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THE STRICT TOPOLOGY ON BOUNDED SETS

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If B is a Banach algebra with approximate identity and the Banach space X is a left B-module, the strict topology β on X is the topology given by the seminorms $x \to \|Tx\|$, one for each $T \in B$. It is shown that β is the finest locally convex topology on X agreeing with itself on the bounded sets in X, and that in certain circumstances a single semi-norm $x \to \|Ax\|$ determines β on each bounded set. It is then natural to investigate the sufficiency of sequences in determining the strict topology. A study is made of the finest locally convex topology on X having the same convergent sequences as β, and sufficient conditions are given which place the strict topology in the context of earlier sequential studies of other authors.

In [17] a study is made of the strict topology as defined above. In [16] some partial results are given which make β the Mackey topology on X_β. A crucial result is the extension of [7] to the general setting of [17]. In this paper we present the proofs of this and other results needed for [16] along with some improvements and more application and exploitation, particularly to a study of sequences in X_β, where for example it will be shown that in the case of a countable approximate identity for B, a sequentially continuous linear operator on X_β is continuous.

2. The main result on bounded sets. For each $r > 0$ let $B_r = \{x \in X: \|x\| \leqq r\}$. It follows readily from [14, Th. 2, p. 10] that $\mathscr{W} = \{W \subset X: W$ is absolutely convex, absorbent and for each $r > 0$ there is a β-neighborhood V_r of 0 such that $W \cap B_r \supset B_r \cap V_r\}$ forms a base of neighborhoods of 0 for a locally convex topology on X which, following Dorroh [7], we will denote by β'. In the special case of $X = C(S)$ and $B = C_0(S)$, β' has proven useful in [4], [6] and [15] and it was finally shown in [7] that $\beta = \beta'$ on $C(S)$. We will extend this to the general setting in [17]. Notice that β' is the finest locally convex topology on X agreeing with β on each set B_r and that $\beta \leqq \beta'$. A general study of topologies defined in this way is made in [2].

Before going further, the reader familiar with [17] will recall the introduction of another norm (a more natural one) defined by $\|x\|' = \sup\{\|Tx\|: T \in B, \|T\| \leqq 1\}$. These norms are equivalent if and only if the β bounded sets are bounded in the given norm on X [17, Th. 4.6] and [17, Corollary 4.7] are equivalent whenever X_β is complete Let β'' denote the finest locally convex topology on X agreeing with

β on each set $B'_r = \{x \in X: ||x||' \leqq r\}$. Since $||x||' \leqq ||x||$ clearly $\beta \leqq \beta'' \leqq \beta'$. All further notation is taken from [17].

THEOREM 2.1. *If* $W \in \mathscr{W}$ *is* β*-closed, then* W *is a* β*-neighborhood of* 0.

Proof. By hypothesis and [17, Th. 3.1], for each positive integer n there is a $T_n \in B$ such that $W \cap B_n \supset B_n \cap V_n$ where $V_n = \{x \in X: || T_n x || \leqq 1\}$. Hence $W \supset \bigcup_{n=1}^{\infty} (B_n \cap V_n)$ and therefore

$$W^\circ = \{x' \in X'_\beta: |\langle x, x' \rangle| \leqq 1 \text{ for all } x \in W\} \subset \bigcap_{n=1}^{\infty} (B_n \cap V_n)^\circ .$$

We will show that $\lim_\lambda \sup \{|\langle x, x' \cdot E_\lambda - x' \rangle|: x \in X_e, ||x|| \leqq 1\} = 0$ uniformly on $x' \in W^\circ$.

Let $\varepsilon > 0$ and choose n with $1/n < \varepsilon/2$. Then there is a λ_0 such that $\lambda \geqq \lambda_0$ implies $|| T_n E_\lambda - T_n || < 1/n$. For $x \in X_e$, $||x|| \leqq 1$ let $y_\lambda = n(E_\lambda x - x)$ for $\lambda \geqq \lambda_0$. Then $(1/2)y_\lambda \in B_n \cap V_n \subset W$ since $||E_\lambda|| \leqq 1$. Therefore if $x' \in W^\circ$, then $|\langle (1/2)y_\lambda, x' \rangle| \leqq 1$ or $|\langle E_\lambda x - x, x' \rangle| = |\langle x, x' \cdot E_\lambda - x' \rangle| < 2/n < \varepsilon$ for all $\lambda \geqq \lambda_0$ and $||x|| \leqq 1$.

Finally since W is absorbent in X then W° is bounded in X'_β and applying [17, Th. 4.8(2)] one has that W° is equicontinuous in X'_β and hence that $W^{\circ\circ}$ is a β-neighborhood of 0. Since W is β-closed and absolutely convex, $W = W^{\circ\circ}$ and the proof is complete.

