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SPHERE TRANSITIVE STRUCTURES AND THE TRIALITY AUTOMORPHISM

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Let G be a compact connected Lie group which acts transitively and effectively on a sphere S^{n-1}. A manifold M is said to have a sphere transitive structure if the structure group of the tangent bundle of M can be reduced from $O(n)$ to G. The study of the existence of such structures is a generalization of the well-known problem of the existence of almost complex structures. We completely solve the question of existence of sphere transitive structures on spheres.

For our study of sphere transitive structures we need to know some facts about the triality automorphism λ of Spin (8). We completely determine the cohomology homomorphism induced by λ on the cohomology of the classifying space of Spin (8).

Berger [1] has classified the holonomy groups of manifolds having an affine connection with zero torsion. Either from this classification or directly from Simons [11], it follows that the holonomy group of an irreducible Riemannian manifold which is not a symmetric space acts transitively on a sphere.

On the other hand we have the following elementary fact: if the holonomy group of a Riemannian manifold M is G, then the structure group of the tangent bundle of M can be reduced to G. Therefore a more fundamental question than whether or not a Riemannian manifold M has a given Lie group G as its holonomy group is the question of the reduction of the structure group of the tangent bundle of M to G. In this paper we consider the latter question and give some necessary conditions and some sufficient conditions in terms of characteristic classes. From the remarks above it suffices to consider the case when G is a connected Lie group which acts transitively and effectively on a sphere.

We introduce the following notions.

DEFINITIONS. Let $\xi = (E, M, p, F)$ be a vector bundle where M is a CW-complex and $\dim F = n$. Then a *sphere transitive reduction* is a reduction of the structure group $O(n)$ of ξ to a connected Lie subgroup G of $O(n)$ which acts transitively and effectively on the sphere S^{n-1}. In the special case when ξ is the tangent bundle of M we call the reduction a *sphere transitive structure* on M.

According to [10] the connected Lie groups G which act effectively and transitively on spheres are the following: $SO(n)$, $U(n)$,

$SU(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot SO(2)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, G_2, $\mathrm{Spin}(7)$, and $\mathrm{Spin}(9)$. We have

$$\begin{aligned} SO(n)/SO(n-1) &= S^{n-1},\ U(n)/U(n-1) = SU(n)/SU(n-1) = S^{2n-1}\,, \\ \mathrm{Sp}(n)/\mathrm{Sp}(n-1) &= \mathrm{Sp}(n)\cdot SO(2)/\mathrm{Sp}(n-1)\cdot SO(2) \\ &= \mathrm{Sp}(n)\cdot\mathrm{Sp}(1)/\mathrm{Sp}(n-1)\cdot\mathrm{Sp}(1) = S^{4n-1}\,, \\ G_2/SU(3) = S^6\,,\quad \mathrm{Spin}(7)/G_2 &= S^7\,,\quad \mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}\,. \end{aligned}$$

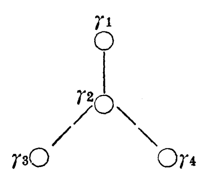
In § 2 we discuss the triality automorphism λ of Spin (8) and the cohomology of the self homeomorphism of the classifying space induced by λ. The results of § 2 are then used in § 3 to determine the cohomology of the classifying space $B\,\mathrm{Spin}(n)$ ($n = 7, 8, 9$) and a good deal of the cohomology of BG_2. Then we determine some necessary conditions for sphere transitive reductions for the cases $G = G_2$, Spin (7), Spin (9). In § 4 we discuss the existence of sphere transitive structures on certain homogeneous spaces. In particular we completely solve the problem of the existence of sphere transitive structures on spheres.

2. The cohomology of the triality automorphism. Spin (8) is the simply connected compact Lie group whose Lie algebra is of type D_4. Now D_4 is the unique simple Lie algebra with an outer automorphism of order 3. In fact, if $\mathrm{Aut}(D_4)$ (resp. $\mathrm{Inn}(D_4)$) denotes the group of all (resp. inner) automorphisms of D_4, then the factor group $\mathrm{Aut}(D_4)/\mathrm{Inn}(D_4)$ is isomorphic to the symmetric group on 3 letters. Let $\kappa, \lambda \in \mathrm{Aut}(D_4)$ be such that their images in $\mathrm{Aut}(D_4)/\mathrm{Inn}(D_4)$ generate this group and satisfy the relations $\lambda^3 = 1$, $\kappa^2 = 1$, $\kappa\lambda\kappa = \lambda^2$.

According to [7] it is possible to choose κ and λ so that the *principle of triality* holds. This means the following. Let V be the 8-dimensional algebra of Cayley numbers and denote the product of $x, y \in V$ by xy. Then for $A \in D_4$, $x, y \in V$ we have

$$(Ax)y + x(\lambda(A)y) = ((\lambda\kappa)(A))(xy)\,.$$

The Dynkin diagram of D_4 is



where $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is a simple system of roots of D_4. Since κ and λ are outer, they give rise to symmetries of the Dynkin diagram of D_4. It may be checked that $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ may be chosen so that

$\kappa(\gamma_1) = \gamma_1$, $\kappa(\gamma_2) = \gamma_2$, $\kappa(\gamma_3) = \gamma_4$, $\kappa(\gamma_4) = \gamma_3$, $\lambda(\gamma_1) = \gamma_3$, $\lambda(\gamma_2) = \gamma_2$, $\lambda(\gamma_3) = \gamma_4$, $\lambda(\gamma_4) = \gamma_1$. Henceforth we assume that the principal of triality holds and that the above choice of simple roots has been made.

