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REPRESENTATION AND SERIES SUMMABILITY OF COMPLETE BIORTHOGONAL SEQUENCES

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Let $\{x_j, x'_j\}$ be a complete biorthogonal sequence with each x_j in a Banach space X and each x'_j in the conjugate space X^*. A study is made of the summability properties of the series $\sum_j x'_j(x)x_j$ and $\sum_j x'_j(x)x'(x_j)$ as x ranges over X and x' over X^*. Conditions are given for matrix and abstract methods of series summability for these series in terms of certain sequence spaces which arise naturally in connection with the biorthogonal sequence.

Fundamental work on complete biorthogonal sequences in Banach spaces was done by Banach in Chapter VII of [1] and by Frink in [3]. Further contributions were made by others including Kadets and Pełczynski in [6] where the concept of a norming biorthogonal sequence was defined. There are various types of biorthogonal sequences which generalize the Schauder basis and these are discussed in [11].

Let $\{x_j, x'_j\}$ be a complete biorthogonal sequence with each x_j in a Banach space X and each x'_j in the conjugate space X^*. In § 3 four spaces of scalar sequences are defined in connection with this biorthogonal sequence: (a) S, the associated sequence space, consisting of all sequences $\{x'_j(x)\}$ as x ranges over X; (b) S^f which consists of all sequences $\{x'(x_j)\}$ as x' ranges over X^*; (c) $M(S)$, the multiplier algebra; $(a_j) \in M(S)$ if and only if $\{a_j x'_j(x)\} \in S$ whenever $x \in X$; (d) $\mathscr{S}(S)$, the series space, which is, roughly speaking, the smallest Banach sequence space which contains all sequences of the form $\{x'_j(x)x'(x_j)\}$ as x ranges over X and x' over X^*. Note that these sequences are the terms of the numerical expansion

$$x'(x) \sim \sum_j x'_j(x)x'(x_j) \; . \tag{1.1}$$

The object of this paper is to represent these four sequence spaces, determine their relationship and apply them to determine properties of the biorthogonal sequence $\{x_j, x'_j\}$ and the space X. Particular attention is paid to the series summability of (1.1) and

$$x \sim \sum_j x'_j(x)x_j \; . \tag{1.2}$$

The most important results are Theorem 6.4 where conditions are noted under which there is a continuous linear functional on $\mathscr{S}(S)$ which gives to each series $\{x'_j(x)x'(x_j)\}$ its "correct" sum and Theorem

7.2 where conditions are given for the existence of a row finite matrix with $u_n(j)$ in the nth row jth column such that

$$x = \lim_n \sum_j u_n(j)x'_j(x)x_j$$

for each x in X.

In § 2 it is noted that a norming complete biorthogonal sequence can always be constructed in a separable Banach space. This strengthens a result of Markushevitch [7]. In § 4 we derive necessary machinery for what follows and discuss consistency for a subset of φ.

Means of constructing all types of spaces S and S^f are given in § 5. Since S^f is isomorphic to X^* and every separable Banach space admits a complete biorthogonal sequence, 5.4 is a representation for the dual of every separable Banach space. Also conditions are given which differentiate between norming and nonnorming biorthogonal sequences in terms of S^f.

The series space is studied in § 6 and the multiplier algebra in § 7. For instance, it is noted in 6.2 that $\mathscr{S}(S)$ consists of the diagonals of nuclear operators in X with respect to the biorthogonal sequence $\{x_j, x'_j\}$. Proposition 7.1 (a) generalizes Theorem 4.2 of [8] where multiplier algebras are also studied. Finally these final two sections contain the summability theorems as noted above.

2. Complete biorthogonal sequences.

A double sequence $\{x_j, x'_j\}$ with each x in a linear topological space X and each x'_j in the conjugate space X^* is called a *complete biorthogonal sequence* ("in X") if

(a) For each i and each j, $x_j(x_k) = \delta_{jk}$ (Kronecker δ);

(b) $\{x'_j\}$ is total on X, i.e., $x'_j(x) = 0$ for x in X only if $x = 0$;

(c) $\{x_j\}$ is total on X^*, i.e., $x'(x_j) = 0$ for x' in X^* only when $x' = 0$.

This terminology differs somewhat from that used by Banach in note § 1 to Chapter VII of [1] in that he uses a pair of sequences $\{x_j\}$, $\{x'_j\}$ instead of a double sequence. A biorthogonal sequence $\{x_j, x'_j\}$ in a Banach space X is called *norming* if there is a subset A of $[\{x_j\}]$, the span of $\{x_j\}$, such that the norm

$$||x||_A = \sup \{|x'(x)|: x' \in A\}$$

determines the topology of X.

THEOREM 2.1. *There is a complete norming biorthogonal sequence in every separable Banach space.*

Proof. Let $\{y_n\}$ be a sequence dense in X. By the Hahn-Banach theorem there is a sequence $\{y'_n\}$ in X^* such that $y'_n(y_n) = ||y_n||$ anb $||y'_n|| = 1$ for each n.

