PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 2, 1970

A NOTE ON HANF NUMBERS

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We show that for every $\xi < (2^\kappa)^+$, there is a theory T and set of types P in a language of power κ, such that there is a model of T which omits every $p \in P$ of power λ if and only if $\lambda \leqq \beth_\xi$. We also disprove a conjecture of Morley on the existence of algebraic elements.

The results which are proved here appear in [5].

1. On η_κ.

DEFINITION 1.1. η_κ will be the first cardinal such that for every language L, $|L| \leqq \kappa$, and set of types $\{p: p \in P\}$ (in L) if T has a model of power $\geqq \eta_\kappa$ which omits all the types in P, then T has such models in every power $\geqq |T|$. (A type is a set of formulas with the variables $x_0, \cdots, x_n$ only for some $n < \omega$. A model omits p if there does not exist $a_0, \cdots, a_n$ in the model such that $\varphi(x_0, \cdots, x_n) \in p$ implies $M \models \varphi[a_0, \cdots, a_n]$.)

Chang showed in [2], by methods of Morley from [4] that $\eta_\kappa \leqq \beth[(2^{|T|})^+]$. He also in [1] asked what is η_κ. We shall show that $\eta_\kappa = \beth[(2^\kappa)^+]$. For this it is sufficient to prove that for every $\xi < (2^\kappa)^+$ there exists a theory T and a set of types P (in a language $L = L(T)$ of power $\leqq \kappa$) such that T has a model of power λ which omits all the types in P if and only if $\lambda \leqq \beth_\xi$.

The following theorem appears in many articles which deals with finding lower bounds for Hanf numbers.

THEOREM 1.1. *If there exists a theory T, $|L(T)| \leqq \kappa$, and a set of types P in $L(T)$, such that every model of T which omits every $p \in P$ is well ordered in an order type $\leqq \xi$, and it has such a model whose order type is ξ, then $\eta_\kappa > \beth_\xi$.*

Proof. We adjoin to L the predicates $Q_1(x)$, $Q(x)$, $x \in y$, the constants c_n, $n < \omega$ and the function $F(x)$, and we get a language L_1, $|L_1| \leqq \kappa$. We define $T_1 = \{\psi^Q: \psi \in T\}$ [ψ^Q is ψ relativized to Q, that is instead of $(\exists x)\varphi$ we write $(\exists x)(Q(x) \wedge \varphi)$ and instead of $(\forall x)\varphi$ we write $(\forall x)[(Q(x) \rightarrow \varphi)]$. We also define $P_1 = \{p^Q: p \in P\} \cup \{q\}$, $p^Q = \{\varphi^Q: \varphi \in p\}$, $q = \{Q_1(x)\} \cup \{x \neq c_n: n < \omega\}$.

We add to T_1 an axiom of extensionality

$$\varphi_1 = (\forall xy)[(\forall z)[z \in x \leftrightarrow z \in y] \rightarrow x = y]$$

and an axiom saying that $F(x)$ is the rank of x

$$\varphi_2 = (\forall x)Q(f(x)),\ \varphi_3 = (\forall xy)[x \in y \rightarrow F(x) < F(y)]$$

and an axiom saying that $Q_1(x)$ if and only if the rank of x is minimal

$$\varphi_4 = (\forall x)[Q_1(x) \leftrightarrow \neg (\exists y)(F(y) < F(x))]$$

and $T_2 = T_1 \cup \{\varphi_i: i = 1, 4\}$.

Let M be a model of T_2 which omits every type in P_1. It is clear that Q^M is well ordered by $<^M$ in an order type $\leqq \xi$. Assume $Q^M = \{a_i: i < i_0 \leqq \xi\}$, where $i < j$ implies $a_i <^M a_j$. Let us define $A_i = \{a: F^M[a] = a_i\}$, and a function f, $f(a) = \{b \in M: b \in^M a\}$. As M is a model of φ_1; $f(a) = f(b)$ if and only if $a = b$, and as M is a model of φ_2 and φ_3, if $a \in A_i$ then $f(a) \subset \bigcup_{j<i} A_j$. From this it is clear that $|A_i| = |\{f(a): a \in A_i\}| \leqq 2^{|\bigcup_{j<i} A_j|}$. It is also clear that $|A_0| = \beth_0$. From this it is easy to prove by induction that $|\bigcup_{j<i} A_j| \leqq \beth_i$, and so $\|M\| \leqq |\bigcup_{i<i_1} A_i| \leqq \beth_\xi$.

