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LOCALLY COMPACT SPACES AND TWO CLASSES OF C*-ALGEBRAS

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Let X be a topological space which is second countable, locally compact, and T_0. Fell has defined a compact Hausdorff topology on the collection $\mathscr{C}(X)$ of closed subsets of X. X may be identified with a subset of $\mathscr{C}(X)$, and in the first part of this paper, the original topology on X is related to that induced from $\mathscr{C}(X)$. The main result is a necessary and sufficient condition for X to be almost strongly separated. In the second part, these results are applied to the primitive ideal space Prim (A) of a separable C^*-algebra A, giving in particular a necessary and sufficient condition for Prim (A) to be almost separated. Further information concerning ideals in A which are central as C^*-algebras is obtained.

Most of the theorems in the paper were suggested by the results for simplex spaces recently obtained by Effros [10], Effros and Gleit [11], Gleit [14], and Taylor [17]. The notion of a simplex space was introduced by Effros in [9]. If $\mathfrak{A}$ is a simplex space, then max $\mathfrak{A}$, $P_1(\mathfrak{A})$, and $EP_1(\mathfrak{A})$ denote the closed maximal ideals in $\mathfrak{A}$, the bounded positive linear functionals on $\mathfrak{A}$ of norm at most one, and its set of extreme points, resp., the first set provided with the hull-kernel topology and the latter two sets with the weak* topology. The sets max $\mathfrak{A}$ and $EP_1(\mathfrak{A})$-{0} are in a natural one-to-one correspondence, but the topologies do not agree in general. Information about the simplex space $\mathfrak{A}$ can be obtained by comparing these two topologies (see [11], [14], [17]).

In trying to develop an analogous theory for a C^*-algebra A, the first problem is to decide on replacements for max $\mathfrak{A}$, $P_1(\mathfrak{A})$, and $EP_1(\mathfrak{A})$. For simplicity, assume that A is separable and has a T_1 structure space. An obvious substitute for max $\mathfrak{A}$ is the structure space of A, Prim (A) (the primitive ideals in A, or in this case the maximal proper closed two-sided ideals in A, with the hull-kernel topology). To replace $P_1(\mathfrak{A})$ and $EP_1(\mathfrak{A})$ by the corresponding sets of linear functionals on A does not seem to lead to a fruitful theory. Instead, $P_1(\mathfrak{A})$ and $EP_1(\mathfrak{A})$-{0} are replaced by $N(A)$ and $EN(A)$-{0}, resp., where $N(A)$ is the compact Hausdorff space of C^*-semi-norms on A, and $EN(A)$ is the set of "extreme" points of $N(A)$ (see [4; § 1. 9. 13], [8], [12]). Then Prim (A) and $EN(A)$-{0} are in a natural one-to-one correspondence which is in general not a homeomorphism. By identifying these sets, the primitive ideals in A are endowed with two topologies. Regarding Prim (A) as a subset of $\mathscr{C}(\text{Prim}(A))$, the identification of Prim (A) and $EN(A)$-$\{0\}$ extends naturally to a homeomorphism of $\mathscr{C}(\text{Prim}(A))$ and $N(A)$. Thus the second topology on Prim (A) is just its relative topology in $\mathscr{C}(\text{Prim}(A))$. It is therefore natural to attempt to formulate those theorems about a simplex space $\mathfrak{A}$ which involve only the two topologies on $\max \mathfrak{A}$ in terms of a locally compact space X and the associated space $\mathscr{C}(X)$.

The paper is organized as follows. § 2 contains theorems which relate the topology of X to that of $\mathscr{C}(X)$. The applications to C^*-algebras are in § 3. Two classes of C^*-algebras, called GM- and GC-algebras, are investigated; they correspond to the GM- and GC-simplex spaces of [11]. A C^*-algebra is a GM-algebra if its structure space is almost strongly separated, and a GC-algebra if it has a composition series (I_α) of closed two-sided ideals such that the $I_{\alpha+1}/I_\alpha$ are all central C^*-algebras. These algebras were studied by Delaroche [2], who in particular showed that the GC-algebras are just the GM-algebras with only modular primitive ideals. A new proof of this fact (Theorem 3.7) is included. Finally, § 4 points out how the GM- and GC-algebras are related to some of the classes of C^*-algebras in the literature.

2. Locally compact spaces. Throughout this section X is assumed to be a locally compact topological space satisfying the T_0 separation axiom. Recall that X is T_0 means that if $x, y \in X$ are such that $\{x\}^- = \{y\}^-$ (bar indicates closure), then $x = y$, and that X is locally compact means that if $x \in X$, then each neighborhood of x contains a compact neighborhood of x. It is important to remember that although a closed subset of a compact set must be compact, the converse need not be true in a non-Hausdorff space. Let X_1 denote the *closed* points in X, i.e., those x for which $\{x\}^- = \{x\}$. If $X = X_1$, then X is said to be T_1.

The following construction is due to J. M. G. Fell [13]. Let $\mathscr{C}(X)$ denote the collection of all closed subsets of X. The function $\lambda = \lambda_X: X \to \mathscr{C}(X): x \to \{x\}^-$ is one-to-one. If C is a compact subset of X and if $\mathscr{F}$ is a (possibly empty) finite collection of open subsets of X, then $\mathscr{U}(C; \mathscr{F})$ will denote the collection of all those $F \in \mathscr{C}(X)$ such that $F \cap C = \emptyset$ and $F \cap G \neq \emptyset$ for each $G \in \mathscr{F}$. The sets $\mathscr{U}(C; \mathscr{F})$ form a basis for a compact Hausdorff topology on $\mathscr{C}(X)$ [13]. It is readily verified that a net (F_α) in $\mathscr{C}(X)$ will converge to an element F in $\mathscr{C}(X)$ if and only if (1) for each x in F and neighborhood N of x, eventually $F_\alpha \cap N \neq \emptyset$, and (2) if P is the complement of a compact set with $F \subset P$, then eventually $F_\alpha \subset P$. This topology is metrizable whenever X is second countable [6; Lemma 2] (see Corollary 2.7 for a partial converse). A simple argument will prove

LEMMA 2.1. (1) λ *is open onto its image, and* (2) X *is Hausdorff if and only if* $\lambda: X \to \lambda(X)$ *is a homeomorphism.*

The first object is to find sets on which λ restricts to a homeomorphism. A set $\mathscr{T} \subset \mathscr{C}(X)$ will be called *dilated* if $x \in F$ for some $F \in \mathscr{T}$ implies that $\lambda(x) \in \mathscr{T}$. In particular, if $F \in \mathscr{C}(X)$, the set $F^{\perp} = \{E \in \mathscr{C}(X): E \subset F\}$ is compact and dilated.