Since Spin (8) is simply connected, λ and κ induce outer automorphisms of Spin (8); these in turn induce homeomorphisms of $B\,\mathrm{Spin}\,(8)$, which we continue to denote by λ and κ. In order to determine the cohomology of λ and κ, it will be convenient to use some cohomology classes introduced by Thomas [12]. Let p: $B\,\mathrm{Spin}\,(n) \rightarrow BSO(n)$ be the map defined by the covering homomorphism of Spin (n) over $SO(n)$. Denote by w_i the universal Stiefel-Whitney classes, by P_i the universal Pontryagin classes, and by X the Euler class of $BSO(8)$. Then $H^*(BSO(8), \boldsymbol{Z}) = \boldsymbol{Z}[P_1, P_2, P_3, X] +$ 2-torsion and $H^*(BSO(8), \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_2, \cdots, w_8]$. According to Thomas [12] there exist cohomology classes $Q_i \in H^*(B\,\mathrm{Spin}, \boldsymbol{Z})$ $(i = 1, 2, 3, 4)$ and $w_i^* \in H^*(B\,\mathrm{Spin}, \boldsymbol{Z}_2)$ $(i = 4, 6, 7, 8)$ (where Spin denotes the stable Spin group) such that

$$\begin{aligned} &p^*(P_1) = 2Q_1 && p^*(w_i) = w_i^* \ (i = 4, 6, 7, 8) \\ &p^*(P_2) = 2Q_2 + Q_1^2 && p^*(w_i) = 0 \quad (i = 2, 3, 5) \\ &p^*(P_3) = Q_3 && \rho_2(Q_1) = w_4^*, \ \rho_2(Q_2) = w_8^* \\ &p^*(P_4) = 2Q_4 + Q_2^2 && \rho_2(Q_3) = w_6^{*2}, \ \rho_4(Q_4) = w_{16}^* \,. \end{aligned}$$

The cohomology classes $Q_1, Q_2, Q_3, w_4^*, w_6^*, w_7^*, w_8^*$ give rise to the cohomology classes in $H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z})$ and $H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}_2)$ which we denote by the same letters.

THEOREM 2.1. (i) *There exist*

$$Y \in H^8(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}) \quad \textit{and} \quad \omega \in H^8(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}_2)$$

such that

$$\begin{aligned} H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}) &= \boldsymbol{Z}[Q_1, Q_2, Q_3, Y] + \text{2-torsion} \\ H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}_2) &= \boldsymbol{Z}_2[w_4^*, w_6^*, w_7^*, w_8^*, \omega] \,. \end{aligned}$$

Furthermore Y and ω can be chosen so that $p^*(X) = 2Y - Q_2$ and $\rho_2(Y) = \omega$.

(ii) *The cohomology homomorphisms λ^* and κ^* are given as follows:*

$$\begin{aligned} &\lambda^*(Q_1) = Q_1 \,, && \lambda^*(w_i^*) = w_i^* \ (i = 4, 6, 7) \,, \\ &\lambda^*(Q_2) = 3Y - 2Q_2 \,, && \lambda^*(w_8^*) = \omega \,, \\ &\lambda^*(Y) = Y - Q_2 \,, && \lambda^*(\omega) = w_8^* + \omega \,, \\ &\lambda^*(Q_3) = Q_3 + 2Q_1Y - 2Q_1Q_2 \,, && \kappa^*(w_i^*) = w_i^* \ (i = 4, 6, 7, 8) \,, \\ &\kappa^*(Q_i) = Q_i \ (i = 1, 2, 3) \,, && \kappa^*(\omega) = w_8^* + \omega \,, \\ &\kappa^*(Y) = -Y + Q_2 \,. && \end{aligned}$$

Before proving this theorem we state without proof a lemma which we shall need.

LEMMA 2.2. *Let $s: K \to L$ be a p^n-fold covering of a compact connected Lie group where p is a prime, and denote by*

$$s^*: H^*(BL, \boldsymbol{Z}) \longrightarrow H^*(BK, \boldsymbol{Z})$$

the corresponding cohomology homomorphism of classifying spaces. Let S be a subset of $H^*(BK, \boldsymbol{Z})$ such that S generates $s^*(H^*(BL, \boldsymbol{Z}))$ as a group (ring) and $\rho_p(S)$ generates $\rho_p(H^*(BK), \boldsymbol{Z}) \subsetneqq H^*(BK, \boldsymbol{Z}_p)$ as a group (ring). (ρ_p denotes reduction mod p.) *Then S generates $H^*(BK, \boldsymbol{Z})$ as a group (ring).*

Proof of Theorem 2.1. Using a result of Borel [2] it is not hard to see that w_4^*, w_6^*, w_7^*, and w_8^* are generators of

$$H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}_2)\ .$$

Furthermore if $\rho_0: \boldsymbol{Z} \to \boldsymbol{R}_0$ denotes the inclusion, where $\boldsymbol{R}_0$ is the rationals, then it is obvious that

$$H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{R}_0) = \boldsymbol{R}_0[\rho_0(Q_1), \rho_0(Q_2), \rho_0(Q_3), \rho_0(p^*(X))]\ .$$

We first establish part of (ii). The automorphism κ of Spin (8) gives rise to an outer automorphism $\tilde{\kappa}$ of $SO(8)$; this is the ordinary orientation reversing automorphism of $SO(8)$. The induced homomorphism $\tilde{\kappa}^*$ is the identity on $H^*(BSO(8), \boldsymbol{Z}_2)$ and satisfies $\tilde{\kappa}^*(P_i) = P_i$ $(i = 1, 2, 3)$, $\tilde{\kappa}^*(X) = -X$. Hence $\kappa^*(w_i^*) = w_i^*$ $(i = 4, 6, 7, 8)$, and $\kappa^*(Q_i) = Q_i$ $(i = 1, 2, 3)$. It is also easy to see that $\lambda^*(Q_1) = Q_1$ and $\lambda^*(w_i^*) = w_i^*$ for $i = 4, 6, 7$.