THEOREM 2.2. $\beta = \beta'' = \beta'$.

Proof. It suffices to show $\beta = \beta'$. From [14, p. 12] β' has a base of β'-closed absolutely convex neighborhoods of 0. If we can show that the β'-closed convex sets are β-closed then by Theorem 2.1 we are through. By [14, p. 34] the closed convex sets are the same in any topology of a dual pair. Hence it suffices to show that the β'-continuous linear functionals on X are β-continuous.

Let f be a β'-continuous linear functional. We can consider $f \in X'$ and X' is a right Banach B-module under the multiplication $(x' \cdot T)(x) = x'(Tx)$. By [17, Th. 4.1(1)] and [13, Proposition 3.4] it suffices to show that $\lim_\lambda \sup \{|f(E_\lambda x - x)|: x \in X, ||x|| \leqq 1\} = 0$. If this were not so there would exist an $\varepsilon > 0$ such that for each λ there is a $\lambda' \geqq \lambda$ and an $x_{\lambda'} \in X$, $||x_{\lambda'}|| \leqq 1$, such that $|f(E_{\lambda'} x_{\lambda'} - x_{\lambda'})| \geqq \varepsilon$. Let $\Gamma' = \{\lambda' \in \Gamma$: there is an $x \in B_1$ such that $|f(E_{\lambda'} x - x)| \geqq \varepsilon\}$. Then Γ' is a nonempty directed set and if $x_{\lambda'} \in \{x: |f(E_{\lambda'} x - x)| \geqq \varepsilon\}$ for each $\lambda' \in \Gamma'$, then $||x'_\lambda|| \leqq 1$ and $y_{\lambda'} = E_{\lambda'} x_{\lambda'} - x_{\lambda'}$ is a net in X for which $||E_{\lambda'} x_{\lambda'} - x_{\lambda'}|| \leqq 2$. Furthermore if $\delta > 0$ and $T \in B$ then there is a λ_0 and a $\lambda'_0 \in \Gamma'$ such that $\lambda \geqq \lambda'_0 \geqq \lambda_0$ implies $|| TE_\lambda - T|| < \delta$. Thus $\lambda' \in \Gamma'$, $\lambda' \geqq \lambda'_0$ implies $|| T(E_{\lambda'} x_{\lambda'} - x_{\lambda'}) || < \delta$. Therefore $y_{\lambda'} \xrightarrow{\beta} 0$. But being bounded, $y_{\lambda'} \xrightarrow{\beta'} 0$

and therefore $|f(y_{\lambda'})| \geqq \varepsilon \to 0$, a contradiction.

Consequently β is the finest locally convex topology on X agreeing with β on each set B_r, extending the result of [7] to the general setting. This is an improvement over [16] which was not apparent until the topics in §4 were considered. From this we can obtain a kind of minimal Banach algebra B defining the strict topology of a given B on X. Let B_0 denote the minimal closed subalgebra of B containing all the E_λ. Let β_0 denote the strict topology on X defined by B_0.

COROLLARY 2.3. $\beta = \beta_0$.

Proof. Clearly $\beta_0 \leqq \beta$ and $\kappa \leqq \beta_0$ where κ is defined in [17]. But by [17, Th. 3.3(2)], $\kappa = \beta$ on each set B_r so that $\beta = \beta_0$ on each set B_r. Hence $\beta \leqq \beta_0'$ and by Theorem 2.2 $\beta = \beta_0$.

COROLLARY 2.4. *If $E_\lambda E_\mu = E_\mu E_\lambda$ for all $\lambda, \mu \in \Gamma$, then β is defined by the commutative Banach algebra B_0.*

Regarding Corollary 2.4 we point out the result in [11] that if B is a C^*-algebra with a positive element then B has a countable commutative approximate identity and conversely. Finally from the definition of β' we have

COROLLARY 2.5. *If E is a locally convex space and L is a linear operator on X into E then L is β-continuous if and only if L is β-continuous at 0 on each B_r. Consequently if E is complete, then $\mathscr{L}(X_\beta, E) = \{L: X_\beta \to E: L \text{ is continuous}\}$ is complete under the topology of uniform convergence on B_1.*

3. The case of a countable approximate identity. In this section we obtain the very useful result that the strict topology on each B_r is determined by a single element $A \in B$ when, for example, B has a countable approximate identity.