LEMMA 2.2. *If* $\mathscr{T}$ *is a compact and dilated subset of* $\mathscr{C}(X)$, *then* $\lambda^{-1}(\mathscr{T})$ *is closed.*

Proof. Suppose that $x_0 \in X$ and $x_0 \notin \lambda^{-1}(\mathscr{T})$. Say $F \in \mathscr{T}$. As $\mathscr{T}$ is dilated, $x_0 \notin F$, and so there is a compact neighborhood $C(F)$ of x_0 which is disjoint from F. The sets $\mathscr{U}(C(F); \varnothing)$, $F \in \mathscr{T}$, form an open covering for $\mathscr{T}$; hence there are sets $F_1, \cdots, F_n \in \mathscr{T}$ such that

$$\mathscr{T} \subset \bigcup_{i=1}^{n} \mathscr{U}(C(F_i); \varnothing) .$$

Suppose $x \in C = \bigcap_{i=1}^{n} C(F_i)$ and $\lambda(x) \in \mathscr{T}$. Then $\lambda(x) \cap C(F_i) = \varnothing$ for some i, hence $x \notin C(F_i)$, a contradiction. This shows that C is a neighborhood of x_0 which is disjoint from $\lambda^{-1}(\mathscr{T})$.

If T is a subset of X_1, then $\lambda(T)$ is dilated; hence

COROLLARY 2.3. *If* T *is a subset of* X_1 *for which* $\lambda(T)$ *is compact, then* λ *restricts to a homeomorphism of* T *onto* $\lambda(T)$.

The following shows that convergence in X is closely related to that in $\mathscr{C}(X)$. The trick employed in the proof of (ii) was used by both Gleit [14] and Taylor [17].

THEOREM 2.4. (i) *Let* (x_α) *be a net in* X *such that* $\lambda(x_\alpha) \to F$ *for some* $F \in \mathscr{C}(X)$. *Then* $x_\alpha \to x$ *for any* $x \in F$.

(ii) *Let* (x_n) *be a sequence in* X_1 *such that* $\lambda(x_n) \to F$ *for some* $F \in \mathscr{C}(X)$. *Then the limit points of the set* $\{x_n: x \geqq 1\}$ *lie in* F.

Proof. (i) Say $x \in F$, and let G be an open set containing x. Then since $F \cap G \neq \varnothing$, eventually $\lambda(x_\alpha) \cap G \neq \varnothing$, hence $x_\alpha \in G$.

(ii) For each m the set $\{\lambda(x_n): n \geqq m\} \cup F^{\perp}$ is both closed and dilated, hence its inverse image $F_m = \{x_n: n \geqq m\} \cup F$ is closed. If x

is a limit point of $\{x_n: n \geqq 1\}$, it must lie in each of the sets F_m, and thus is an element of F.

COROLLARY 2.5. *Suppose that X is second countable. If $\emptyset \in \lambda(X_1)^-$, then neither X_1 nor X can be compact.*

Proof. $\mathscr{C}(X)$ is metrizable, hence there is a sequence (x_n) in X_1 with $\lambda(x_n) \rightarrow \emptyset$. It follows from Theorem 2.4 (ii) that no subsequence of (x_n) can converge to a point in X.

COROLLARY 2.6. *Suppose that $\lambda(X)^-$ is first countable (this is the case if X is second countable), and that T is a compact subset of X_1. If $F \in \mathscr{C}(X)$ and $T \cap F = \emptyset$, then $\lambda(T)^- \cap F^\perp = \emptyset$.*

Proof. If $E \in \lambda(T)^- \cap F^\perp$, there is a sequence (x_n) in T with $\lambda(x_n) \rightarrow E$. Since T is compact, the set $\{x_n: n \geqq 1\}$ has a limit point x in T. Then $x \in E$ from Theorem 2.4 (ii), and since $E \in F^\perp$, $x \in F$. But this is a contradiction.

COROLLARY 2.7. *Suppose that X is locally compact and T_1. If $\lambda(X)^-$ is second countable, then so is X.*

Proof. Let $\mathscr{T}_1, \mathscr{T}_2, \cdots$ be a basis of open sets for the topology of $\lambda(X)^-$; with no loss in generality, the sets $\mathscr{T}_n$ may be assumed to be closed under finite unions. Suppose that an $x \in X$ and an $F \in \mathscr{C}(X)$ with $x \notin F$ are given. It is sufficient to show that for some n, $\lambda^{-1}(\mathscr{T}_n)$ contains x in its interior and is disjoint from F. Using the local compactness of X, choose a compact neighborhood C of x disjoint from F. Corollary 2.6 and the fact that $F^\perp$ is closed give

$$\lambda(C)^- \subset \lambda(X)^- - F^\perp = \bigcup_k \mathscr{T}_{n_k}$$

for suitable integers n_k. As $\lambda(C)^-$ is compact and as the $\mathscr{T}_n$ are closed under finite unions, there is an n for which $\mathscr{T}_n \cap F^\perp = \emptyset$ and $\lambda(C) \subset \mathscr{T}_n$. This completes the proof.

The following will be useful in § 3.