We may write

$$\lambda^*(P^*(\rho_0(X))) = a\rho_0(X) + b\rho_0(Q_2) + c\rho_0(Q_1^2)\ ,$$
$$\lambda^*(\rho_0(Q_2)) = d\rho_0(X) + e\rho_0(Q_2) + f\rho_0(Q_1^2)\ ,$$

where a, b, c, d, e, f are rational numbers. Using the facts that $\lambda^*(Q_1^2) = Q_1^2$, $\lambda^3 = 1$, $\kappa\lambda\kappa = \lambda^2$, and the knowledge of κ^*, we calculate that $c = f = 0$, $a = e = -1/2$, and $bd = -3/4$.

To compute b, d, and $\lambda^*(\rho_0(Q_3))$ we must resort to some calculations with roots. Let $\widetilde{Q}_1$, $\widetilde{Q}_2$, $\widetilde{Q}_3$, and $\widetilde{X}$ denote the real cohomology classes corresponding to Q_1, Q_2, Q_3, and $p^*(X)$. Then we may regard $\widetilde{Q}_1$, $\widetilde{Q}_2$, $\widetilde{Q}_3$ and $\widetilde{X}$ as polynominals on the Lie algebra of a maximal torus of Spin (8), i.e., polynomials in the roots of Spin (8). A calculation shows in fact that (if we write $\gamma_0 = -\gamma_1 - 2\gamma_2 - \gamma_3 - \gamma_4$),

$$\begin{aligned}
\widetilde{Q}_1 &= -2\varepsilon(\gamma_0^2 + \gamma_1^2 + \gamma_3^2 + \gamma_4^2) , \\
\widetilde{Q}_2 &= \varepsilon^2(-\gamma_0^2\gamma_1^2 - 2\gamma_3^2\gamma_4^2 + \gamma_0^2\gamma_3^2 + \gamma_0^2\gamma_4^2 + \gamma_1^2\gamma_3^2 + \gamma_1^2\gamma_4^2) , \\
\widetilde{X} &= \varepsilon^2(-\gamma_0^2\gamma_3^2 - \gamma_1^2\gamma_4^2 + \gamma_0^2\gamma_4^2 + \gamma_1^2\gamma_3^2) , \\
\widetilde{Q}_3 &= -2\varepsilon^3(\gamma_0^4\gamma_3^2 + \gamma_0^4\gamma_4^2 + \gamma_1^4\gamma_3^2 + \gamma_1^4\gamma_4^2 + \gamma_3^4\gamma_0^2 + \gamma_3^4\gamma_1^2 + \gamma_4^4\gamma_0^2 + \gamma_4^2\gamma_1^2 \\
&\quad -2\gamma_0^2\gamma_1^2\gamma_3^2 - 2\gamma_0^2\gamma_1^2\gamma_4^2 - 2\gamma_0^2\gamma_3^2\gamma_4^2 - 2\gamma_1^2\gamma_3^2\gamma_4^2) .
\end{aligned}$$

Thus we obtain

$$(*) \qquad \begin{cases}
\lambda^*(\rho_0(X)) = -\dfrac{1}{2}\rho_0(X) - \dfrac{1}{2}\rho_0(Q_2) , \\
\lambda^*(\rho_0(Q_2)) = \dfrac{3}{2}\rho_0(X) - \dfrac{1}{2}\rho_0(Q_2) , \\
\lambda^*(\rho_0(Q_3)) = \rho_0(Q_3) + \rho_0(Q_1X) - \rho_0(Q_1Q_2) .
\end{cases}$$

Define $Y = -\lambda^*(p^*(X))$ and $\omega = \rho_2(Y)$. Then $\lambda^*(w_8^*) = \omega$. From this, equations (*), and the fact that $H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z})$ has only 2-torsion, we obtain the rest of (ii).

From (ii) and Borel [2] we see that ω may be taken to be the remaining generator of $H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}_2)$. This fact together with (ii) and Lemma 2.2 imply (i).

3. The cohomology of $B\,\mathrm{Spin}\,(7)$, $B\,\mathrm{Spin}\,(9)$, and BG_2. We first compute the cohomology of $B\,\mathrm{Spin}\,(7)$ and its inclusion in $BSO(8)$. Actually there are two natural 8-dimensional representations of Spin (7) according to [8]. These are equivalent in $O(8)$ but not in $SO(8)$. Denote these representations by j_+ and j_-. In the terminology of [8] j_+ and j_- give rise to the two distinct 3-fold vector cross products on R^8. Let $i\colon \mathrm{Spin}\,(7) \to \mathrm{Spin}\,(8)$ be the natural inclusion. The following lemma [8], [13] will be necessary.