In [17] it is seen that $X_e = \{Tx: T \in B, x \in X\} = \{x \in X: ||E_\lambda x - x|| \to 0\}$ is a norm closed, β-dense subspace of X. Let $U = \{x \in X_e: ||x|| \leqq 1\}$ and $U^\circ = \{x' \in X_e': |\langle x, x'\rangle| \leqq 1 \text{ for all } x \in U\}$. If $T \in B$, then thinking of T as a continuous linear operator on X_β into X_e, we have $T'(X_e') \subset X_\beta'$ where $T'x'(x) = x'(Tx) = x' \cdot T(x)$ in the notation of [17].

THEOREM 3.1. *Let F_n be a bounded sequence in B such that $F_n x \to x$ strictly for each $x \in X$ and let $1 \leqq a_n \to \infty$. Then*

(a) *$H = \bigcup_{n=1}^\infty (1/a_n) F_n'(U^\circ)$ is β-equicontinuous*

(b) *There is an $A \in B$ such that $x \to Ax$ is one-to-one on X and*

$||F_nA^{-1}|| = \sup\{||F_nA^{-1}y||: y \in A(X), ||y|| \leqq 1\} \leqq a_n$.

Proof. It is apparent that we can assume $||F_n|| \leqq 1$ for each n. Furthermore since H is bounded, then $H^\circ = \{x \in X: |\langle x, x'\rangle| \leqq 1$ for all $x' \in H\}$ is β-closed, absolutely convex and absorbent in X. If $r > 0$ and $a_k > r$ for $k \geqq N$, then it quickly follows that $H^\circ \cap B_r \supset B_r \cap V$, where V is the β-neighborhood of 0, $V = \{x: ||F_nx|| \leqq a_n$ for $n = 1, 2, \cdots, N\}$. By Theorem 2.1 H° is a β-neighborhood of 0 and hence H is equicontinuous.

Furthermore since H° is a β-neighborhood then by [17, Th. 3.1] there is an $A \in B$ such that $H^\circ \supset \{x: ||Ax|| \leqq 1\}$. Thus if $Ax = 0$ then $\alpha x \in H^\circ$ for any $\alpha > 0$ and consequently for all $\alpha > 0$ one has $|\langle \alpha x, (1/a_n)F_n'x'\rangle| \leqq 1$ for all $x' \in U^\circ$, or $|\langle F_nx, x'\rangle| \leqq a_n/\alpha$, which implies $F_nx = 0$ for each n, since $F_nx \in X_e$. But since $F_nx \to x$ in the strict topology (which as defined in [17] is Hausdorff) one has $x = 0$ and A is one-to-one.

Finally if $y = Ax$, $||y|| \leqq 1$, then $x \in H^\circ$ and again $|\langle x, (1/a_n)F_n'x'\rangle| = |\langle F_nA^{-1}y, x'\rangle| \leqq a_n$ for all $x' \in U^\circ$. Consequently $||F_nA^{-1}|| \leqq a_n$ on $A(X)$.

Consequently if B has a countable approximate unit $\{E_n\}$, then there is a one-to-one $A \in B$ such that $||E_nA^{-1}|| \leqq a_n$ on $A(X)$ for a given sequence $a_n \to \infty$. Conversely, suppose there is an $A \in B$ which is one-to-one and for which $a_\lambda = ||E_\lambda A^{-1}|| = \sup\{||E_\lambda A^{-1}y||: y \in A(X), ||y|| \leqq 1\} < \infty$ for each λ. Then since $\{E_\lambda\}$ is an approximate identity for B one can choose a subsequence $F_n = E_{\lambda_n}$ with $\lambda_n \geqq \lambda_{n-1}$ such that $||AF_n - A|| \to 0$ and $||F_nA - A|| \to 0$. We then have

THEOREM 3.2. (a) *$F_nx \to x$ in the strict topology,* (b) *If $A(X)$ is dense in X_e, then $||F_nx - x|| \to 0$ for all $x \in X_e$.*

THEOREM 3.3. *If ω denotes the norm topology on X defined by the norm $x \to ||Ax||$ then*

(a) $\kappa \leqq \omega \leqq \beta$ (see [17, Th. 3.3]).

(b) *$\kappa = \omega = \beta$ on each set B_r'.*

(c) *If X is ω complete or X_β is complete and* $\sup_\lambda ||E_\lambda A^{-1}|| < \infty$, *then β is the given norm topology on X.*

Proof of 3.2. (a) For a fixed λ, $||E_\lambda(F_nx - x)|| = ||E_\lambda A^{-1}(AF_n - A)x|| \leqq a_\lambda||AF_n - A||\,||x|| \to 0$ as $n \to \infty$. Since $\{F_nx\}$ is bounded in X then $F_nx \xrightarrow{\beta} x$ by [17, Th. 3.3(4)].

