6]. If $N = \text{kernel } T \subseteq \phi(L)$, then equality holds by Proposition 2 in [6]. Suppose $N \not\subseteq \phi(L)$. Let $L = N + K$ be a reduced partial sum. Using Lemma 2, $\phi(T(L)) = \phi(L/N) \simeq N + \phi(K)/N = T(\phi(K))$. Since $T(N + \phi(L)) = T(\phi(L)) \subseteq \phi(T(L)) = \phi(L/N) = T(N + \phi(K))$, $N + \phi(L) \subseteq N + \phi(K) \subseteq N + \phi(L)$. Hence $N + \phi(L) = N + \phi(K)$ and $\phi(T(L)) = T(\phi(K)) = T(\phi(L))$.

THEOREM 4. *Let $L \in \mathfrak{X}$. Necessary conditions that an ideal N of L be the Frattini subalgebra of L are that*

- (1) $\phi(\text{Ad}_N(L)) = \text{Ad}_N(\phi(L))$.
- (2) *There exists a subalgebra M of L such that $M/N \simeq \text{Ad}_N(L)/\text{Ad}_N(\phi(L))$.*

Proof. (1) Let T be the mapping from L into the derivation algebra of N by $T(x) = \text{ad } x$ restricted to N for all $x \in L$. Then $T(\phi(L)) = \text{Ad}_N(\phi(L)) = \phi(T(L)) = \phi(\text{Ad}_N(L))$.

(2) Let $M = Z_L(\phi(L))$. Suppose that $M \not\subseteq \phi(L)$ and let $F = L/\phi(L)$ and $A = (M + \phi(L))/\phi(L)$. Since $\text{Ad}_{\phi(L)}(L) \simeq L/M$ and $\text{Ad}_{\phi(L)}(\phi(L)) \simeq \phi(L)/Z(\phi(L)) = \phi(L)/M \cap \phi(L) = (M + \phi(L))/M$, $F/A \simeq (L/\phi(L))/(M + \phi(L))/\phi(L) \simeq L/(M + \phi(L)) \simeq (L/M)/((M + \phi(L))/M) \simeq \text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L))$. Since $\phi(F) = 0$, there exists a subalgebra D in F such that F is the reduced partial sum of A and D . Using Proposition 2 and Lemma 1, $A \cap D \subseteq \phi(D) = 0$, hence $A \cap D = 0$. Let E be the subalgebra of L which contains $\phi(L)$ and corresponds to D . Then $E/\phi(L) \simeq D \simeq F/A \simeq \text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L))$. If $M \subseteq \phi(L)$, then $\text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L)) \simeq (L/M)/(\phi(L)/Z(\phi(L))) = (L/M)/(\phi(L)/M \cap \phi(L)) = (L/M)/(\phi(L)/M) \simeq L/\phi(L)$.

Related to part (1) of Theorem 4 are the following results.

THEOREM 5. *Let $L \in \mathfrak{X}$ and let K be an ideal of L containing $\phi(L)$. Then $\phi(\text{Ad}_K(L)) \simeq \text{Ad}_K(K)$ if and only if $K = \phi(L) + Z(K)$.*

Proof. Let T be the Lie homomorphism from L into the derivation algebra of K given by $T(x) = \text{ad } x$ restricted to K for each $x \in L$. Then $\phi(\text{Ad}_K(L)) = \phi(T(L)) = T(\phi(L)) = \text{Ad}_K(\phi(L)) \simeq \phi(L)/Z_{\phi(L)}(K) = \phi(L)/(Z(K) \cap \phi(L)) \simeq (\phi(L) + Z(K))/Z(K)$. If $\phi(L) + Z(K) = K$, then $\text{Ad}_K(K) \simeq K/Z(K) = (\phi(L) + Z(K))/Z(K) \simeq \phi(\text{Ad}_K(L))$. If $\phi(L) + Z(K) \subset K$, then $\text{Ad}_K(K) \simeq K/Z(K) \supset (\phi(L) + Z(K))/Z(K) = \phi(\text{Ad}_K(L))$.

THEOREM 6. *Let $L \in \mathfrak{X}$ and let A be an ideal of L contained in $\phi(L)$. Then $\phi(\text{Ad}_A(L)) \simeq \text{Ad}_A(A)$ if and only if $\phi(L) = A + Z_{\phi(L)}(A)$.*

Proof. If $\phi(L) = A + Z_{\phi(L)}(A)$, then $\text{Ad}_A(A) = \text{Ad}_A(\phi(L)) = T(\phi(L)) = \phi(T(L)) = \phi(\text{Ad}_A(L))$.

Conversely, $\text{Ad}_A(L) \simeq L/Z_L(A)$ and $Z_L(A) + A/Z_L(A) \simeq A/Z(A) = \text{Ad}_A(A)$. Then $L/Z_L(A) + A \simeq \text{Ad}_A(L)/\text{Ad}_A(A)$ and $\phi(L/Z_L(A) + A) \simeq \phi(\text{Ad}_A(L)/\text{Ad}_A(A)) = \phi(\text{Ad}_A(L))/\text{Ad}_A(A) = 0$. Hence $\phi(L) \subseteq Z_L(A) + A$ and $\phi(L) = Z_{\phi(L)}(A) + A$.

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