COROLLARY 2.8. *Suppose that X is second countable and that $f: \mathscr{C}(X) \rightarrow [0, \infty)$ is continuous and monotone in the sense that E, $F \in \mathscr{C}(X)$ and $E \subset F$ imply $f(E) \leqq f(F)$. Suppose further that $f(\lambda(x)) > 0$ for all x in some compact subset T of X_1. Then there is an $\alpha > 0$ such that $f(\lambda(x)) \geqq \alpha$ for all $x \in T$.*

Proof. If there is no such α, choose a sequence (x_n) in T such that $f(\lambda(x_n)) \to 0$. Using first the compactness of $\mathscr{C}(X)$ and then that of T, it may be assumed that $\lambda(x_n) \to F$ for some $F \in \mathscr{C}(X)$ and that $x_n \to x$ for some $x \in T$. From Lemma 2.4 (ii), it follows that $x \in F$. Consequently, $0 < f(\lambda(x)) \leqq f(F)$ and $f(F) = 0$, a contradiction.

For simplex spaces, the following result is due to P. D. Taylor.

COROLLARY 2.9. *Suppose that X is second countable and that f is a continuous complex-valued function on $\lambda(X_1)^-$. For each $x \in X_1$, let $c(x)$ denote the set of all those $F \in \lambda(X_1)^-$ which contain x. Then $f \circ \lambda$ is continuous on X_1 if and only if f is constant on the sets $c(x)$, $x \in X_1$.*

Proof. Notice that $\lambda(x) \in c(x)$ for each $x \in X_1$. Suppose that $f \circ \lambda$ is continuous on X_1. Say $x \in X_1$ and $F \in c(x)$. Then there is a sequence (x_n) in X_1 such that $\lambda(x_n) \to F$. From Theorem 2.4 (i), $x_n \to x$, and

$$f(F) = \lim_{n \to \infty} f(\lambda(x_n)) = f(\lambda(x)) .$$

Conversely, suppose that f is constant on the $c(x)$, $x \in X_1$. Let (x_n) be a sequence in X_1 converging to an $x \in X_1$. To show that

$$f(\lambda(x_n)) \to f(\lambda(x)) ,$$

it is sufficient (since $f(\lambda(X_1))$ lies in the compact set $f(\lambda(X_1)^-)$) to show that every convergent subsequence of $f(\lambda(x_n))$ converges to $f(\lambda(x))$. Passing to a subsequence, suppose that $f(\lambda(x_n)) \to \alpha$ for some complex number α. Using the fact that $\mathscr{C}(X)$ is a compact metric space and passing to a further subsequence, it may even be assumed that $\lambda(x_n) \to F$ for some $F \in \lambda(X_1)^-$. Then from Theorem 2.4, (ii), $x \in F$, i.e., $F \in c(x)$, and therefore

$$f(\lambda(x)) = f(F) = \lim_{n \to \infty} f(\lambda(x_n)) = \alpha .$$

If G is a nonempty open subset of X, then G is locally compact and T_0 in its relative topology. Let ρ_G be the map $F \to F \cap G$ of $\mathscr{C}(X)$ onto $\mathscr{C}(G)$, and let σ_G be its restriction to $\lambda_X(G)$. Then $\sigma_G \circ \lambda_X = \lambda_G$ and σ_G is a bijection of $\lambda_X(G)$ onto $\lambda_G(G)$. Using the fact that G is open in X, it is easily checked that ρ_G is continuous; however, σ_G is in general not a homeomorphism.

LEMMA 2.10. *Let G be a nonempty open subset of X, and suppose that $\lambda(X)^- \subset \lambda(X) \cup (X - G)^\perp$. If $\mathscr{T}$ is a subset of $\lambda_X(G)$ and if $\sigma_G(\mathscr{T})$ is compact, then so is $\mathscr{T}$.*

Proof. As ρ_G is continuous,

$$\rho_G(\mathscr{T}^-) \subset [\rho_G(\mathscr{T})]^- = [\sigma_G(\mathscr{T})]^- = \sigma_G(\mathscr{T}) \subset \lambda_G(G) ,$$

and since $\emptyset \notin \lambda_G(G)$, $\mathscr{T}^- \cap (X - G)^\perp = \emptyset$. But

$$\mathscr{T}^- \subset \lambda(X)^- \subset \lambda(X) \cup (X - G)^\perp \subset \lambda_X(G) \cup (X - G)^\perp ,$$

so that $\mathscr{T}^-$ is contained in $\lambda_X(G)$, the domain of σ_G. Since

$$\sigma_G(\mathscr{T}^-) = \rho_G(\mathscr{T}^-) \subset \sigma_G(\mathscr{T})$$

and σ_G is one-to-one, $\mathscr{T}$ must be closed in $\mathscr{C}(X)$.

A point x in X will be said to be *strongly separated* in X if for each $y \neq x$, there are disjoint neighborhoods of x and y (i.e., x is closed, and separated in the sense of [3; § 1]). A nonempty subset Y of X will be called *strongly separated* in X provided each of its points is strongly separated in X. Finally, X will be called *almost strongly separated* if each nonempty closed subset F of X contains a nonempty relatively open subset G which is strongly separated in F (equivalently, every open subset U of X distinct from X is properly contained in an open subset V such that $V - U$ is strongly separated in $X - U$).

PROPOSITION 2.11. *A nonempty open subset G of X is strongly separated in X if and only if* $\lambda(X)^- \subset \lambda(X_1) \cup (X - G)^\perp$.