LEMMA 3.1. *We have the following commutative diagrams*

$$\begin{array}{ccc}
\mathrm{Spin}\,(8) & \xrightarrow{\lambda} & \mathrm{Spin}\,(8) \\
{\scriptstyle i}\uparrow & & \downarrow{\scriptstyle p} \\
\mathrm{Spin}\,(7) & \xrightarrow[j_+]{} & SO\,(8)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Spin}\,(8) & \xrightarrow{\lambda^2} & \mathrm{Spin}\,(8) \\
{\scriptstyle i}\uparrow & & \downarrow{\scriptstyle p} \\
\mathrm{Spin}\,(7) & \xrightarrow[j_-]{} & SO(8) .
\end{array}$$

Where it is convenient we write $j_\pm$ to mean either j_+ or j_-. Let $i^*\colon H^*(B\,\mathrm{Spin}\,(8)) \to H^*(B\,\mathrm{Spin}\,(7))$ and

$$j_\pm^*\colon H^*(BSO\,(8)) \to H^*(B\,\mathrm{Spin}\,(7))$$

be the induced cohomology homomorphisms of i and $j_\pm$ on classifying spaces.

THEOREM 3.2. *Identify* $i^*(w_i^*)$ *with* w_i^* $(i = 4, 6, 7)$, $i^*(\omega)$ *with* ω, $i^*(Q_i)$ *with* Q_i $(i = 1, 3)$ *and* $i^*(Y)$ *with* Y. *Then we have*

(i) $H^*(B\,\mathrm{Spin}\,(7), \boldsymbol{Z}) = \boldsymbol{Z}[Q_1, Q_3, Y] + 2$-*torsion*,
$H^*(B\,\mathrm{Spin}\,(7), \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_4^*, w_6^*, w_7^*, \omega]$;

(ii) $i^*(w_8^*) = 0$ *and* $i^*(Q_2) = 2Y$;

(iii) $j_\pm^*(P_1) = 2Q_1$, $\quad j_\pm^*(w_i) = w_i^*$ $(i = 4, 6, 7)$
$j_\pm^*(P_2) = -2Y + Q_1^2$, $\quad j_\pm^*(w_8^*) = \omega$;
$j_\pm^*(P_3) = Q_3 - 2Q_1Y$,
$j_\pm^*(X) = \mp Y$,

(iv) *The kernel of* $j_\pm^*$ *on integral cohomology is the ideal generated by* $4P_2 - P_1^2 \mp 8X$.

Proof. Since i: Spin (7) $\to$ Spin (8) covers the ordinary inclusion of $SO(7)$ in $SO(8)$, we have $(i^* \circ p^*)(X) = 0$. Thus $i^*(Q_2) = 2Y$. From this fact, Theorem 2.1 and Lemma 2.2 we obtain (i) and (ii). Furthermore (iii) follows from (i), (ii), and Lemma 3.1; finally (iv) is an easy calculation from (iii).

Let M be a CW-complex and let ξ be an oriented vector bundle over M with fiber dimension 8. Denote by $f\colon M \to BSO(8)$ the classifying map determined by ξ. We shall say that ξ admits a *nontransitive* Spin (7) reduction if $f = p \circ i \circ g$ for some $g\colon M \to B\,\mathrm{Spin}\,(7)$:

$$\begin{array}{ccc} B\,\mathrm{Spin}\,(7) & \xrightarrow{\;i\;} & B\,\mathrm{Spin}\,(8) \\ {\scriptstyle g}\big\uparrow & & \big\downarrow{\scriptstyle p} \\ M & \xrightarrow[\;f\;]{} & BSO\,(8)\,. \end{array}$$

(Here i and p denote the maps induced by the maps Spin (7) $\to$ Spin (8) and Spin (8) $\to SO(8)$ which we also designate by i and p.) On the other hand by Lemma 3.1, M admits a sphere transitive Spin (7) reduction in the sense of this paper if and only if for some $g\colon M \to B\,\mathrm{Spin}\,(7)$ we have $f = p \circ \lambda \circ i \circ g$ or $f = p \circ \lambda^2 \circ i \circ g$. Therefore we have the following lemma.

LEMMA 3.3. *Assume* $w_2(\xi) = 0$. *Then* ξ *has a transitive* Spin (7) *reduction* (*that is a reduction of* $SO(8)$ *to* $j_\pm$ (Spin (7))) *if and only if* $\lambda^{\mp 1}(\xi)$ *has a nontransitive* Spin (7) *reduction.*

Next we determine the primary and secondary obstructions to the existence of sphere transitive Spin (7) structures.

THEOREM 3.4. *Let* M *be a* CW-*complex and let* ξ *be an oriented vector bundle over* ξ *with fiber dimension* 8. *Denote by* $c^2(\xi)$ *and*

$c^8(\xi)$ *the primary and secondary obstructions to the existence of a transitive* $j_\pm$ (Spin (7)) *structure. Then* $c^2(\xi) \in H^2(M, Z_2)$, $c^8(\xi) \in H^8(M, Z)$, *and we have*

$$c^2(\xi) = w_2(\xi) ,$$
$$16c^8(\xi) = 4P_2(\xi) - P_1^2(\xi) \pm 8X(\xi) .$$

Proof. We first note that $SO(8)/\mathrm{Spin}(7)$ is diffeomorphic to real projective space P^7. Hence $c^2(\xi) \in H^2(M, \pi_1(P^7)) = H^2(M, Z_2)$ and $c^8(\xi) \in H^8(M, \pi_7(P^7)) = H^8(M, Z)$. A transgression argument given in [8] shows that $w_2(\xi) = c^2(\xi)$.