On the other hand it is not hard to see that T_2 has a model of power $\beth_\xi$ which omits every $p \in p_1$.

So it is clear that $\eta_\kappa > \beth_\xi$.

THEOREM 1.2. *For every $\xi < (2^\kappa)^+$, there is a theory T, $|L(T)| \leqq \kappa$, and a set of types P (in the language L) such that for every model M of T which omits every $p \in P$ its set of elements is well ordered by $<^M$, and its order type is $\leqq \xi$. Also T has a model which omits every $p \in P$, and the order type of the set of its elements is ξ.*

Proof. For simplicity suppose $|\xi| = 2^\kappa$ (it is clear that this is sufficient for proving $\eta_\kappa = \beth_{(2^\kappa)^+}$).

Let S be the set of subsets of $\kappa = \{i: i < \kappa\}$. As $|S| = 2^\kappa = |\xi|$ we can order S in an order of type ξ. $S = \{a_i: i < \xi\}$.

Let us define the language L. It will have κ one-place predicates Q_i, $i < \kappa$, and an order predicate $<$, and the equality sign. We define

$$p_0 = \{(Q_i(x_0) \leftrightarrow Q_i(x_1)): i < \kappa\} \cup \{x_0 \neq x_1\}\ .$$

For every j, $i < \xi$,

$$p^{i,j} = \{x_0 \leqq x_1\} \cup \{Q_h(x_0): h \in s_i\} \cup \{\neg Q_h(x_0): h \notin s_i,\ h < \kappa\} \cup \{Q_h(x_1): h \in s_j\} \cup \{\neg Q_h(x_1): h \notin s_j,\ h < \kappa\}\ .$$

We define $P = \{p_0\} \cup \{p^{i,j}: j < i < \xi\}$.

If M is a model, which omits every $p \in P$, we define a function f from the set of elements of M to S by $f(a) = \{h: h < \kappa,\ a \in Q_h^M\}$. As M omits p_0, $a \neq b \Rightarrow f(a) \neq f(b)$, and as M omits $p^{i,j}$ for every $j < i < \xi$, it is clear that $a <^M b$ if and only if $f(a) <^M f(b)$. So it is clear that $T = \{\ \}$ and P satisfies the conclusion of the theorem.

THEOREM 1.3. $\eta_\kappa = (2^\kappa)^+$.

Proof. Immediate.

2. On algebraic elements. Morley in [4] conjectured that if T is a complete denumerable theory in a language L, p a type in L, and T has a model omitting p of power κ if and only if $\kappa_0 > \kappa \geqq \aleph_0$, and $\kappa_0 > \aleph_1$, then T has exactly $\aleph_0$ algebraic elements, where:

DEFINITION 2.1. (1) In a model M an element a is algebraic if there is a formula $\varphi(x)$ such that $M \models \varphi[a]$ and $|\{b \in M: M \models \varphi[b]\}| < \aleph_0$.

(2) A complete theory T has λ algebraic elements if every model of T has λ algebraic elements.

We shall disprove this conjecture.

DEFINITION 2.2. $K(T, p)$ is an infinite cardinal such that T has a model of power κ which omits p if $\kappa < K(T, p)$, $\kappa \geqq |T|$, and has no such model of power $\geqq K(T, p)$, $K(T, p) = \infty$ if there is no such cardinal.

Claim 2.1. Let T be a complete theory, p_i is a type in the variables $x_0, \cdots, x_{n_i-1}$ for $i = 0, \cdots, m$, and T has a model of power κ omitting $p_0, \cdots, p_m$ if and only if $\kappa_0 > \kappa \geqq |T|$.

Then there exists a complete theory T_1, $|T_1| = |T| + \aleph_0$ and a type p in the variable x_0, such that $K(T_1, p) = \kappa_0$ and T_1 has algebraic elements if and only if T has algebraic elements.