Proof. Assume first that G is strongly separated in X. Suppose that there is a net (x_α) in X and an $F \notin \lambda(X_1) \cup (X - G)^\perp$ such that $\lambda(x_\alpha)$ converges to F. Then F must contain two distinct points, at least one of which is in G, which is impossible by Theorem 2.4 (i). Conversely, suppose that $\lambda(X)^- \subset \lambda(X_1) \cup (X - G)^\perp$. From this inclusion it is immediate that $G \subset X_1$. As $\rho_G(\lambda(X)^-)$ is compact and contains $\lambda_G(G)$,

$$\lambda_G(G)^- \subset \rho_G(\lambda(X)^-) \subset \lambda_G(G) \cup \{\emptyset\} ,$$

and therefore $\lambda_G(G) \cup \{\emptyset\}$ is compact. For any relatively closed subset $\mathscr{T}$ of $\lambda_G(G)$, $\mathscr{T} \cup \{\emptyset\}$ is compact and dilated, hence $\lambda_G^{-1}(\mathscr{T})$ is a closed subset of G in the relative topology (Lemma 2.2). This shows that λ_G is continuous; since it is always open onto its image, λ_G is a homeomorphism and G is Hausdorff. To show that G is strongly separated, suppose $x \in G$ and $y \notin G$ are given. Let $U \subset G$ be a compact neighborhood of x; it will suffice to show that U is closed in X. As $\lambda_G(U)$ is compact and as $\lambda_G(U) = \sigma_G(\lambda_X(U))$, $\lambda_X(U)$ is compact (Lemma 2.10). $\lambda_X(U)$ is dilated since $U \subset X_1$, and so $U =$

$\lambda_X^{-1}(\lambda_X(U))$ is closed, by Lemma 2.2.

A topological space which is a countable union of compact sets will be called a K_σ.

LEMMA 2.12. *If X is second countable and if G is an open nonempty strongly separated subset of X, then $\lambda_X(G)$ is K_σ.*

Proof. Since G is Hausdorff, $\lambda_G(G)^- \subset \lambda_G(G) \cup \{\varnothing\}$ by Proposition 2.11, and $\lambda_G(G)$ is locally compact. Now $\mathscr{C}(G)$ is second countable, for as G is second countable, $\mathscr{C}(G)$ is a compact metric space [6; Lemma 2]. Therefore $\lambda_G(G)$ is K_σ. The equality $\lambda_G(G) = \sigma_G(\lambda_X(G))$, Lemma 2.10 and Proposition 2.11 now imply that $\lambda_X(G)$ is K_σ.

LEMMA 2.13. *Let E be a nonempty closed subset of X. Then the map $\theta: E^\perp \to \mathscr{C}(E)$ defined by $\theta(F) = F$ for all $F \in E^\perp$ is a homeomorphism onto, where $E^\perp$ has the relative topology from $\mathscr{C}(X)$.*

Proof. That θ is a bijection is clear. Since $E^\perp$ is compact Hausdorff, it is enough to show that θ is continuous. But this follows from the definition of the topologies and the fact that E is closed.

LEMMA 2.14. *If X is almost strongly separated, so is any nonempty subset of X which is either open or closed.*

Proof. See [11; § 3].

THEOREM 2.15. *Suppose that X is second countable, locally compact, and T_0. Then X is almost strongly separated if and only if*

(1) *X is T_1,*

(2) *$\lambda(X)$ is K_σ, and*

(3) *every nonempty closed subset of X is second category in itself.*

Proof. Say that (1)-(3) hold. Let F be a nonempty closed subset of X. Then F is T_1 and second category, and $\lambda_F(F)$ is K_σ by Lemma 2.13. Replacing F by X, it is therefore sufficient to show that if X satisfies (1) and (2) and is second category, then X contains a nonempty open strongly separated set. Write $\lambda(X) = \bigcup_{n=1}^\infty \mathscr{T}_n$, where each $\mathscr{T}_n$ is compact. Since the $\mathscr{T}_n$ are dilated, the $\lambda^{-1}(\mathscr{T}_n)$ are closed by Lemma 2.2. X is second category, hence for some n, $\lambda^{-1}(\mathscr{T}_n)$ contains a nonempty set G which is open in X. As $\lambda^{-1}(\mathscr{T}_n)$ is closed in X and is Hausdorff in the relative topology (Corollary

2.3), G is strongly separated in X.

Conversely, suppose that X is almost strongly separated. By a transfinite induction (see [11; Proposition 3.1]), there is an ordinal α_0 and a family (G_α) of open subsets of X, indexed by those ordinals α with $0 \leqq \alpha \leqq \alpha_0$, such that: (i) $G_0 = \emptyset$, $G_{\alpha_0} = X$; (ii) if $\alpha \leqq \alpha_0$ is a limit ordinal, then $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$; and (iii) if $\alpha < \alpha_0$, then $G_\alpha \subset G_{\alpha+1}$ and $G_{\alpha+1} - G_\alpha$ is a nonempty strongly separated subset of $X - G_\alpha$. To see that (1) holds, say $x \in X$. Let β be the least ordinal such that $x \in G_\beta$. By (ii), β cannot be a limit ordinal; let $\alpha + 1 = \beta$. Then $x \in G_{\alpha+1} - G_\alpha$, so that $\{x\}$ is closed in $X - G_\alpha$, and therefore in X.

The natural map θ_α of $(X - G_\alpha)^\perp$ onto $\mathscr{C}(X - G_\alpha)$ is a homeomorphism, where $(X - G_\alpha)^\perp$ has the relative topology from $\mathscr{C}(X)$ (Lemma 2.13). Since θ_α carries $\lambda_X(G_{\alpha+1} - G_\alpha)$ onto $\lambda_{X-G_\alpha}(G_{\alpha+1} - G_\alpha)$ and since the latter is K_σ by (iii) and Lemma 2.12, $\lambda_X(G_{\alpha+1} - G_\alpha)$ must be K_σ. Now

$$X = \bigcup_{\alpha<\alpha_0}(G_{\alpha+1} - G_\alpha)$$

by the above and α_0 is countable (see [16; § 19, II]), so (2) holds. If $F_1, F_2, \cdots$ are closed and nowhere dense subsets of X, then $F_1 \cap G_1, F_2 \cap G_1, \cdots$ are closed and nowhere dense in the relative topology of G_1. Being locally compact and Hausdorff, G_1 is Baire, so the $F_n \cap G_1$ do not cover G_1. Thus X is second category. By Lemma 2.14, this is enough to show that (3) holds.

COROLLARY 2.16. *If X is second countable and almost strongly separated, then all nonempty closed and all nonempty open subsets of X are Baire.*

Proof. This follows from Lemma 2.14 and Theorem 2.15.