(b) If $y \in X_e$, $\varepsilon > 0$ and $||Ax - y|| < \varepsilon/3$, then for $n \geqq N$ such that $||F_nAx - Ax|| < \varepsilon/3$, we have $||F_ny - y|| < \varepsilon$.

Proof of 3.3. (a) Clearly $\omega \leqq \beta$. Since κ is defined by the seminorms $x \rightarrow ||E_\lambda x||$ and the sets $\{x: ||E_{\lambda_i} x|| \leqq \varepsilon, \lambda_1, \cdots, \lambda_n \in \Gamma\}$ form a base of neighborhoods for κ and $||Ax|| \leqq \min\{\varepsilon/(a_{\lambda_i} + 1): 1 \leqq i \leqq n\}$ implies $||E_{\lambda_i} x|| \leqq ||E_{\lambda_i} A^{-1}|| \, ||Ax|| < \varepsilon$, then $\kappa \leqq \omega$.

(b) This follows from [17, Th. 3.3(2)].

(c) If X is ω-complete and $Ax_n \rightarrow y \in X$ then $\{x_n\}$ is ω-cauchy and there is an $x \in X$ such that $Ax = y$. But then A has closed range and by [17, Th. 2.4 and 3.2], β is the given norm topology on X. In the case that $M = \sup_\lambda ||E_\lambda A^{-1}|| < \infty$, if $||Ax|| \leqq 1$ then $||E_\lambda x|| \leqq M$ for all λ. Hence the β-neighborhood $V_A = \{x: ||Ax|| \leqq 1\}$ is β-bounded and it quickly follows that $p(x) = \inf\{\lambda: x \in \lambda A\}$ is a norm giving the strict topology on X. Since $p(x) \leqq ||A|| \, ||x||$ and X is complete it follows from the open-mapping theorem that the β and norm topologies are equivalent.

The sequence $\{F_n\}$ in 3.2 need not be an approximate identity for B. For example if S is the union of two disjoint σ-compact spaces S_1 and S_2 and if $B = C_0(S)$ with $X = \{f \in C(S): f \equiv 0 \text{ on } S_2\}$ (where $f \in C(S)$ if and only if f is bounded and continuous) then there is a $\phi \in C_0(S)$ such that $\phi \equiv 0$ on S_2, $f \rightarrow \phi f$ is one-to-one on X and $\{F_n\}$ would only be an approximate identity for $C_0(S_1)$.

The results above, particularly 3.3(b) were crucial to the proof of the main theorem in [16] because of a particular use of the following observation.

COROLLARY 3.4. *Under the conditions of* 3.2, *if E is a locally convex space such that continuity of a linear mapping on E is determined by continuity on bounded sets in E and $L: E \rightarrow X$ is bounded, then $L: E \rightarrow X_\beta$ is continuous if and only if $L: E \rightarrow X_\omega$ is continuous.*

Consequently although the strict topology is in general not barelled, bornological or Fréchet (see [17]), continuity on X_β is determined on bounded sets and continuity into X_β can be determined in the case just described by a single $A \in B$.

4. Sequences in X_β. The above results along with [17] indicate that the strict topology has some rather nice properties. In particular in the light of 3.3(b) and §2 one naturally wonders—when are sequences enough? In considering this question we were fortunate to come upon the work of Webb [18] and Dudley [8]. Following Webb's notation we denote by β^+ the finest locally convex topology on X having the same convergent sequences as β. By [18, Proposition 1.1], $\mathscr{U} = \{V \subset X$: V is absolutely convex and each β-null sequence is eventually in $V\}$ is a base at 0 for the topology β^+. Clearly $\beta \leqq \beta^+$ so that $X'_\beta \subset X'_{\beta^+}$. (Note also [17, Th. 3.3] that a sequence $\{x_n\}$ is β-null if and only if it is $||\cdot||'$-bounded and $E_\lambda x_n \rightarrow 0$ for each λ.)

Furthermore, if $X_\beta^+ = \{f \in X' : f(x_n) \to 0$ for each β-null sequence $\{x_n\}\}$, then $X'_{\beta^+} = X_\beta^+ = X_{\beta^+}^+ = \{f \in X' : f(x_n) \to 0$ for each β^+-null sequence $\{x_n\}\}$. Also by [18, Proposition 1.9], $f \in X_\beta^+$ if and only if $N(f) = \{x : f(x) = 0\}$ is β-sequentially closed. Finally, a set $K \subset X_\beta^+$ is called β-limited if every β-null sequence $\{x_n\}$ converges to zero uniformly on K and by [18, Proposition 1.3], β^+ is the topology of uniform convergence on the β-limited subsets of X_β^+.