Let $x \in X$ and $\varepsilon > 0$ be given. There is y_n such that $||x - y_n|| < \varepsilon/2$. Then

$$\begin{aligned} 0 &\leqq ||x|| - |y'_n(x)| \leqq |\,||x|| - y'_n(x)| \\ &\leqq |\,||x|| - ||y_n||\,| + |\,||y_n|| - y'_n(x)| \\ &\leqq ||x - y_n|| + ||y'_n||\,||y_n - x|| < \varepsilon \end{aligned}$$

which implies that

$$||x|| = \sup_n |y'_n(x)| \ .$$

Since $\{y_n\}$ being dense in X is total on X^* and $\{y'_n\}$ is total on X there is by Theorem 4 of [4] a complete biorthogonal sequence $\{x_n, x'_n\}$ in X such that $[\{x_n\}] = [\{y_n\}]$ and $[\{x'_n\}] = [\{y'_n\}]$. If A consists of all $x' \in X^*$ such that $||x'|| \leqq 1$ then $\{y'_n\} \subset A$ so that

$$||x||_A = ||x||$$

for each $x \in X$. Therefore, $\{x_n, x'_n\}$ is norming.

The first part of the preceeding argument is frequently encountered; see, for instance, Lemma 4 of [6].

3. Sequence spaces associated with a complete biorthogonal sequence. In the following, the letters s, t, u, v will always denote sequences of scalars. If s is the sequence $\{a_1, a_2, \cdots\}$, $s(j)$ denotes the jth coordinate a_j. Addition, scalar multiplication and product of sequences is defined coordinatewise. The sequence each of whose coordinates equals one is denoted by e; the sequence $\{\delta_{nj}: j = 1, 2, \cdots\}$ by e_n. A linear space of sequences on which there is a locally convex topology is called a *K-space* if each coordinate functional given by

$$E_j(s) = s(j)$$

is continuous. A K-space which is a Banach space is called a *BK-space*.

For $\{x_j, x'_j\}$ a complete biorthogonal sequence in a Banach space X the *associated sequence space* written $S\{x_j, x'_j\}$ or simply S consists of all sequences

$$s_x = \{x'_j(x): j = 1, 2, \cdots\}$$

as x range over X. The correspondence of x in X to s_x is called the *canonical isomorphism* of X onto S. If S is given the topology of identification, i.e., the norm

$$||s_x|| = ||x||$$

then S becomes a BK-space. The canonical isomorphism is one to one since $\{x'_j\}$ is total on X. Under the canonical isomorphism x_j corresponds to e_j and E_j corresponds to x'_j in the sense that

$$E_j(s_x) = x'_j(x) .$$

The *dual associated sequence space* written $S^f\{x_j, x'_j\}$ or simply S^f consists of all sequences

$$t_y = \{y(x_j): j = 1, 2, \cdots\}$$

as y ranges over X^* or equivalently of all sequences

$$t_y = \{y(e_j): j = 1, 2, \cdots\}$$

as y ranges over S^*. With the norm

$$||t_y||_f = ||y||$$

S^f is isometric to X^* (or S^*) under the canonical isomorphism of y to t_y since $\{x_j\}$ is total on X^*.

The *multiplier algebra* denoted by $M(\{x_j, x'_j\})$ or more usually by $M(S)$ consists of all sequences u such that

$$us \in S \quad \text{whenever} \quad s \in S .$$

According to 3.3 of [8], $M(S)$ with the norm

$$||u||_M = \sup\{||us||: ||s|| \leqq 1\}$$

is a BK-algebra isometric to the subalgebra $\mathscr{D}$ of continuous operators F from X into X such that

$$x'_j(Fx_k) = 0$$

for $j \neq k$.

The *series space* of $\{x_j, x'_j\}$ denoted by $\mathscr{S}(S)$ consists of all sequences v having the form

$$v = \sum_{n=1}^{\infty} s_n t_n \tag{3.1}$$

such that each s_n in φ, each t_n is in S^f and

$$\sum_{n=1}^{\infty} ||s_n||\, ||t_n||_f < \infty . \tag{3.2}$$

The infinite series in (3.1) converges coordinatewise since

$$\sum_{n=1}^{\infty} |s_n(j)|\, |t_n(j)| \leqq ||e_j||\, ||e_j||_f \sum_{n=1}^{\infty} ||s_n||\, ||t_n||_f .$$

Throughout this paper the familiar properties of BK-space discussed in 11.3 and 12.4 of [12] are used without citation.

4. Absolutely convex subsets and *BK*-spaces. The space of all sequences which are eventually 0 is denoted by φ. The space of all sequences is denoted by ω. The space ω given the Tychonoff topology as a countable product of scalar fields has many familiar properties which are assumed in the sequel. For A a subset of φ and S an arbitrary space of sequences $A^{(S)}$ is equal to the set of all $s \in S$ such that $|\sum_j s(j)t(j)| \leqq 1$ for each t in A. Thus $A^{(S)}$ is the absolute polar of A in S when S and φ are placed in duality by means of the bilinear form

$$(s, t) = \sum_j s(j)t(j) .$$

For B a subset of ω, $B^{(\varphi)}$ denotes the absolute polar of B in φ, namely the set of all t in φ such that $|(s, t)| \leqq 1$ for each s in B.

4.1. Let A be a bounded subset of ω which is balanced but not necessarily convex, and let p be the Minkowski gauge of A, i.e.,

$$p(s) = \inf\{a > 0 : s \in aA\} .$$

(a) If $\{s_n\}$ is a sequence in ω such that $\sum_n p(s_n) < \infty$ then $\sum_n |s_n(j)| < \infty$ for each j.