Proof. Suppose M is a model of T. We define a model M_1 whose elements will be the elements of M and sequences of length $n = \sum_{i<m} n_i < \aleph_0$ of elements of M. The relations will be the relations in M, and Q^{M_1} which will be the set of elements of M, the functions $F_i^{M_1}$ for $i < n$ such that $F_i^{M_1}(\langle a_0, \cdots, a_{n-1}\rangle) = a_i$ (when $a_0, \cdots, a_{n-1} \in M$) and $F_i^{M_1}(a) = a$ (when $a \in M$). The theory T_1 will be the set of sentences which hold for M_1. It is easily seen that T_1 is a complete theory, $|T_1| = |T| + \aleph_0$, and that T_1 has algebraic elements if and only if T has algebraic elements.

We shall also define

$$p = \left\{\bigvee_{h=0}^{m} \varphi_h(F_{l_h}(x), \cdots, F_{l_h+n_h-1}(x)) : \varphi_h(x_0, \cdots, x_{n_i-1}) \in p_h,\ l_h = \sum_{j<h} n_j\right\}.$$

It is easily seen that T_1 and p satisfy our demands.

THEOREM 2.2. *If T is a complete theory, p a type, then there exists a complete theory T^1 and a type p^1 such that $|T_1| = |T| + \aleph_0$ and $K(T, p) = K(T^1, p^1)$, and T^1 has no algebraic elements.*

REMARK. Clearly this disproves Morley's conjecture.

Morley told me that between 1963 and 1966 he disproved his conjecture. Later some people wrote him that they disproved the conjecture, but he did not remember their names. Seemingly, the review [3] is the first place the disproof was mentioned, but the proof does not appear anywhere.

Proof. Let N be a model of T. We shall define M the elements of M will be pairs of the form $\langle a, i\rangle$ where $a \in N$, and i is an integer. If R^N is a relation in N, then

$$R^M = \{\langle\langle a_i, l\rangle, \cdots, \langle a_n, l\rangle\rangle : \langle a_i, \cdots, a_n\rangle \in R^N, l \text{ is integer}\} .$$

We define $\leqq^M : \langle a_1, i_1\rangle \leqq^M \langle a_2, i_2\rangle$ if and only if $i_1 \leqq i_2$ (as integers.). We define F^M, $F^M(\langle a_1, i_1\rangle, \langle a_2, i_2\rangle) = \langle a_1, i_2\rangle$.

T_1 will be the set of sentences that M satisfies.

Let us define

$$p_2 = \Big\{(\forall z_1, \cdots, z_n)(\exists z_{n+1})(x_0 \leqq z_{n+1} \wedge z_{n+1} \leqq x_1 \wedge \bigwedge_{i=1}^{n} \neg(z_{n+1} \leqq z_i \wedge z_i \leqq z_{n+1})) : n < \omega\Big\}$$

W. l.o.g. let y be the only unbound variable which appears in the formulas of p. We define ψ^* by induction for subformulas of formulas of p: if in φ no quantifiers appear, then $\varphi^* = \varphi$, and $((\exists x)\varphi)^* = (\exists x)[x \leqq y \wedge y \leqq x \wedge \varphi^*]$.

We define $p_1 = \{\psi^* : \psi \in p\}$.

It is clear that for every integer i_0, the mapping $\langle a, i\rangle \rightarrow \langle a, i + i_0\rangle$ is an automorphism of M. So for every element of M there exists an infinite number of elements which are its image by some automorphism of M_1. So M has no algebraic elements. It is clear that if M_2 is a model of T_1 which omits p_1, then for every $a \in M_2$, $\kappa_1 = |\{b \in M_2 : M_2 \models b \leqq a \wedge a \leqq b\}| < K(T, p)$. If M_2 also omits p_2, then the power of M_2 is $\kappa_1 \aleph_0 = \kappa_1 < K(T, p)$. On the other hand, for every $\kappa < K(T, p)$, $\kappa \geqq |T|$, it is easy to construct a model of T_1 omitting p_1 and p_2. By Theorem 2.1 the conclusion of 2.2 follows immediately.

The referee has informed me that a little later than I, James

Schmerl (U.B.C.) independently discovered the same proof of Theorem 1.3. $-\eta_\kappa = \beth[(2^\kappa)^+]$. After writing this paper, I find in a review on an article of Morley, that Morley has already disproved this conjecture (see [3]).

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Received July 28, 1969.

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