Dudley [8] takes a more general approach to sequential properties and we list his definitions in our context for purposes of discussion. If $\{x_n\} \subset X$ and $x_n \xrightarrow{\beta} x$ (or equivalently $x_n \xrightarrow{\beta^+} x$), we will write $x_n \xrightarrow{C} x$ where $C = C(\beta)$ [8, p. 484]. Then $T(C) = \{U \subset X : x \in U$ and $x_n \xrightarrow{C} x$ implies $x_n \in U$ for $n \geqq$ (some) $N\}$, while $T_c(C) = \{V \subset X : x \in V$ implies there is a convex $U \in T(C)$ such that $x \in U \subset V\}$. Both $T(C)$ and $T_c(C)$ are topologies on X. While $(X, T(C))$ need not be a topological vector space it is straightforward to verify that $\beta^+ = T_c(C)$ since $T_c(C)$ is a locally convex linear topology for X [8, pp. 492–3]. Among other special spaces, Dudley goes on to single out spaces which he calls CS. From what we have noted and [8, p. 493], X_{β^+} is a CS-space since $T_c(C(\beta^+)) = T_c(C) = \beta^+$ and so X_β is a CS-space when $\beta = \beta^+$ and conversely. Hence when $\beta = \beta^+$ [8, §6] applies.

We begin with a study of when $\beta = \beta^+$ and then go on to show that the ideas developed in §3 fit nicely into another general structure considered by Dudley.

The next result can be proven for arbitrary locally convex spaces E with suitable definitions, as is apparent from the proof, but we will state it only for the case $E = X_\beta$.

THEOREM 4.1. *If $\beta = \beta^+$, then every β-sequentially continuous linear operator L on X into a locally convex space F is continuous on X_β. Conversely, if every β-sequentially continuous linear operator L on X into any space $C(T)$ of all bounded continuous functions on T with the sup norm topology is continuous, then $\beta = \beta^+$.*

Proof. If V is an absolutely convex neighborhood of 0 in E, then $L^{-1}(V)$ is a β^+-neighborhood of 0 when L is sequentially continuous. Hence L is β-continuous when $\beta = \beta^+$. Conversely, let T be a β-limited subset of X'_β and give T the weak* topology. If $x \in X$, then the restriction Lx of x to T is a bounded continuous function on T. The correspondence $x \to Lx$ defines a β-sequentially continuous linear operator on X into $C(T)$ because T is β-limited. If L is β-continuous then $\{x : \sup_{x' \in T} |\langle x, x' \rangle| \leqq 1\}$ is a β-neighborhood of 0, and at the same time is the polar of T in X. By [18, Proposition 1.3], this means $\beta = \beta^+$.

At this point we will give some results on special cases of the strict topology with interjections of more general results; hopefully the two will illuminate one another. In the sequel S is a locally compact, Hausdorff space and the strict topology on $C(S)$ is defined by the algebra $C_0(S)$.

THEOREM 4.2. *S is pseudo-compact if and only if β^+ is the norm topology on $C(S)$.*

Proof. If S is pseudo-compact and $f_n \to 0$ in the strict topology then by [1, Th. 2], $||f_n|| \to 0$ since $f_n \to 0$ uniformly on compacta. Hence β^+ is finer than the norm topology and thus equivalent. Conversely, suppose β^+ is the norm topology on $C(S)$. By [1, Th. 3 and 1] it suffices to show that if $\mathscr{U}$ is a countable, locally finite disjoint collection of open sets in S, then $\mathscr{U}$ is finite.

For each $U_n \in \mathscr{U}$ there is a $g_n \in C_0(S)$ such that $0 \leqq g_n \leqq 1$, $g_n(x) = 1$ for at least one $x \in U_n$ and $g_n \equiv 0$ on $S \backslash U_n$. Let $f_n = \max\{g_k: 1 \leqq k \leqq n\}$ and $f = \max\{g_k: k = 1, 2, \cdots\}$. Because $\mathscr{U}$ is locally finite, $f_n \to f$ in the compact open topology and since the f_n are all bounded by 1, $f_n \to f$ in the strict topology. This also means that $f \in C(S)$. Since β^+ is the norm topology, then $||f_n - f|| \to 0$ and if $\mathscr{U}$ were not finite, this would lead to a contradiction.

The next few results consider the general case and indicate that the relationship between β and β^+ is intimately related with the topological structure of S in the case of $C(S)$, while in the general case, it appears that a characterization of equality for these two topologies when B does not have a countable approximate identity must involve the topological relationship of X_e to X.