(b) Let $S(A)$ denote the collection of all sequences s in ω such that

$$s = \sum_n s_n, \sum_n p(s_n) < \infty .$$

Then $S(A)$ is a *BK*-space with the norm

$$||s||^A = \inf\{\sum_n p(s_n) : \sum_n s_n = s\} . \tag{4.1}$$

(c) The absolutely convex hull of A denoted by $\kappa(A)$ is norm dense in the unit ball of $S(A)$. Thus $[A]$, the span of A, is dense in $S(A)$.

(d) Every *BK*-space S is of the form $S(A)$ where A is a balanced subset of ω.

Proof. (a) Since A is bounded in ω,

$$a_j = \sup\{|s(j)| : s \in A\} < \infty \tag{4.2}$$

for each j. Thus if $\sum_n p(s_n) < \infty$

$$\sum_n |s_n(j)| \leqq \sum_n a_j p(s_n) < \infty .$$

(b) The routine proof that $|| \ ||^A$ is a seminorm and that $S(A)$ is a linear space is omitted.

If $||s||^A = 0$ then for each $\varepsilon > 0$ there is a sequence $\{s_n\}$ in ω such that $\sum_n s_n = s$ and $\sum_n p(s_n) < \varepsilon$. Thus if a_j is given by (4.2)

$$|s(j)| \leqq a_j \varepsilon$$

for each j. This implies that $||\quad||^A$ is a norm since $s(j) = 0$ for each j.

For $s \in S(A)$ and each j

$$|E_j(s)| \leqq a_j \sum_n p(s_n)$$

whenever $\sum_n s_n = s$. Thus

$$|E_j(s)| \leqq a_j \, ||s||^A$$

so that $S(A)$ is a normed K-space.

Let $\{s_n\}$ be a sequence in $S(A)$ such that $\sum_n ||s_n||^A < \infty$. For each n let

$$s_n = \sum_{j=1}^{\infty} s_{nj}, \; \sum_{j=1}^{\infty} p(s_{nj}) < ||s_n||^A + \varepsilon/2n \; ,$$

and let

$$s = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} s_{nj} \; .$$

Then s is in $S(A)$ and $\{\sum_{n=1}^{k} s_n\}$ converges to s since

$$\left\| s - \sum_{n=1}^{k} s_n \right\|^A \leqq \sum_{n=k+1}^{\infty} \sum_{j=1}^{\infty} p(s_{nj})$$
$$\leqq \varepsilon/2^k + \sum_{n=k+1}^{\infty} ||s_n||^A \; .$$

Therefore, $S(A)$ is a complete space.

(c) If $||s||^A < 1$ there is a sequence $\{s_n\} \subset A$ such that

$$s = \sum_{n=1}^{\infty} s_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} p(s_n) < 1 \; .$$

Since A is balanced $\sum_{n=1}^{k} s_n$ is in $\kappa(A)$ for each k and

$$\lim_k \left\| \sum_{n=1}^{k} s_n - s \right\|^A = 0 \; .$$

Thus $\kappa(A)$ is dense in the unit ball of $S(A)$.

(d) Trivial. Let A be the unit ball of S.

Let A and B be two subsets of ω. If there is a number $a > 0$ such that $A \subset aB$ then B is said to *absorb* A. If A absorbs B and B absorbs A then A and B are called *equivalent*. If p is the Minkowski gauge of a balanced set A and $||\quad||^A$ the norm given by (4.1) then the set

$$B = \{s \in [A]\colon ||s||^A \leqq 1\}$$

contains $\kappa(A)$, but $\kappa(A)$ may not absorb B. For example, let t denote the sequence $(1, 1/2, 1/4, \cdots)$ and define p on $\varphi \oplus [\{t\}]$ by

$$p(s + at) = \sum_j |s(j)| + |a| .$$

Then $2^n(t - \sum_{j=1}^{n+1} t(j)e_j)$ is in B for each n whereas

$$p\Big(2^n\Big(t - \sum_{j=1}^{n+1} t(j)e_j\Big)\Big) > 2^n .$$

4.2. Let A be a bounded absolutely convex subset of ω; let $T = [A]$; let p be the Minkowski gauge of A; and let $\|\ \|^A$ be given by (4.1). Then the following statements are equivalent

(a) If $\sum_{n=1}^\infty s_n = s$ coordinatewise then $p(s) \leqq \sum_{n=1}^\infty p(s_n)$.

(b) There exists K such that $p(s) \leqq K \sum_{n=1}^\infty p(s_n)$ whenever $\sum_{n=1}^\infty s_n = s$ coordinatewise.

(c) There is K such that $p(s) \leqq K \|s\|^A$ for s in T.

(d) There is a BK-space S of which the normed space (T, p) is a topological subspace.

(e) If $\hat{E}_j$ denotes the extension of E_j to the completion $\hat{T}$ of (T, p) then $\{\hat{E}_j\}$ is total on $\hat{T}$.

(f) If $\{s_n: n = 1, 2, \cdots\}$ is a Cauchy sequence in (T, p) such that $s_n(j) \to 0$ for each j then $\lim_n p(s_n) = 0$.

(g) $p(s) = \|s\|^A$ for s in T.

Proof. First note that (T, p) is a K-space since A is bounded in ω.

(a)⇒(b)⇒(c). Obvious.

(c)⇒(d). Let $S = S(A)$. By definition $\|s\|^A \leqq p(s)$ for s in T so that (c) implies (T, p) is a topological subspace of $S(A)$.