Assume that $w_2(\xi) = 0$. By Lemma 3.3, ξ has a sphere transitive $j_\pm$ (Spin (7)) structure if and only if $\lambda^{\mp 1}(\xi)$ has a nontransitive Spin (7) structure. The first obstruction to the latter is $X(\lambda^{\mp 1}(\xi))$, as is well-known. On the other hand by Theorem 2.1 and 3.2 we have

$$16X^{\pm 1}(\lambda(\xi)) = 4P_2(\xi) - P_1^2(\xi) \mp X(\xi) .$$

Hence the theorem follows.

COROLLARY 3.5. *Let ξ be an oriented vector bundle with fiber dimension* 8 *over a CW-complex M. Assume that* $\dim M \leqq 8$ *and that* $H_8(M, Z)$ *has no 2-torsion. Then ξ has a sphere transitive* $j_\pm$ (Spin (7)) *structure if and only if* $w_2(\xi) = 0$ *and*

$$4P_2(\xi) - P_1^2(\xi) \pm X(\xi) = 0 .$$

Theorems 2.1 and 3.4 and Corollary 3.5 correct an error in [8]. We now turn to Spin (9). First we need a lemma.

LEMMA 3.6. *We have the following commutative diagram:*

$$\begin{array}{ccccc} & & \mathrm{Spin}(8) \times \mathrm{Spin}(8) & \xrightarrow{\Delta} & \mathrm{Spin}(16) \\ & \nearrow^{\lambda \times \lambda^2} & & & \downarrow p \\ \mathrm{Spin}(8) & \xrightarrow[k]{} & \mathrm{Spin}(9) & \xrightarrow[l]{} & SO(16) \end{array}$$

where Δ is the standard map of Spin (8) $\times$ Spin (8) *into* Spin (16), *p is the covering projection, k is the standard inclusion of* Spin (8) *in* Spin (9), *and l is the sphere transitive* 16-*dimensional representation of* Spin (9).

Proof. Let F_4 denote the automorphism group of the exceptional Jordan algebra of 3×3 Hermitian matrices of Cayley numbers. Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The subgroup H_i of F_4 which leaves E_i fixed is isomorphic to Spin (9) (see [7]). On the other hand Spin (8) is isomorphic to $H_1 \cap H_2 \cap H_3$. Let

$$V_1 = \text{matrices of the form } \begin{pmatrix} 0 & z & w \\ \bar{z} & 0 & 0 \\ \bar{w} & 0 & 0 \end{pmatrix},$$

$$V_2 = \text{matrices of the form } \begin{pmatrix} 0 & z & 0 \\ \bar{z} & 0 & w \\ 0 & \bar{w} & 0 \end{pmatrix},$$

$$V_3 = \text{matrices of the form } \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & w \\ \bar{z} & \bar{w} & 0 \end{pmatrix}.$$

Then V_i is an irreducible representation space for H_i. Since there is only one irreducible 16-dimensional representation of Spin (9), each representation of H_i on V_i is just l. Now the representation of Spin (8) on V_1 is $\lambda \times 1$, on V_2 is $\lambda \times \lambda^2$, and on V_3 is $1 \times \lambda^2$. Hence we get the commutative diagram

$$\begin{array}{ccccc} & & \text{Spin}(8) \times \text{Spin}(8) & \longrightarrow & \text{Spin}(16) \\ & \nearrow^{\lambda^{i-1} \times \lambda^i} & & & \downarrow p \\ \text{Spin}(8) & \xrightarrow[k_i]{} & \text{Spin}(9) & \xrightarrow[l]{} & SO(16)\,. \end{array}$$

We claim that k_2 is the standard inclusion of Spin (8) in Spin (9) while k_1 and k_3 are not. This may be proved by showing that $k_i^*(H^*(B\,\text{Spin}\,(9), \boldsymbol{R}_0))$ is $\boldsymbol{R}_0[P_1, P_2, P_3, X^2] \subsetneqq H^*(B\,\text{Spin}\,(8), \boldsymbol{R}_0)$ for $i = 2$, but not for $i = 1$ or 3. (See the proof of the next theorem.) This completes the proof of the lemma.

THEOREM 3.7. (i) *There exist cohomology classes*

$$Z \in H^{16}(B\,\text{Spin}\,(9), \boldsymbol{Z})$$

and $\phi \in H^{16}(B\,\text{Spin}\,(9), \boldsymbol{Z}_2)$ *such that*

$$H^*(B\,\text{Spin}\,(9), \boldsymbol{Z}) = \boldsymbol{Z}[Q_1, Q_2, Q_3, Z] + 2\text{-}torsion$$
$$H^*(B\,\text{Spin}\,(9), \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_4, w_6, w_7, w_8, \phi]\,.$$

Here $k_2^*(Q_i) = Q_i$ $(i = 1, 2, 3)$, $k_2^*(4Z) = p^*(X^2 - P_2^2)$, $k_2^*(w_i^*) = w_i^*$ $(i = 4, 6, 7, 8)$, *and* $k_2^*(\phi) = \omega^2 + \omega w_8^*$.