Suppose that X is second countable. If all nonempty closed subsets of X are Baire, then $\lambda(X)$ is G_δ [6; Th. 7]; in view of [16; § 30, VI], this fact may be useful in deciding whether X satisfies (2) of Theorem 2.15. As examples in § 4 will show, (1) and (2) are independent of one another even if all nonempty closed subsets of X are Baire. The set of integers with the Zariski (or cofinite) topology is second countable, locally compact, T_0, and satisfies conditions (1) and (2), but not (3), of Theorem 2.15.

3. C*-Algebras. Let A be a C^*-algebra. Throughout this section and the next, an ideal in A will always mean a closed two-sided ideal. Let $Z(A)$ be the center of A, and let Id(A) [resp., Prim (A), Max (A), and Mod (A)] donote the set of all ideals [primitive ideals, maximal ideals, and modular ideals] in A. For $a \in A$ and $I \in \mathrm{Id}(A)$, define $a(\mathrm{I})$ as the canonical image of a in A/I and $I^{\perp}$ as the set of all those ideals J in A which contain I. Prim (A) with the hull-kernel topology (sometimes called the structure, or Jacobson, topology) is the *structure space* of A. The following facts about the structure space (see [4]) will be used frequently without explicit mention: its closed points are the elements of Max (A); it is locally compact and T_0; it is second countable whenever A is separable; and $I \rightarrow \mathrm{Prim}(A) \cap I^{\perp}$ is a one-to-one correspondence between Id (A) and the closed subsets of Prim (A). The weakest topology on Id (A) making each of the maps $I \rightarrow \| a(I) \|$, $a \in A$, continuous will be called the weak* topology on Id (A). It is not hard to show that $I \rightarrow \mathrm{Prim}(A) \cap I^{\perp}$ is a homeomorphism of Id (A) onto $\mathscr{C}(\mathrm{Prim}(A))$ which restricts to λ on Prim (A) and carries $I^{\perp}$ onto $(\mathrm{Prim}(A) \cap I^{\perp})^{\perp}$ (where the second $\perp$ is taken in the sense of § 2) [12, Th. 2.2]. In what follows, Id (A) and $\mathscr{C}(\mathrm{Prim}(A))$ will be identified. Recall that if A is separable, Id (A) and Prim (A) with the weak* topology may be identified with the spaces $N(A)$ and $EN(A)$-$\{0\}$ of § 1.

In view of the above, the results of § 2 may be applied to C^*-algebras. Save for one, these will not be explicitly mentioned. For any $a \in A$, $I \rightarrow \| a(I) \|$ is a function of the type described in Corollary 2.8. This has the following amusing consequence: If A is separable and if T is a structurally compact subset of Max (A), then $\bigcup\{P: P \in T\}$ is a norm-closed subset of A.

A nonzero ideal I in A will be called an *M-ideal* in A if Prim $(A) - I^{\perp}$ is a strongly separated subset of the structure space of A, and A will be called an *M-algebra* [resp., a *GM-algebra*] if the structure space of A is Hausdorff [almost strongly separated]. Clearly A is an M-algebra if and only if A is an M-ideal in itself. Using [4; § 3.2], it is easily verified that A is a GM-algebra if and only if every nonzero quotient of A contains a nonzero M-ideal.

PROPOSITION 3.1. *The following are equivalent for a nonzero ideal I in a C^*-algebra A:*

(1) *I is an M-ideal*

(2) Prim $(A)^- \subset \mathrm{Max}(A) \cup I^{\perp}$, *where* Prim $(A)^-$ *is the weak* closure of* Prim (A) *in* Id (A)

(3) *for each $a \in I$, $P \rightarrow \| a(P) \|$ is continuous on* Prim (A) *in the structure topology.*

Proof. (1) $\Leftrightarrow$ (2): This is Proposition 2.11.

(1), (2) $\Rightarrow$ (3): Suppose that an $a \in I$ and an $\alpha > 0$ are given. The map $p \rightarrow \| a(P) \|$ is lower semi-continuous on Prim (A) with the structure topology, so it is enough to show that $T = \{P \in \text{Prim}(A): \|a(P)\| \geqq \alpha\}$ is structurally closed. Now T is a structurally compact subset of $\text{Prim}(A) - I^{\perp}$, and as I is an M-ideal in A, $\text{Prim}(A) - I^{\perp}$ is Hausdorff in the relative structure topology. The map σ which sends P into $P \cap I$ is a homeomorphism of $\text{Prim}(A) - I^{\perp}$ onto $\text{Prim}(I)$ for the structure topologies, hence the structure space of I is Hausdorff. From Lemma 2.1, this means that the structure and weak* topologies coincide on $\text{Prim}(I)$. Then $\sigma(T)$ is a weak* compact subset of $\text{Prim}(I)$, and T is a weak* compact subset of $\text{Prim}(A)$ (Lemma 2.10). Since T is contained in $\text{Max}(A)$, it is dilated and therefore structurally closed by Lemma 2.2.

(3) $\Rightarrow$ (1): Say $P \in \text{Prim}(A) - I^{\perp}$ and $Q \in \text{Prim}(A)$ are distinct. If $Q \in I^{\perp}$, choose an $a \in I$ with $\|a(P)\| = 2$. Then $\{R \in \text{Prim}(A): \|a(R)\| > 1\}$ and $\{R \in \text{Prim}(A): \|a(R)\| < 1\}$ are disjoint structurally open sets containing P and Q, resp. Now suppose that $Q \notin I^{\perp}$. For $R \in \text{Prim}(A) - I^{\perp}$ and $a \in I$, $R \cap I \in \text{Prim}(I)$ and

$$\|a(R \cap I)\| = \max\{\|a(R)\|, \|a(I)\|\} = \|a(R)\| .$$

This equality together with the homeomorphism σ of the previous paragraph implies that the structure and weak* topologies on $\text{Prim}(I)$ coincide, and therefore that $\text{Prim}(A) - I^{\perp}$ is Hausdorff in the relative structure topology. As $\text{Prim}(A) - I^{\perp}$ is a structurally open subset of $\text{Prim}(A)$, there are disjoint structure neighborhoods of P and Q.