(d)⇒(e). By 4.1(c), T is dense in $S(A)$, and if (d) holds, (T, P) is a topological subspace of $S(A)$. Thus $S(A)$ is isomorphic to $\hat{T}$ and the set of functionals $\{E_j\}$ defined on $S(A)$ as total.

(e)⇒(f). Let $x = \lim_n s_n$ in $\hat{T}$. If $\{E_j\}$ is total on $\hat{T}$ then $x = 0$ since $\hat{E}_j(x) = \lim_n E_j(s_n) = 0$ for each j. Thus $\lim_n p(s_n) = p(x) = 0$.

(f)⇒(a). It may be assumed that $\sum_{n=1}^\infty p(s_n) < \infty$. Then

$$\Big\{\sum_{n=k}^\infty s_n: k = 1, 2, \cdots\Big\}$$

is a Cauchy sequence in (T, p) such that $\sum_{n=k}^\infty s_n(j) \to 0$ for each j. Thus $\lim_k p(\sum_{n=k}^\infty s_n) = 0$ by (f) so that (a) holds by passing to the limit in the inequality

$$p(s) \leqq \sum_{n=1}^k p(s_n) + p\Big(\sum_{n=k+1}^\infty s_n\Big) .$$

(a)⇒(g)⇒(c). Obvious.

DEFINITION 4.3. A bounded absolutely convex subset A of ω will be called *consistent* if one, hence all, of the conditions of 4.2 is satisfied. A bounded subset A of ω which is balanced will be called *consistent* if $\kappa(A)$ is consistent.

See 3.1 of [10] in connection with the idea of consistency.

5. Representation.

THEOREM 5.1. (a) *A sequence space is associated with a complete biorthogonal sequence in a Banach space if and only if it is of the form* $S(A)$ *where* A *is a balanced subset of* φ *which absorbs each point of* φ.

(b) *In addition it may be assumed that* $A^{(\omega)(\varphi)} = A$ *and* A *is consistent.*

(c) *The complete biorthogonal sequence is norming if and only if the associated sequence space is of the form* $S(A)$ *where* $A^{(\varphi)(\varphi)} = A$.

Proof. (a) If A is a subset of φ having the properties listed in (a) then by 4.1, $S(A)$ is a BK-space in which $\{e_j, E_j\}$ is a complete biorthogonal sequence.

If S is a sequence space associated with a complete biorthogonal sequence in a Banach space let A consist of all $s \in \varphi$ such that $||s|| \leqq 1$ where $||\ \ ||$ is the identity norm on S obtained in §3. If p is the Minkowski gauge of A then $p(s) = ||s||$ for s in φ so that A is consistent by 4.2. Since φ is dense in S and in $S(A)$ the spaces are equal.

(b) It has already been noted that the set A defined in the previous paragraph is consistent. Furthermore A is absolutely convex and closed in the relative topology of S on φ so that $A^{(\omega)(\varphi)} = A$.

(c) If S has the form $S(A)$ where $A = A^{(\varphi)(\varphi)}$ then for s in φ

$$p(s) = \sup\{|(s, t)|: t \in A^{(\varphi)}\} . \tag{3.3}$$

Here p is the Minkowski gauge of A. Thus by 4.2(a) it follows that A is consistent and for $s \in S(A)$

$$||s||^A = \sup\{|(s, t)|: t \in A^{(\varphi)}\}$$

so that $\{e_j, E_j\}$ is a norming complete biorthogonal sequence in $S(A)$.

If S is associated with a complete norming biorthogonal sequence in a Banach space let B be a subset of φ such that the norm

$$||s|| = \sup\{|(s, t)|: t \in B\}$$

determines the topology of S. If $A = B^{(\varphi)}$ then $S = S(A)$ and $A^{(\varphi)(\varphi)} = B^{(\varphi)(\varphi)(\varphi)} = A$ because of the bipolar theorem.

For S a BK-space containing φ, S^o denotes the closure of φ in S and S^f denotes the space of all sequences of the form $\{x'(e_j): j = 1, 2, \cdots\}$, where $x' \in S^*$. Note that if $S = S^o$ then S^f has the same meaning as in §3.

5.2. Let S be a BK-space containing φ and having norm $\|\ \|$, and let $A = \{s \in \varphi: \|s\| \leqq 1\}$.

(a) $S^f = S^{of}$.

(b) With the norm

$$\|t\|_A = \sup\{|(t, s)|: s \in A\} \tag{5.1}$$

S^f is a BK-space isometric to S^{o*} under the correspondence of x' in S^{o*} to $\{x'(e_j)\}$ in S^f.

(c) $S^f = \bigcup_{n=1}^{\infty} nA^{(\omega)}$.

(d) $\{e_j, E_j\}$ is a norming complete biorthogonal sequence in S^{fo}.

Proof. (a) Since each continuous linear functional on S is continuous and linear when restricted to S^o, $S^f \subset S^{of}$. Since each continuous linear functional on S^o can be extended to S by the Hahn-Banach theorem $S^{of} \subset S^f$.

(b) Since $\{e_j, E_j\}$ is a complete biorthogonal sequence in S^o, S^f is a isometric to S^{o*} with the norm

$$\begin{aligned} \|t_y\|_f &= \sup\{|y(s)|: s \in S^o \text{ and } \|s\| \leqq 1\} \\ &= \sup\{|y(s)|: y \in A\} \end{aligned}$$

since A is dense in the unit ball of S^o. But for s in φ

$$y(s) = \sum_j s(j)y(e_j) = (s, t_y)\,.$$

(c) Because of (b) the unit ball of S^f is part of $A^{(\omega)}$ so that $S^f \subset \bigcup_{n=1}^{\infty} nA^{(\omega)}$.