(ii) *We have (modulo elements of order* 2)

$$\begin{aligned}
l^*(P_1) &= 4Q_1 \\
l^*(P_2) &= -2Q_2 + 6Q_1^2 \\
l^*(P_3) &= 2Q_3 - 6Q_1Q_2 + 4Q_1^2 \\
l^*(P_4) &= -34Z - 7Q_2^2 + 4Q_1Q_3 - 6Q_1^2Q_2 + Q_1^4 \\
l^*(P_5) &= 28Q_1Z - 2Q_2Q_3 + 2Q_1^2Q_3 + 10Q_1Q_2^2 - 2Q_1^3Q_2 \\
l^*(P_6) &= 22Q_2Z - 2Q_1^2Z + Q_3^2 - 2Q_1Q_2Q_3 + 5Q_2^3 + Q_1^2Q_2^2 \\
l^*(P_7) &= 2Q_3Z - 10Q_1Q_2Z + Q_2^2Q_3 - 3Q_1Q_2^3 \\
l^*(X) &= Z .
\end{aligned}$$

and

$$\begin{aligned}
l^*(w_i) &= 0 \text{ for } i = 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13 \\
l^*(w_8) &= w_4^{*2} + w_8^* \\
l^*(w_{12}) &= w_6^{*2} + w_4^*w_8^* \\
l^*(w_{14}) &= w_7^{*2} + w_6^*w_8^* \\
l^*(w_{15}) &= w_7^*w_8^* \\
l^*(w_{16}) &= \phi .
\end{aligned}$$

Proof. Let ξ be an 8-dimensional vector bundle with $w_2(\xi) = 0$ and set $\nu = \lambda(\xi) \oplus \lambda^2(\xi)$. Then the Pontryagin, Euler, and Stiefel-classes of ν may be computed by means of the Whitney sum formula together with Theorem 2.1. On the other hand any maximal torus (maximal 2-subgroup) of Spin (8) is also a maximal torus (maximal 2-subgroup) of Spin (9). Therefore the formulas for above mentioned characteristic classes are the most general possible.

Set $Z = l^*(X)$ and $\phi = l^*(w_{16})$. Then we obtain (ii). Finally (i) follows from (ii) and Lemma 2.2.

Theoretically the kernel of l^* can be determined from Theorem 3.7 (ii). This yields some necessary conditions that a 16-dimensional vector bundle have a transitive Spin (9) reduction. However, we omit the details. In the only example we consider in § 4, namely the Cayley plane, it is simpler to use Theorem 3.7 itself.

We conclude this section by noting a few facts about the cohomology of BG_2 and its inclusion in $B\,\mathrm{Spin}\,(7)$.

LEMMA 3.8. *Let g be the standard inclusion of G_2 in $SO(7)$, and denote by h the lifting of g into* Spin (7):

$$\begin{array}{ccccc} & & \mathrm{Spin}\,(7) & \xrightarrow{\;i\;} & \mathrm{Spin}\,(8) \\ & {}^{h}\nearrow & \downarrow & & \\ G_2 & \xrightarrow[g]{} & SO(7)\,. & & \end{array}$$

If i denotes the standard inclusion of Spin (7) *in* Spin (8), *then we have*

$$\lambda \circ i \circ h = i \circ h \ .$$

Proof. This follows from the fact that G_2 is the fixed point set of λ.

THEOREM 3.9. (i) *We have*

$$\begin{aligned} H^*(BG_2, \boldsymbol{R}_0) &= \boldsymbol{R}_0[g^*(P_1), g^*(P_3)] \\ &= \boldsymbol{R}_0[h^*(Q_1), h^*(Q_3)] \ , \end{aligned}$$

where g^* and h^* are induced by g and h defined in the previous lemma and $\boldsymbol{R}_0$ denotes the rationals.

(ii) *In integral cohomology, the kernel of g^* is the ideal generated by $4P_2 - P_1^2$ and the kernel of h^* is the ideal generated by Y.*

Proof. The proof of (i) and the fact that $g^*(4P_2 - P_1^2) = 0$ consists of identifying the Pontryagin classes with polynomials in the roots of $SO(7)$, computing the images of these polynomials under g^*, and using the fact that there are two generators of $M^*(BG_2, \boldsymbol{R}_0)$, one 4-dimensional, and the other 12-dimensional. We omit the details. From Lemma 3.8, Theorem 2.1 and Theorem 3.2, we have $h^*(Y) = 0$ and $h^*(\omega) = 0$. An easy calculation shows that $g^*(4P_2 - P_1^2) = 0$. That Y and $4P_2 - P_1^2$ generate the kernels of h^* and g^* follows from (i).

4. Sphere transitive structures on spheres and other homogeneous spaces. The study of the existence of almost complex structures on spheres is a well-known problem in algebraic topology; it was solved by Borel and Serre [4]. Thus the results of this section can be viewed as a generalization of this problem. Many of the results we present are not new. However, we give them in order that we may write down in an organized fashion the complete solution to the problem of the existence of sphere-transitive structures on spheres.

We shall need two preliminary results.

LEMMA 4.1. *Let G act transitively and linearly on S^{2n-1} with* *isotropy subgroup H. Then if the tangent bundle of S^{2n} can be reduced to G, the subgroup of elements of $\pi_{2n-2}(H)$ which are inessential in G has order at most 2.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc} H & \xrightarrow{k} & G & \xrightarrow{h} & SO(2n) & \xrightarrow{j} & SO(2n+1) \\ & & \downarrow p_1 & \swarrow p_2 & & & \downarrow p_3 \\ & & S^{2n-1} & & & & S^{2n} . \end{array}$$