THEOREM 3.2. *If A is a separable C^*-algebra, then* $\text{Prim}(A)$ *is a G_δ in the weak* topology, and A is a GM-algebra if and only if*

(1) $\text{Max}(A) = \text{Prim}(A)$, *i.e., the structure space of A is T_1, and*

(2) $\text{Prim}(A)$ *is K_σ in the weak* topology.*

Proof. This is an immediate consequence of Theorem 2.15, [6; Th. 7], and the fact that all nonempty closed subsets of the structure space are Baire [4; Corollaire 3.4.13].

Section 4 contains examples which show that neither (1) nor (2) is a consequence of the other, even for separable C^*-algebras. This completes the analogy between GM-simplex spaces and GM-C^*-algebras. In studying the second class of C^*-algebras, the following two lemmas will be useful.

LEMMA 3.3. *For any ideal I in a C^*-algebra A, $Z(I) = I \cap Z(A)$.*

Proof. See [1; Lemma 6].

LEMMA 3.4. *The following are equivalent for a C^*-algebra A:*

(i) *$Z(A) \not\subset P$ for each $P \in \mathrm{Prim}(A)$ and the structure space of A is Hausdorff, and*

(ii) *$P \to P \cap Z(A)$ is a one-to-one map from* $\mathrm{Prim}(A)$ *into* $\mathrm{Prim}(Z(A))$.

If these conditions are satisfied, then the map in (ii) *is a homeomorphism of* $\mathrm{Prim}(A)$ *onto* $\mathrm{Prim}(Z(A))$ *for the structure topologies.*

Proof. For the equivalence of (i) and (ii), see [1; Proposition 3] or [18; Corollary 3.1.2]. The last statement is contained in [15; Th. 9.1].

A C^*-algebra satisfying one of the equivalent conditions of the last lemma is called *central;* for other equivalent definitions, see [1; Proposition 3].

Several results from [7; § 4] will now be recalled. Consider an $a \in Z(A)$ and a primitive ideal P in A. Choose an irreducible representation π of A with kernel P. As $\pi(a)$ is in the center of $\pi(A)$, it must be a multiple α of the identity operator on the space of π. Then $\pi(a)\pi(b) = \alpha\pi(b)$, i.e., $ab - \alpha b \in P$, for all $b \in A$. This last condition determines α uniquely, and shows that it depends only on P (and not on π). Set $f_a(P) = \alpha$. The function f_a is clearly bounded on $\mathrm{Prim}(A)$. It is easy to show that $\varphi(a) = f_a(P)$ for any $\varphi \in \theta^{-1}(P)$, where θ is the natural mapping of $P(A)$, the pure states on A, onto $\mathrm{Prim}(A)$. Because θ is an open map,

$$\begin{aligned} f_a^{-1}(U) &= \{P \in \mathrm{Prim}(A) \colon f_a(P) \in U\} \\ &= \theta(\{\varphi \in P(A) \colon f_a(\theta(\varphi)) \in U\}) \\ &= \theta(\{\varphi \in P(A) \colon \varphi(a) \in U\}) \end{aligned}$$

is structurally open for any open set U of complex numbers. This shows that f_a is structurally continuous. If A is central, then $P \in \mathrm{Prim}(A)$ implies $P \cap Z(A) \in \mathrm{Max}(Z(A)) = \mathrm{Prim}(Z(A))$, and regarding $a \in Z(A)$ as a function on $\mathrm{Max}(Z(A))$, $f_a(P) = a(P \cap Z(A))$. Since $Z(A) \cong C_0(\mathrm{Max}\, Z(A))$, we may identify the functions f_a with $C_0(\mathrm{Prim}(A))$.

A C^*-algebra A will be said to have *local identities* if given $P_0 \in \mathrm{Prim}(A)$, there is an $a \in A$ such that $a(P)$ is an identity in A/P for all P in some structure neighbourhood of P_0. A nonzero ideal I in A will be called a *C-ideal* in A if I is a central C^*-algebra. A will be called a *C-algebra* if it is a C-ideal in itself (i.e., is central), and a *GC-algebra* if every nonzero quotient of A contains a nonzero C-ideal.

PROPOSITION 3.5. *A nonzero ideal I in A is a C-ideal if and* *only if it is an M-ideal with local identities.*

Proof. Suppose that I is a C-ideal. Let P and Q be distinct primitive ideals in A with $P \notin I^{\perp}$. If $Q \notin I^{\perp}$, then since I is central, $P \cap Z(I)$ and $Q \cap Z(I)$ are distinct maximal ideals in $Z(I)$ hence there is an $a \in Z(I) \subset Z(A)$ with $f_a(P) \neq 0$ and $f_a(Q) = 0$. If $Q \in I^{\perp}$, let a be any element of $Z(I)$ with $a(P) \neq 0$. Then f_a will provide disjoint neighborhoods for P and Q, and A is an M-ideal.

Thus it suffices to show that a C^*-algebra A is a C-algebra if and only if it is an M-algebra with local identities. If A is a C-algebra, $Z(A)$ may be identified with $C_0(\mathrm{Prim}\,(A))$, hence it is trivial that A has local identities. Conversely, suppose that A is an M-algebra with local identities. Say $P_0 \in \mathrm{Prim}\,(A)$, and choose an $a \in A$ such that $a(P)$ is an identity in A/P for all P in some neighborhood T of P_0. Consider a continuous bounded complex-valued function f on $\mathrm{Prim}\,(A)$ with $f(P_0) = 1$ and whose support is contained in T. From the Dauns-Hofmann theorem (see [7; § 7]), there is a $b \in A$ such that $b(P) = f(P)a(P)$ for all $P \in \mathrm{Prim}\,(A)$. Then $(bc - cb)(P) = 0$ if $c \in A$ and $P \in \mathrm{Prim}\,(A)$, so that $b \in Z(A)$. Since $b \notin P_0$, A must be a C-algebra.