On the other hand let $t \in nA^{(\omega)}$ for some n. Define the functional x'_t on φ by

$$x'_t(s) = (s, t) \qquad s \in \varphi\,.$$

Since $t \in nA^{(\omega)}$

$$|x'_t(s)| \leqq n\,\|s\|$$

for $s \in \varphi$ so that x'_t can be continuously extended to S^o. Since $x'_t(e_j) = t(j)$ for each j, $t \in S^f$.

(d) This follows from (b).

Since

$$\{t \in \varphi: \|t\|_A \leqq 1\} = A^{(\omega)} \cap \varphi = A^{(\varphi)}$$

5.2(c) can be reapplied to determine that

$$S^{ff} = \bigcup_{n=1}^{\infty} nA^{(\varphi)(\omega)}$$

$$S^{fff} = \bigcup_{n=1}^{\infty} nA^{(\varphi)(\varphi)(\omega)}$$

$$S^{ffff} = \bigcup_{n=1}^{\infty} nA^{(\varphi)(\varphi)(\varphi)(\omega)} = S^{ff}$$

since $A^{(\varphi)(\varphi)(\varphi)} = A^{(\varphi)}$ for a subset A of φ because of the bipolar theorem.

5.3. For S a BK-space in which φ is dense the following statements are equivalent:

(a) $\{e_j, E_j\}$ is a norming complete biorthogonal sequence in S.
(b) $S^{ffo} = S$.
(c) $S^{fff} = S^f$.

Proof. (a)$\Rightarrow$(b). If $\{e_j, E_j\}$ is a norming biorthogonal sequence there is a subset A of φ such that $A = A^{(\varphi)(\varphi)}$ and $S = S^{(A)}$. By 5.2(c) the norm $\| \ \|_{ff}$ on S^{ff} is given by

$$\|t\|_{ff} = \sup \{|(t, s)|: s \in A^{(\varphi)(\varphi)}\} .$$

But since $A^{(\varphi)(\varphi)} = A$, $\|t\|_{ff} = \|t\|_A$ for $t \in S$. Thus $S = S^{ffo}$ since φ is dense in both spaces.

(b)$\Rightarrow$(c). If $S^{ffo} = S$ then their BK-topologies can be determined by the same norm $\| \ \|$. If $A = \{s \in \varphi: \|s\| \leqq 1\}$ then both S^{fff} and S^f are equal to $\bigcup_{n=1}^{\infty} nA^{(\omega)}$.

(c)$\Rightarrow$(a). If $S^f = S^{fff}$ then the absolutely convex sets $A^{(\omega)}$ and $A^{(\varphi)(\varphi)(\omega)}$ are equivalent since the first is the closed unit ball of S^f and the second is the closed unit ball of S^{fff}. Thus $A^{(\omega)(\varphi)}$ is equivalent to $A^{(\varphi)(\varphi)(\omega)(\varphi)} = A^{(\varphi)(\varphi)}$ and so $S = S(A) = S(A^{(\varphi)(\varphi)})$ which implies that $\{e_j, E_j\}$ is norming in S by 5.1.

The following representation theorem is an immediate consequence of 5.2 and 5.3.

THEOREM 5.4. *Let S be a sequence space associated with a complete biorthogonal sequence in a Banach space X. If S is equal to $S(A)$ where A is a consistent balanced subset of φ then X^* is isomorphic to $S^f = S(A^{(\omega)}) = \bigcup_{n=1}^{\infty} nA^{(\omega)}$. The given complete biorthogonal sequence is norming if and only if $S^{fff} = S^f$.*

6. The series space and series summable biorthogonal sequences. Let S be a sequence space associated with a complete biorthogonal sequence in a Banach space and let A consist of all s in

φ with $||s|| \leqq 1$. Since the Minkowski gauge of the balanced set $AA^{(\omega)}$ is

$$p(v) = \inf\{||s||\,||t||_f\colon s \in \varphi,\ t \in S^f \text{ and } st = v\}$$

$\mathscr{S}(S)$ as defined by (3.1) and (3.2) is the space $S(AA^{(\omega)})$ in the sense of Proposition 4.1. The following statement thus follows immediately.

6.1. With the norm

$$||v|| = \inf\left\{\sum_{n=1}^{\infty} ||s_n||\,||t_n||_f\colon \sum_{n=1}^{\infty} s_n t_n = v,\ s_n \in \varphi,\ t_n \in S^f\right\},$$

$\mathscr{S}(S)$ is a BK-space in which φ is dense.

Since φ is dense in S it can be shown that $\mathscr{S}(S)$ consists of all sequences v having the form

$$v = \sum_j s_j t_j,\ s_j \in S,\ t_j \in S^f;\ \sum_j ||s_j||\,||t_j||_f < \infty \tag{6.1}$$

and that

$$||v||_{\mathscr{S}} = \inf\left\{\sum_j ||s_j||^A||t_j||_A\colon \sum_j s_j t_j = v,\ s_j \in S,\ t_j \in S^f\right\}.$$

For X a Banach space let $\mathscr{N}(X)$ denote the Banach space of all nuclear operators F from X into X with the norm

$$||F|| = \inf\left\{\sum_j ||y_j'||\,||y_j||\colon y_j \in X,\ y_j' \in X^*,\ \sum_j y_j'(x)y_j = F(x) \text{ for } x \in X\right\}.$$

See 3.1.3 of [9].