THEOREM 4.3. *If there is a norm η on X which gives the strict topology at 0 on each set B_r, then $\beta = \beta^+$.*

Proof. Let U be an absolutely convex β^+-neighborhood of 0 in X and let $W = \{x: \eta(x) \leqq 1\}$. Let $r > 0$ be fixed. If there is no $a > 0$ such that $U \cap B_r \supset B_r \cap aW$, then for each n there is an $x_n \in B_r \cap (1/n)W$ such that $x_n \notin U$. But then $\eta(x_n) \to 0$ and hence $x_n \in U$ eventually. By Theorem 2.2 U is a β-neighborhood of 0.

COROLLARY 4.4. *If B has a countable approximate identity or the hypothesis of Theorem 3.2 holds, then $\beta = \beta^+$ and any β-sequentially continuous linear operator on X is β-continuous.*

Proof. For the norm $\eta(x) = ||Ax||$ satisfies the conditions of 4.3 according to 3.3(b).

The next corollary is another version of the result given in [4, Corollary 6.2].

COROLLARY 4.5. *If S is σ-compact, then $\beta = \beta^+$.*

This brings up an interesting problem. Characterize those S for which $\beta = \beta^+$. In particular, is $\beta = \beta^+$ when S is paracompact (and perhaps even metrizable)? This case falls in between the extremes of S σ-compact and S pseudo-compact. Recalling [5, Th. 2.6], that β is the Mackey topology on $C(S)$ when S is paracompact, it is sufficient to show that $C(S)^+_\beta = C(S)'_\beta$ in order to obtain $\beta = \beta^+$. Referring to [17], [4, Th. 4.2], and the usual decomposition of a linear functional into its positive and negative parts, one needs to prove or disprove that a positive β^+-continuous linear functional F has the property that $F(\phi_K - 1) \to 0$ where $\{\phi_K\}$ is a β-totally bounded approximate (net) identity for $C_0(S)$. After some consideration of even the case of $C(S)_\beta$, S discrete (studied by Collins [3]) there appears to be no obvious answer. The work of Glicksburg [10] is related to this problem but it too does not appear to provide a definitive conclusion. Finally, Theorem 4.1 above and [16, Th. 2.1] indicates a relationship of sorts between the β^+ and Mackey topologies on $C(S)_\beta$.

Returning to the general case, the converse of Theorem 4.2 does not hold. To see this let X be a Hilbert space and let B be the algebra of compact operators in X. The strict topology is then the topology of uniform convergence on compacta in X and is the finest locally convex topology agreeing with the weak topology on each B_r [17], [12]. Consequently a strictly convergent sequence is bounded and weakly convergent and conversely. Hence β^+ is the finest locally convex topology on X having the same convergent sequences as the weak topology and

THEOREM 4.6. *For B and X defined as above $\beta = \beta^+$. When H is not separable there is no norm giving the strict topology on each B_r.*

Proof. Since β^+ is coarser than the norm topology on the reflexive space X and since β is finer than the weak topology on X, then $X'_\beta = X^+_\beta = X$. Let K be a β-limited subset of X^+_β. From our remarks above and [18, Proposition 1.3] it suffices to establish that K is norm relatively compact.

If this were not so then there is an $\varepsilon > 0$ and a sequence $\{x_n\} \subset K$ such that $||x_n - x_k|| \geqq \varepsilon$ for $k < n$. Since K is norm bounded and $X = X'$, then by the Eberlein-Smulian theorem [9, V. 6.1] there is an $x \in X$ and subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x$ weakly. Since $||x_{n_{k+1}} - x_{n_k}|| \geqq \varepsilon$, there is a $y_k \in X$ such that $||yk|| \leqq 1$ and

$|(x_{n_{k+1}} - x_n, y_k)| > \varepsilon/2$. Again by the Eberlein-Smulian theorem there is a $y \in X$ and a subsequence $\{yk_j\} \subset \{y_k\}$ such that $y_{k_j} \to y$ weakly. Hence $y_{k_j} \to y$ uniformly on the β-limited set K and there is a j_0 such that $j \geqq j_0$ implies $|(x_{n_i}, y_{k_j} - y)| < \varepsilon/8$ for all i. Hence $\varepsilon/4 > |(x_{n_{i+1}} - x_{n_i}, y_{k_j})| - |(x_{n_{i+1}} - x_{n_i}, y)|$. Then there is a k_0 such that $k \geqq k_0$ implies $|(x_{n_{k+1}} - x_{n_k}, y)| < \delta$ for a given $\delta > 0$. If $j \geqq j_0$ such that $k_j > k_0$, then $\varepsilon/4 > |(x_{n_{k_j+1}} - x_{n_{k_j}}, y_{k_j})| - \delta > \varepsilon/2 - \delta$. Hence $\varepsilon/4 > \varepsilon/2 - \delta$ for all $\delta > 0$ and this is a contradiction.