LEMMA 3.6. *For a nonzero C-ideal I in A,*

(1) $P \rightarrow \|a(P)\|$ *is structurally continuous on* $\mathrm{Prim}\,(A) - I^{\perp}$ *for each* $a \in A$, *and*

(2) $\mathrm{Prim}\,(A)^{-} \subset [\mathrm{Max}\,(A) \cap \mathrm{Mod}\,(A)] \cup I^{\perp}$.

Proof. To prove (1), fix $a \in A$, and suppose $P_0 \in \mathrm{Prim}\,(A) - I^{\perp}$ is given. It is sufficient to show that $P \rightarrow \|a(P)\|$ is structurally continuous on some structure neighborhood of P_0. From the structure homeomorphism of $\mathrm{Prim}\,(A) - I^{\perp}$ onto $\mathrm{Prim}\,(I)$ and the fact that I has local identities, there is a structure neighborhood T of P_0 contained in $\mathrm{Prim}\,(A) - I^{\perp}$ and a $b \in I$ such that $b(P \cap I)$ is an identity in $I/(P \cap I)$ for each $P \in T$. As I is an M-ideal in A, each $P \in T$ is a structurally closed point in $\mathrm{Prim}\,(A)$, and so is a maximal ideal. Therefore $P + I = A$ and there is a *-isomorphism of A/P onto $I/(I \cap P)$ which carries $c(P)$ into $c(I \cap P)$, $c \in I$ [4; Corollaire 1.8.4]. Hence $b(P)$ is an identity in A/P for each $P \in T$, and since $ab \in I$, Proposition 3.1 implies that $P \rightarrow \|(ab)(P)\| = \|a(P)\|$ is structurally continuous on T. Turning to (2), suppose $P \in \mathrm{Prim}\,(A)^{-}$, $P \notin I^{\perp}$. Since I is an M-ideal in A, Proposition 3.1 gives $P \in \mathrm{Max}\,(A)$. As I is central, there is an $a \in Z(I) \subset Z(A)$ with $a \notin P$. Since $a(P)$ is a nonzero central element of A/P, P must be modular.

In the case of simplex spaces, the analogues of (1) and (2) of the previous lemma are each equivalent to I being a C-ideal. This is not the case for C^*-algebras. In fact, there is an example of a noncentral C^*-algebra A which satisfies (1) and (2) with I replaced by A, viz, the algebra of all functions a from $\{1, 2, \cdots\}$ into the two-by-two matrices with complex entries such that $\lim_{n\to\infty} a_{ij}(n)$ exists and is equal to zero unless $i = j = 1$ (this example was also used by Delaroche in [2; § 6]).

The following result is due to Delaroche [2, Proposition, 14].

THEOREM 3.7. *A separable C^*-algebra A is a GC-algebra if and only if*

(1) *A is a GM-algebra, and*

(2) *every primitive ideal in A is modular.*

Proof. Suppose that A is a GC-algebra. Then by Proposition 3.5, A is a GM-algebra. If $P \in \text{Prim}(A)$, then since P is a maximal ideal in A (Theorem 3.2), A/P must be central. But then A/P is primitive and has a nontrivial center, implying that P is modular.

Conversely, suppose that (1) and (2) hold, and let $I \neq A$ be an ideal in A. From Lemma 2.14, A/I is a GM-algebra. Since any primitive ideal in A/I is of the form P/I for some $P \in \text{Prim}(A) \cap I^\perp$ [4; Proposition 2.11.5 (i)], and since $(A/I)/(P/I) \cong A/P$ for such P, every primitive ideal in A/I is modular. So to show that A is a GC-algebra, it is only necessary to show that A possesses a nonzero C-ideal. Let I be a nonzero M-ideal in A. The structure space of I, being homeomorphic to $\text{Prim}(A) - I^\perp$ with the relative structure topology [4; Proposition 3.2.1], is Hausdorff. Since any $P \in \text{Prim}(A) - I^\perp$ is a maximal ideal in A, $P + I = A$ and $I/(P \cap I) \cong (P + I)/P = A/P$ [4; Corollaire 1.8.4]. So any primitive ideal in I, being of the form $P \cap I$ for some $P \in \text{Prim}(A) - I^\perp$, must be modular. This and [4; Proposition 1.8.5] show that it is sufficient to establish the following: If A is a separable C^*-algebra all of whose primitive ideals are modular and whose structure space is Hausdorff, then A has a nonzero C-ideal.

For such a C^*-algebra A, the structure and weak* topologies coincide on $\text{Prim}(A)$ (Lemma 2.1). Let 1_P be the identity in A/P, $P \in \text{Prim}(A)$. Let (u_n) be an approximate identity in A indexed on the positive integers, and set

$$T_n = \{P \in \text{Prim}(A) \colon \| u_n(P) - 1_P \| \leqq 1/2\} ,$$

$n = 1, 2, \cdots$. Since $u_n(P) \to 1_P$ as $n \to \infty$ for each P, $\text{Prim}(A) = \bigcup_{n=1}^\infty T_n$. Let A' be the C^*-algebra obtained by adjoining an identity 1 to A. Then $\text{Prim}(A') \cong \text{Prim}(A) \cup \{A\}$ and $A^\perp = \{A\}$. Fix a $P' \in \text{Prim}(A') - A^\perp$, and set $P = P' \cap A$. Then $a(P) \to a(P')$, $a \in A$, is an isomorphism of A/P onto $(A + P')/P'$. Choose a $b \in A$ such that

$b(P) = 1_P$. Then $b(P')$ must be an identity in $(A + P')/P'$. The latter is an ideal in A'/P', and from Lemma 3.3, $b(P')$ is a central idempotent in A'/P'. Since A'/P' is primitive, $b(P') = 1(P')$. Consequently,

$$\|(u_n - 1)(P')\| = \|(u_n - b)(P')\| = \|(u_n - b)(P)\| \\ = \|u_n(P) - 1_P\| .$$