Here the p_i are evaluation maps, j and k denote the inclusion of the respective isotropy subgroups, and h denotes the representation of G arising from the action on S^{2n-1}. Let ∂_1 be the boundary operator in the homotopy sequence of the fibration $H \xrightarrow{k} G \xrightarrow{p} S^{2n-1}$ and ∂_2 the boundary operator in the homotopy sequence of fibration $SO(2n) \to SO(2n+1) \to S^{2n}$. Let $\iota_k \in \pi_k(S^k)$ denote the homotopy class of the identity map of S^k. A reduction of the structure group of the tangent bundle of S^{2n} to G is equivalent to the existence of an element $\alpha \in \pi_{2n-1}(G)$ such that $h_*(\alpha) = \partial_2(\iota_{2n})$. Then $p_{1*}(\alpha) = p_{2*}h_*(\alpha) = p_{2*}\partial_2(\iota_{2n}) = 2\iota_{2n-1}$ and so $\partial_1(2\iota_{2n-1}) = \partial_1(p_{1*}\alpha) = 0$. Hence $\partial_1(\pi_{2n-1}(S^{2n-1}) \subseteqq \pi_{2n-2}(H)$ has order at most 2. By the exactness of the homotopy sequence this subgroup is equal to $\ker(k\colon \pi_{2n-2}(H) \to \pi_{2n-2}(G))$.

LEMMA 4.2. *We have $\pi_{4n-2}(\mathrm{Sp}(n)) = 0$ and $(2n-1)!$ divides the order of $\pi_{4n-2}(\mathrm{Sp}(n-1))$ for $n \geqq 2$.*

Proof. $\pi_{4n-2}(\mathrm{Sp}(n))$ is in the stable range and is 0 by Bott periodicity. To prove the other assertion we consider the homomorphism of homotopy sequences of fibrations induced by the commutative diagram

$$\begin{array}{ccccc} \mathrm{Sp}(n-1) & \longrightarrow & \mathrm{Sp}(n) & \longrightarrow & S^{4n-1} \\ \downarrow \iota & & \downarrow & & \downarrow \text{identity} \\ U(2n-1) & \longrightarrow & U(2n) & \longrightarrow & S^{4n-1} \end{array}$$

where the horizontal lines are fibrations. Let ∂_1 and ∂_2 be the boundary maps of the homotopy sequences of the upper and lower lines, respectively. Then $\iota_* \circ \partial_1 = \partial_2$; hence the order of $\mathrm{Im}(\partial_1)$ is a multiple of the order of $\mathrm{Im}(\partial_2)$. But $Z_{(2n-1)!} = \pi_{4n-2}(U(2n-1)) \subset \mathrm{Im}\,\partial_2$ see [5]. Hence $(2n-1)!$ divides the order of $\pi_{4n-2}(\mathrm{Sp}(n-1))$.

THEOREM 4.3. *Let $\tau(S^n)$ denote the tangent bundle of S^n. The following is a complete list of sphere transitive structures on* *spheres*:

(i) $SO(n)$ *on* $\tau(S^n)$,
(ii) $U(3)$ *on* $\tau(S^6)$,
(iii) $SU(3)$ *on* $\tau(S^6)$,
(iv) G_2 *on* $\tau(S^7)$.

Proof. We have (i) because S^n is orientable and (iv) because S^7 is parallelizable. (ii) is a consequence of the fact that S^6 has an almost complex structure. Actually, however, it turns out that structure group of the tangent bundle $\tau(S^6)$ can be reduced to $SU(3)$ (see [8]) so that (iii) holds.

Next we show that there are no other sphere transitive structures. We do this case by case.

$U(n)$: Borel and Serre proved that for $n \neq 1, 3$ $\tau(S^{2n})$ cannot have a $U(n)$ structure.

$SU(n)$: Since $\tau(S^{2n})$ $(n \neq 1, 3)$ cannot have a $U(n)$ structure, it cannot have an $SU(n)$ structure because $SU(n) \subsetneqq U(n)$.

$\mathrm{Sp}(n)$: Since $\tau(S^{4n})$ $(n \neq 1)$ cannot have a $U(2n)$ structure and $\mathrm{Sp}(n) \subsetneqq U(2n)$, $\tau(S^{4n})$ cannot have a $\mathrm{Sp}(n)$ structure.

$\mathrm{Sp}(n)\cdot SO(2)$: We have $\mathrm{Sp}(n)\cdot SO(2) \subsetneqq U(2n)$. Thus the argument for $\mathrm{Sp}(n)$ applies in this case also.

$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$: For $n \geqq 1$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ is covered by

$$\mathrm{Sp}(n) \times \mathrm{Sp}(1) = \mathrm{Sp}(n) \times S^3 .$$

We have $\pi_k(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)) \cong \pi_k \mathrm{Sp}(n) \oplus \pi_k(S^3)$ for $k > 1$. By the second part of Lemma 4.2, it follows that for $n \geqq 2$, $\pi_{4n-2}(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)) = \pi_{4n-2}(S^3)$ and $\pi_{4n-2}(\mathrm{Sp}(n-1)\cdot\mathrm{Sp}(1))$ is the direct sum of $\pi_{4n-2}(S^3)$ with a group of order at least $(2n-1)!$. Since $\pi_{4n-2}(S^3)$ is finite, it follows that the necessary condition for a $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$-structure on S^{4n} given by Lemma 4.1 fails, for $n > 1$.

$\mathrm{Spin}(7)$: According to Theorem 3.2 (iv) a necessary condition that an 8-dimensional vector bundle ξ have a transitive $\mathrm{Spin}(7)$ reduction is that $4P_2(\xi) - P_1^2(\xi) \mp 8X(\xi) = 0$. The tangent bundle of S^8 (or its negative) does not satisfy this condition.