6.2. $\mathscr{S}(S)$ consists of all sequences of the form

$$v_F = \{E_j(Fe_j)\}$$

as F ranges over $\mathscr{N}(S)$. The function defined by

$$\Delta(F) = v_F$$

is a continuous linear operator from $\mathscr{N}(S)$ onto $\mathscr{S}(S)$.

Proof. It is first shown that if $F \in \mathscr{N}(S)$ then $\Delta(F) \in \mathscr{S}(S)$. Suppose for each $s \in S$

$$F(s) = \sum_n y_n(s)s_n,\ y_n \in S^*,\ s_n \in S,\ \sum_n ||y_n||\,||s_n|| < \infty\,.$$

Let t_n in S^f be such that $t_n(j) = y_n(e_j)$ for each n and each j. Then for each j

$$\sum_n s_n(j)t_n(j) = \sum_n E_j(s_n)y_n(e_j) = E_j(Fe_j) = v_F(j)$$

so that $\sum_n s_n t_n = v_F$. Since $\sum_n \|s_n\| \, \|t_n\|_f < \infty$, $v_F \in \mathscr{S}(S)$.

It is clear that $\varDelta$ is linear. Since $F(x) = \sum_n y'_n(x) y_n$ implies that $E_j(Fe_j) = \sum_n y'_n(e_j) E_j(y_n)$ for each j, $\|v_F\|_{\mathscr{S}} \leqq \|F\|$ because the first norm is defined as an infimum over a larger set. Therefore, $\varDelta$ is continuous.

6.3. $\mathscr{S}(S)^f \subset M(S) \subset M(\mathscr{S}(S))$.

Proof. $\mathscr{S}(S)^f \subset M(S)$. Let B denote the set of all $v \in \varphi$ such that $\|v\|_{\mathscr{S}} \leqq 1$ and let A be the set of all $s \in \varphi$ such that $\|s\| \leqq 1$. Then $AA^{(\omega)} \subset B$ so that

$$\mathscr{S}(S)^f = \bigcup_n nB^{(\omega)} \subset \bigcup_n n(AA^{(\omega)})^{(\omega)} = M(S) .$$

$M(S) \subset M(\mathscr{S}(S))$. If $u \in M(S)$ and $v \in \mathscr{S}(S)$, let $v = \sum_n s_n t_n$. Then $uv = \sum_n us_n t_n$ and since

$$\sum_n \|us_n\| \, \|t_n\|_f \leqq \sum_n \|u\|_M \|s_n\| \, \|t_n\|_f < \infty$$

it follows that $uv \in \mathscr{S}(S)$. Thus $u \in M(\mathscr{S}(S))$.

From equation (6.1) it follows that $SS^f \subset \mathscr{S}(S)$.

THEOREM 6.4. *Let S be a sequence space associated with a complete biorthogonal sequence in a Banach space and let A be a consistent absolutely convex subset of φ such that $\| \ \|^A$ determines the BK-topology on S. Then the following statements are equivalent:*

(a) *$AA^{(\omega)}$ is a consistent subset of φ;*

(b) *$\mathscr{S}(S)^f = M(S)$;*

(c) *$e \in \mathscr{S}(S)^f$.*

(d) *There is a continuous linear functional E on $\mathscr{S}(S)$ such that if $s_x \in S$ and $t'_x \in S^f$ then*

$$E(st) = x'(x) .$$

Proof. (a)$\Rightarrow$(b). If $AA^{(\omega)}$ is consistent then by 5.2(c)

$$\mathscr{S}(S)^f = S(AA^{(\omega)})^f = \bigcup_n n(AA^{(\omega)})^{(\omega)} = M(S) .$$

(b)$\Rightarrow$(c). $e \in M(S)$.

(c)$\Rightarrow$(d). If $e \in \mathscr{S}(S)^f$ there is a continuous linear functional E on $\mathscr{S}(S)$ such that $E(e_j) = 1$ for each j.

Given $t \in S^f$ define F_t from S into $\mathscr{S}(S)$ by $F_t s = st$. Then F_t is a closed, hence a continuous operator. Thus the functional defined by

$$y'(s) = E(F_t s) = E(st)$$

is continuous linear functional on S. For each s in φ

$$y'(s) = E(st) = \sum_j s(j)t(j) = x'_t(s)$$

so that

$$y'(s) = E(st) = x'_t(s)$$

for each s in S because φ is dense in S.

(d)$\Rightarrow$(c). If E is given by (d) then $E(e_j) = 1$ for each j.

(c)$\Rightarrow$(b). If $e \in \mathscr{S}(S)^f$ then $M(\mathscr{S}(S)^f) \subset \mathscr{S}(S)^f$ for if $u \in M(\mathscr{S}(S)^f)$ then $ue = u \in \mathscr{S}(S)^f$. By 6.3 $\mathscr{S}(S)^f \subset M(S)$ and by 3.5 of [8] $M(T) \subset M(T^f)$ for every BK-space. Thus $\mathscr{S}(S)^f = M(S)$.