Therefore

$$T_n = \{P' \cap A : P' \in \mathrm{Prim}\,(A') \quad \text{and} \quad \|(u_n - 1)(P')\| \leqq 1/2\} ,$$

and T_n is a closed subset of Prim (A). Since the structure space of A is Baire [4; Corollaire 3.4.13], some T_n contains a nonempty open set T. Because $u_n \geqq 0$ and $\|u_n\| \leqq 1$, $\mathrm{Sp}\, u_n(P) \subset [1/2,\ 1]$ for each $P \in T$. Choosing a continuous real-valued function f on $[0, 1]$ with $f(0) = 0$ and $f = 1$ on $[1/2, 1]$ and setting $a = f(u_n)$, $a(P) = 1_P$ for each $P \in T$ [4; Proposition 1.5.3]. Let I be the ideal in A with $\mathrm{Prim}\,(A) - I^{\perp} = T$. Say $P \in T$. Since Prim (A) is locally compact and Hausdorff, there is a continuous bounded function g on Prim (A) such that $g(P) = 1$ and g vanishes off T. From the Dauns-Hofmann theorem (see [7; § 7]), there is a $b \in A$ with $b(Q) = g(Q)a(Q)$ for all $Q \in \mathrm{Prim}\,(A)$. Then $b(Q) = 0$ if $I \subset Q \in \mathrm{Prim}\,(A)$ and $(bc - cb)(Q) = 0$ if $c \in A$ and $Q \in \mathrm{Prim}\,(A)$, which imply (by [4; Th. 2.9.7 (ii)] that $b \in Z(I)$. Therefore I satisfies condition (i) of Lemma 3.4, and so is a C-ideal in A. This completes the proof of Theorem 3.7.

It is not known whether the conclusion of Theorem 3.7 is true for nonseparable C^*-algebras.

4. **Concluding remarks.** Let A be a C^*-algebra. Recall that A is a CCR-algebra ("liminaire") if the image of A by any irreducible representation is contained in the algebra of compact operators on the representing Hilbert space. A nonzero ideal I in A is a CCR-ideal in A if it is a CCR-algebra, and A is a GCR-algebra ("postliminaire") if every nonzero quotient of A contains a nonzero CCR-ideal.

The spectrum of A is the set $\hat{A}$ of all equivalence classes of irreducible representations of A provided with the inverse image topology by the natural map $\pi \to \mathrm{Ker}\,\pi$ of $\hat{A}$ onto the structure space of A. Dixmier [4; § 4.5] has shown that the closure $J(A)$ of the finite linear combinations of those $a \in A^+$ for which $\pi \to \mathrm{Tr}\,\pi(a)$ is finite and continuous on $\hat{A}$ is an ideal in A. A nonzero ideal I in A will be called a CTC-ideal in A if $I \subset J(A)$, and A will be called a CTC-algebra [resp., GTC-algebra] if A is a CTC-ideal in itself [every nonzero quotient of A contains a nonzero CTC-ideal]. These algebras have been studied in the literature, where they are sometimes called "C^*-algèbre à trace continue" ["C^*-algèbrea à trace continue géneralisée"]. Recall that a CTC-algebra has Hausdorff structure space and that a GTC-algebra is CCR ([4; § 4]).

A CCR-algebra A with a Hausdorff structure space will be said to satisfy the *Fell condition* if the canonical field of C^*-algebras defined by A satisfies the Fell condition of Dixmier [4; § 10.5]. This amounts to saying that given $P_0 \in \mathrm{Prim}(A)$, there is an $a \in A$ such that $a(P)$ is a one-dimensional projection in A/P for all P in some structure neighborhood of P_0. The following are some of the relations between the various classes of C^*-algebras:

(1) if A is separable, then it is both GM and GCR if and only if it is GTC ([5; Proposition 4.2]),

(2) if A is separable, then it is both GC and GCR if and only if it is GTC and all its irreducible representations are finite-dimensional ((1) and Theorem 3.7),

(3) A is GCR and M and satisfies the Fell condition if and only if it is CTC ([4; Propositions 4.5.3 and 10.5.8]; recall that A is CCR if it is GCR and M),

(4) A is a central GCR-algebra and satisfies the Fell condition if and only if it is a CTC-algebra with local identities ((3) and Proposition 3.7), and

(5) if A is separable, then it is GM if either it is a CCR-algebra with compact structure space or its irreducible representations are all finite-dimensional ([3; § 1]).

Let H be a separable infinite-dimensional Hilbert space. Let B denote the C^*-algebra obtained by adjoining an identity to $CC(H)$, the compact operators on H. The structure space of B (see [4; Exercise 4.7.14 (a)]) fails to be T_1, and therefore is not almost strongly separated. Yet $\mathrm{Prim}(B)$ is K_σ in the weak* topology.

In [3; § 2], Dixmier has constructed a separable CCR-algebra D whose structure space contains no nonempty strongly separated subset. In particular, D is not GM. Nevertheless, there is an open subset of the structure space of D which is homeomorphic to $[0, 1]$, and D contains an ideal C isomorphic to the C^*-algebra of continuous maps of $[0, 1]$ into $CC(H)$. So C is an M-algebra, yet no nonzero ideal in C is an M-ideal in D. Since D is a CCR-algebra, $\mathrm{Prim}(D)$ is T_1 in the structure topology, so that $\mathrm{Prim}(D)$ cannot be K_σ in the weak* topology (Theorem 3.2). These two examples are the ones promised after Theorems 2.15 and 3.2.

Finally, one further point of contact between C^*-algebras and simplex spaces will be mentioned. Fell has shown that a C^*-algebra

A can be described (to within isomorphism) as the set of all functions on Prim $(A)^-$ satisfying certain conditions, the value of such a function at an $I \in \text{Prim}(A)^-$ being an element of A/I [12]. Moreover, the Dauns-Hofmann theorem (see [7; § 7]) may be deduced from this representation theorem [Fell, unpublished]. There is an analogous representation theorem for simplex spaces, due to Effros [10; Corollary 2.5]. The analogue of the Dauns-Hofmann theorem for simplex spaces can be deduced from this representation theorem (however, this is not the manner in which it is proven in the literature; cf. [10; Th. 2.1]).

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