$\mathrm{Spin}(9)$: Suppose the tangent bundle $\tau = \tau(S^{16})$ had a transitive $\mathrm{Spin}(9)$ structure. We have $P_i(\tau) = 0$ $(i = 1, \cdots, 7)$, $X(\tau) = 2$. Hence by Theorem 3.7 (ii), $Q_i(\tau) = 0$ $(i = 1, \cdots, 7)$ and $Z(\tau) = 0$ (at least with rational coefficients). This contradicts the fact that we must have $X(\tau) = Z(\tau)$. The same argument shows that $-\tau$ cannot have a transitive $\mathrm{Spin}(9)$ reduction.

We conclude with some brief remarks about the existence of sphere transitive structures on various simply connected compact homogeneous spaces other than spheres. Denote by $P^n(C)$ and $P^n(\boldsymbol{Q})$

complex and quaternionic projective spaces of real dimension $2n$ and $4n$, respectively. Also let $\bar{\boldsymbol{Q}}_n$ denote the space of all nonoriented 2-planes in $\boldsymbol{R}^{n+2}$.

THEOREM 4.4. *The homogeneous spaces* $S^6\times S^2$, $S^4\times S^4$, $S^4\times S^2\times S^2$, $(S^2)^4$, $P^4(\boldsymbol{C})$, $P^3(\boldsymbol{C})\times S^2$, $P^2(\boldsymbol{C})\times P^2(\boldsymbol{C})$, $P^2(\boldsymbol{C})\times S^4$, $P^2(\boldsymbol{C})\times S^2\times S^2$, $P^1(\boldsymbol{Q})\times P^2(\boldsymbol{C})$, $P^1(\boldsymbol{Q})\times S^4$, $P^1(\boldsymbol{Q})\times S^2\times S^2$, $\bar{\boldsymbol{Q}}_2\times P^1(\boldsymbol{Q})$, $\bar{\boldsymbol{Q}}_2\times P^2(\boldsymbol{C})$, $\bar{\boldsymbol{Q}}_2\times S^4$, $\bar{\boldsymbol{Q}}_2\times S^2\times S^2$ *do not possess sphere transitive* Spin (7) *structures.*

Proof. For each case one computes (see [3]) the Pontryagin and Euler classes and verifies that they do not satisfy $P_2-4P_1^2\pm 8X=0$.

In contrast to Theorem 4.4 we have the following result.

THEOREM 4.5. *Either orientation of the spaces* $P^2(\boldsymbol{Q})$, $\bar{\boldsymbol{Q}}_4$, *and* $G_2/SO(4)$ *possesses a sphere transitive* Spin (7) *structure.*

Proof. According to [3] each of these spaces has integral cohomology $\boldsymbol{Z}[u]/(u^3)$ where u is a 4-dimensional generator. Furthermore $P_1=2u$, $P_2=7u^2$, and $X=\pm 3u^2$ for each of these spaces (with the proper choice of u). Theorem (4.5) now follows from Theorem 3.4.

It would be interesting to construct explicitly a sphere transitive Spin (7) structure (i.e., a 3-fold vector cross product) on $P^2(\boldsymbol{Q})$.

Finally we have the following theorem.

THEOREM 4.6. *Let* $\boldsymbol{C}=F_4/\text{Spin}\,(9)$ *denote the Cayley plane with the canonical orientation. Then* $\boldsymbol{C}$ *does not possess a sphere transitive* Spin (9) *structure, but* $-\boldsymbol{C}$ *does.*

Proof. We have $H^*(\boldsymbol{C},\boldsymbol{Z})=\boldsymbol{Z}[u]/(u^3)$ where u is an 8-dimensional generator. With the proper choice of u we have by [3] that for the Cayley plane, $P_2=6u$, $P_4=39u^2$, $P_1=P_3=0$, and $X=\pm 3u^2$. It is well known that at least one orientation of $\boldsymbol{C}$ possesses a sphere transitive Spin (9) structure. It is not hard to verify that $-\boldsymbol{C}$ satisfies the conclusions of Theorem 3.7 while $\boldsymbol{C}$ does not. Hence we get Theorem 4.6.

REFERENCES

1. M. Berger, *Sur les groupes d'holonomie homogène des variétiés à connexion affine et des variétés riemanniens*, Bull. Soc. Math. France **83** (1955), 279–330.

2. A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math, **76** (1954), 273–342.

3. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458-538.
4. A. Borel and J. Serre, *Groupes de Lie et puissances réduits de Steenrod*, Amer. J. Math. **75** (1953), 409-448.
5. R. Bott, *A report on the unitary group*, Proc. of Symposia Pure Math. vol. 3; *Differential Geometry*, 1-6.
6. R. Brown and A. Gray, *Vector cross products*, Comment. Math. Helv. **42** (1967), 222-236.
7. H. Freudenthal, *Octaven Ausnahmen gruppen, und Oktavengeometrie*, Utrecht, 1951.
8. A. Gray, *Vector cross products on manifolds*, Trans. Amer. Math. Soc. **141** (1969), 465-504.
9. ———, *A note on Riemannian manifolds with holonomy groups* $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, Michigan Math J. **16** (1969), 125-128.
10. D. Montgomery and H. Samelson, *Transformation groups of spheres*, Ann. of Math. **44** (1943), 454-470.
11. J. Simons, *On transitivity of holonomy systems*, Ann. of Math. **76** (1962), 213-234.
12. E. Thomas, *On the cohomology groups of the classifying space for the stable spinor group*, Bol. Soc. Math. Mexicana, **7** (1962), 57-69.
13. H. Toda, Y. Saito and I. Yokota, *Note on the generator of* $\pi_7\,(SO(n))$, Memoirs of the College of Science, University of Kyoto (A) **30** (1957), 227-230.

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