Finally, if there is a norm on X giving the strict topology on each B_r, then by [9, V. 5.2], $X = X'$ is separable.

The next two results are easy consequences of previous work. In both of the special cases considered above, $X = C(S)$ or X a Hilbert space, it is well known that a norm-continuous linear functional on X_e has a unique β-continuous extension to X.

THEOREM 4.7. *Let ξ be a topology on X which is finer than β and having the same bounded sets. If each ξ-continuous linear functional on X_e is β-continuous on X_e, then (in the topology of uniform convergence on the $\xi = \beta$ bounded subsets of X) X'_ξ is the algebraic and topological direct sum of X'_β and the orthogonal complement of X_e in X'_ξ.*

Proof. By [17, Corollary 3.4], X_e is β-dense in X and hence the restriction of an $x' \in X'_\xi$ to X_e has a unique β-continuous functional $J(x')$ on X. Since $J^2 = J$ and J is continuous (because β and ξ have the same bounded sets), then by [14, Proposition 30, p. 96], X'_ξ is the algebraic-topological direct sum of X'_β and $X_e^\circ = J^{-1}(0)$.

COROLLARY 4.8. *If each β^+-continuous linear functional on X_e is β-continuous, then*

(1) $X^+_\beta = X^\circ_e \oplus X'_\beta$ *topologically and algebraically where* $X^\circ_e = \{x' \in X^+_\beta : x' \equiv 0$ *on* $X_e\}$,

and (2) $X^+_\beta = X'_\beta$ *if* X_e *is* β^+*-dense in* X.

Proof. For since $\beta \leqq \beta^+$ any β^+-bounded set is β-bounded while if M is β-bounded and V is a β^+-neighborhood of 0 such that $x_n \in M \backslash nV$ for all n, then $\{(1/n)x_n\}$ is β-null but not eventually in V, a contradiction. Hence β and β^+ have the same bounded sets and 4.7 applies.

The introduction of the idea of a single $A \in B$ which determined β on each B_r was a device for obtaining the main result in [16, Th. 5.1]. This idea dovetails nicely with a structure studied by Dudley

[8, §'s 7 and 8]. Throughout we suppose $A \in B$ has the properties assumed for Theorem 3.2. Let $\rho(x, y) = \|Ax - Ay\|$, $f(x) = \|x\|' = \sup\{\|Tx\|: T \in B, \|T\| \leqq 1\}$. In the notation of [8, §5], $C(\rho, f)$ is, by Theorem 3.3(b) and [17, Th. 3.3], $C(\beta) = \beta$-convergence of sequences, and (X, ρ, f) is a simple quasi-metric space. In the terminology of [8] we will prove

THEOREM 4.9. (a) $(X, C(\beta)) = (X, C(\rho, f))$ *is an* L^**-convex,* L^**-linear space which is also an LS-space (by* (ρ, f)*).*

(b) (X, ρ, f) *is a simple quasi-metric linear space.*

Proof. (a) From [8, p. 492], $(X, C(\rho, f))$ is an L^*-linear space because X_β is a linear topological space and $C(\rho, f) = C(\beta)$ as noted above.

To see that $(X, C(\rho, f))$ is L^*-convex, let $\{x_n\}$ be a $\beta = C(\rho, f)$ null-sequence. By the definition [8, p. 496] it must be shown that if y_k is a convex combination of $\{x_j: j \geqq k\}$, then $\{y_k\}$ is β-null. If $y_k = \sum_{j=k}^{p_k} a_j x_j$, $\sum_{j=k}^{p_k} a_j = 1$, $a_j \geqq 0$ for all j, then $\|A_{y_k}\| \leqq \max\{\|Ax_j\|: k \leqq j \leqq p_k\}$. Since $\|Ax_n\| \to 0$ and A determines β on bounded sets and $\{x_n\}$ and $\{y_n\}$ are bounded, we are through.

Finally $(X, C(\rho, f))$ is an LS-space because ρ is an invariant metric and f is an LS-function [8, p. 496]. This last follows because $f(x) = \|x\|' = \sup\{\|Tx\|: T \in B, \|T\| \leqq 1\}$ and consequently $x_n \to x$ in $C(\rho, f)$ implies that $f(x) \leqq \limsup f(x_n)$.

(b) By [8, p. 496] (X, ρ, f) is simple quasi-metric linear because $(X, C(\rho, f))$ is an L^*-linear space.