(b)$\Rightarrow$(a). Let B consist of all v in φ such that $||v||_{\mathscr{S}} \leqq 1$. If $\mathscr{S}(S)^f = M(S)$ then $M(S) = \bigcup_n nB^{(\omega)}$ by 5.2(c) and $B^{(\omega)}$ is a bounded barrel in $M(S)$. Thus $B^{(\omega)}$ is absorbed by $(AA^{(\omega)})^{(\omega)}$ which implies that $(AA^{(\omega)})^{(\omega)(\varphi)}$ absorbs B. Thus $(AA^{(\omega)})$ is consistent.

A complete biorthogonal sequence in a Banach space will be called *series summable* if the associated sequence space has one, hence all four of the properties mentioned in 6.4.

6.5. Let $\{x_j, x'_j\}$ be a series summable complete biorthogonal sequence in a Banach space X.

(a) If $x \in X$ and $x' \in X^*$ then

$$\sum_j x'(x_j)x'_j(x) = x'(x) \tag{6.2}$$

whenever the sum has finitely many nonzero terms.

(b) If I_1 and I_2 are complementary sets of indices then

$$\mathrm{cl}[\{x_j\colon j \in I_1\}] = \cap\{x \in X\colon x'_j(x) = 0, j \in I_2\} \tag{6.3}$$

(cl means closure).

Proof. (a) Without loss of generality assume the biorthogonal sequence in $\{e_j, E_j\}$ in a BK-space S. Then

$$x'(s) = E(st_{x'}) = \sum_j s(j)t(j)$$

since $st \in \varphi$.

(b) If $j \in I_1$ and $k \in I_2$ then $x'_k(x_j) = 0$ so that the set on the left hand side of (4.6) is contained in that on the right.

If x' in X^* is such that $x'(x_j) = 0$ for j in I_1 and x is in the set on the right hand side of (4.6)

$$x'(x) = \sum_j x'(x_j)x'_j(x) = 0$$

by (a) since each summand is 0. Thus $[x_j\colon j \in I_1\}]$ is dense in the set on the right.

Let Z denote the infinite matrix whose element in the ith row jth column is given by

$$a_{ij} = \begin{cases} 1 & i = j, j+1 \quad j \text{ odd} \\ 1/2^j & i \geqq j \qquad\quad j \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Let (Z) denote the space of all sequences (z_i) such that $\lim_n \sum_{j=1}^{\infty} a_{ij} z_j$ exists. On p. 657 of [13] it is shown that $\{e_j, E_j\}$ is a complete biorthogonal sequence. But, as A. K. Snyder has noted, (6.3) does not hold since $\cap \{s \in (Z): E_{2j}(s) = 0\}$ is c. Thus $\{e_j, E_j\}$ is not series summable.

6.6. Every Banach space in which there is a series summable complete biorthogonal sequence has the approximation property (p. 167 of [5]).

Proof. Without loss of generality it may be assumed that the space is a BK-space S in which $\{e_j, E_j\}$ is a series summable complete biorthogonal sequence. Suppose $\sum_n \|s_n\| \|y_n\| < \infty$, $s_n \in S$, $y_n \in S^*$ and $\sum_n y_n(s) s_n = 0$ for each s in S. Then $\sum_n y_n(e_j) s_n = 0$ for each j so that

$$\sum_n y_n(s_n) = \sum_n E(s_n t_{y_n}) = E\Big(\sum_n s_n t_{y_n}\Big) = E(0) = 0 .$$

Thus the trace of a nuclear operator is well defined on S so that by Proposition 35 of [5] S has the approximation property.

Strongly series summable complete biorthogonal sequences.

7.1. Let S be a sequence space associated with a complete biorthogonal sequence in a Banach space and let A be an absolutely convex consistent subset of φ such that $\| \ \|^A$ determines the topology of S (i.e., $S = S(A)$).

(a) $M(S) = \bigcup_{n=1}^{\infty} n(AA^{(\omega)})^{(\omega)}$.

(b) $\{e_j, E_j\}$ is a norming complete biorthogonal sequence in $M(S)^0$.

(c) $M(S)^{ff} \subset M(S)$.

Proof. (a) As noted in § 3, $M(S)$ is a BK-algebra with the norm

$$\|u\|_M = \sup \{\|us\|^A : \|s\|^A \leqq 1\} .$$

But A is dense in the unit ball of S so that

$$\begin{aligned} \|u\|_M &= \sup \{\|us\|^A : s \in A\} \\ &= \sup \{|(us, t)| : s \in A, t \in A^{(\omega)}\} \\ &= \sup \{|(u, v)| : v \in AA^{(\omega)}\} . \end{aligned}$$

Hence

$$M(S) \subset \bigcup_{n=1}^{\infty} n(AA^{(\omega)})^{(\omega)} . \tag{7.1}$$

If $u \in n(AA^{(\omega)})^{(\omega)}$ for some n, define F'_u on φ by

$$F'_u(s) = us .$$

Then F'_u is continuous on $(\varphi, \| \ \|^A)$ since

$$\|us\|^A = \sup \{|(us, t)|: t \in A^{(\omega)}\} \leqq n \|s\|^A$$

for s in φ. Thus F'_u can be extended continuously to all of S, and if F_u is this extension

$$F_u s = us .$$

Thus $u \in M(S)$ so that equality holds in (7.1).

(b) This follows from the equality $\|u\|_M = \sup \{|(u, v)|: v \in AA^{(\omega)}\}$.