As noted previously $C(\beta) = C(\rho, f)$ and hence $C(\beta^+) = C(\rho, f)$. But also as noted at the beginning of this section $\beta^+ = T_c(C(\beta))$ so that $\beta^+ = T_c(C(\rho, f))$ and [8, Th. 7.3] gives a new characterization of the β^+-neighborhoods of 0 in X. Furthermore in this setting $\beta = \beta^+$, by Corollary 4.4; hence this amounts to a new characterization of the β-neighborhoods of 0 in X. That is, from [8, Th. 7.3].

COROLLARY 4.10. *Under the hypothesis of Theorem* 3.2, *for each sequence of positive numbers* $\{\delta_n\}$, *let* $U\{\delta_n\} = \{\sum_{n=1}^{p} w_n: w_n \in \delta_n V_A \cap B'_n\}$ *where* $V_A = \{x \in X: \|Ax\| < 1\}$ *and* $B'_n = \{x: \|x\|' \leqq n\}$. *Then the collection of all sets* $U\{\delta_n\}$ *is a base for the neighborhood system at* 0 *for the strict topology.*

COROLLARY 4.11. *Under the conditions of Theorem* 3.2, *with* $C = C(\beta)$, $\beta = T_c(C)$ *is the finest topology* T *weaker than* $T(C)$ *such that* X_T *is a topological linear space.*

Proof. Referring to [8, Th. 7.4] we simply recall that under these hypotheses, $\beta = T_c(C)$ by Theorem 4.3.

Finally, Dudley [8] goes on to study complete *LS*-spaces and our final theorem shows that $(X, C(\beta))$ is complete by (ρ, f) when X_β is complete so that the results of [8, §8] apply.

THEOREM 4.12. *If X_β is complete, then $(X, C(\beta))$ is complete by (ρ, f) and conversely.*

Proof. By definition, if $\{x_n\}$ is a $C(\beta) = C(\rho, f)$-cauchy sequence then $x_n - x_{m(n)} \to 0$ in $C(\rho, f)$ for any choice of $m(n) \geqq n$ for all n. Suppose $\{x_n\}$ is not bounded in the $||\cdot||'$. Then there is a sequence $m(n) \geqq n$ such that $f(x_{m(n)}) \geqq f(x_n) + n$ where $f(x) = ||x||'$ as before. Since $x_{m(n)} - x_n \to 0$ in $C(\rho, f)$ then $\{f(x_{m(n)} - x_n)\}$ is bounded by definition. But $f(x_{m(n)} - x_n) \geqq f(x_{m(n)}) - f(x_n) \geqq n$ a contradiction.

Since $\{x_n\}$ is $||\ ||'$-bounded and $\rho(x_{m(n)} - x_n, 0) = ||A(x_{m(n)} - x_n)|| \to 0$, then by Theorem 3.3(b), $\{x_n\}$ is β-cauchy, hence β-convergent to some $x \in X$. But then $\{x_n\}$ is $C(\beta)$-convergent to x and $(X, C(\beta))$ is complete.

Conversely if $(X, C(\beta))$ is complete by (ρ, f) then by Theorem 4.9(a) and [8, Th. 8.1], X is complete for $T_c(C)$. But $\beta = T_c(C)$ as noted above.

Unfortunately, Theorem 4.12 along with 4.6 implies that [8, Th. 8.2] is false. In the notation of [8] and Theorem 4.6, if $\mathscr{N} = \{N(x)$: $N(x) = |(x, y)|$ for some $y \in H$, $||y|| \leqq 1\}$ then $M(x) = \sup\{N(x): N \in \mathscr{N}\} = ||x||$, and hence M cannot be a continuous pseudo-norm for $T_c(C) = \beta^+ = \beta$. Prof. Dudley acknowledges this and has pointed out to me that [8, Th. 8.3] is probably also false, being dependent on 8.2. It does appear however that the strict topology possesses several nice sequential properties and that X_β is a complete *LS*-space for a wide choice of B and X.

Remark added in proof. Because the β and norm topologies on X are locally convex, the mixed-topology defined by these [A. Wiweger, *Linear spaces with mixed topology*, Studia Math. T.XX (1961), 47–68] is locally convex. By Theorem 2.2 and [Wiweger, 2.2.2] β is then the mixed topology and hence is the finest linear topology agreeing with itself on each B_r.

Regarding the paragraph following 4.5, $l_\infty[0, 1]'_{\beta^+} = l_\infty[0, 1]'_\beta = l_1[0, 1]$ from 4.8 and the assumption of the continuum hypothesis, which implies that $[0, 1]$ has nonmeasurable cardinal and hence that $[0, 1]$ has no atomless measure defined on all subsets. Hence the matter appears to ultimately concern the so-called "problem of measure" for which H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals*, Fund. Math. 53 (1964), 225-308, and S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. 16 (1930), 141-150, are appropriate references.

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