(c) By 5.2(c)

$$M(S)^{ff} = \bigcup_{n=1}^{\infty} n(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$$

which is contained in $M(S)$ since

$$(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)} \subset (AA^{(\omega)})^{(\omega)} .$$

Since $AA^{(\omega)} \subset (AA^{(\omega)})^{(\varphi)(\varphi)}$ it follows that

$$SS^f \subset \mathscr{S}(S) \subset M(S)^{fo} .$$

THEOREM 7.2. *Let S be a sequence space associated with a complete biorthogonal sequence in a Banach space, and let A be a consistent absolutely convex subset of φ such that $\| \ \|^A$ determines the topology of S. Then the following statements are equivalent*

(a) $M(S)^{ff} = M(S)$

(b) $e \in M(S)^{ff}$

(c) *There is a sequence $\{u_n\}$ in φ such that $\lim_n u_n(j) = 1$ for each j and $\{u_n\}$ is bounded in $M(S)$.*

(d) *There is a sequence $\{u_n\}$ in φ such that*

$$\lim_n \|s - u_n s\|^A = 0$$

for each $s \in S$.

(e) *There is a sequence $\{u_n\}$ in φ such that $\lim_n (su_n, t_y) = y(s)$ for s in S and $y \in S^*$.*

(f) $(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ *absorbs* $(AA^{(\omega)})^{(\omega)}$.

Proof. (a)$\Rightarrow$(b). Trivial since $e \in M(S)$.

(b)$\Rightarrow$(c). If $e \in M(S)^{ff}$ then e is in $a(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ for some $a > 0$

because $(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ is the unit ball of $M(S)^{ff}$ by 5.2(c). There is thus a sequence $\{u_n\}$ in $(AA^{(\omega)})^{(\varphi)}$ converging coordinatewise to e since $(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ is the closure in ω of $(AA^{(\omega)})^{(\varphi)}$. Also each u_n is in the unit ball of $M(S)$.

(c)⟹(d). Let $\{u_n\}$ be given by (c). For s in φ

$$\lim_n \|su_n - s\|^A = 0 \tag{7.1}$$

since

$$\|s - su_n\|^A \leqq \sum_{j=1}^{N_s} |s(j) - s(j)u_n(j)| \, \|e_j\|^A$$

where N_s is such that $s(j) = 0$ for $j > N_s$. Also

$$\|su_n\|^A \leqq \|u_n\|_M \, \|s\|^A$$

for $s \in S$. Thus by the Banach-Steinhaus theorem (7.1) holds for each s in S.

(d)⟹(e). Let $\{u_n\}$ be given by (d) then for s in S and y in S^*

$$\lim_n (su_n, t_y) = y(\lim_n su_n) = y(s) .$$

(e)⟹(d). Let $\{u_n\}$ be given by (e) then (d) follows from II. 3.2 and II. 3.6 in [2] since $\lim_n u_n e_k = e_k$ for each k.

(d)⟹(f). Let $\{u_n\}$ be given by (d). Given u in $M(S)$ define F_{uu_n} on S by

$$F_{uu_n}s = uu_ns \qquad n = 1, 2, \cdots .$$

Then

$$\lim_n F_{uu_n}s = us$$

for each s in S. By the uniform boundedness principle $\{F_{uu_n}\}$ is a bounded sequence in $\mathscr{D}$. Thus

$$\{uu_n\} \subset a(AA^{(\omega)})^{(\omega)} \cap \varphi = a(AA^{(\omega)})^{(\varphi)}$$

for some $a > 0$. Since $u_ne_j \to e_j$ for each j, $\{uu_n\}$ converges coordinatewise to u. Thus $u \in a(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$. This implies that $(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ absorbs each point in $M(S)$. But $(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ is closed in $M(S)$ since it is closed in ω and $M(S)$ is a BK-space. Thus $(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ absorbs $(AA^{(\omega)})^{(\omega)}$ which is the unit ball of $M(S)$.

(f)⟹(a). By 7.1(c), $M(S)^{ff} \subset M(S)$. But $(AA^{(\omega)})^{(\varphi)(\varphi)(\omega)}$ is the unit ball of $M(S)^{ff}$ so if it absorbs $(AA^{(\omega)})^{(\omega)}$, $M(S) \subset M(S)^{ff}$.

A complete biorthogonal sequence in a Banach space is called *strongly series summable* if the associated sequence space has one, hence all of the properties in 7.2.

7.3. Let $\{x_j, x'_j\}$ be a strongly series summable complete biorthogonal sequence in a Banach space X. Then

(a) $\{x_j, x'_j\}$ is norming;

(b) $\{x_j, x'_j\}$ is a series summable basis;

(c) $\{x_j\}$ is an approximative basis in the sense of Singer ([11], Definition 3.19).

(d) X has the metric approximation property (Definition 10, p. 178 of [5]).

Proof. (a) Let $\{u_n\}$ be a sequence as in 7.2(d) and let

$$B = \left\{\sum_j u_n(j)x'(x_j)x'_j \colon n = 1, 2, \cdots, \|x'\| \leqq 1\right\}.$$

It follows from 7.2(d) that $\|\ \ \|_B$ determines the topology of X.

(b) By 7.1(a) and 5.2(c) the closed unit ball of $M(S)^f$ is $(AA^{(\omega)})^{(\varphi)(\omega)}$ where $A = \{s \in \varphi \colon \|s\| \leqq 1\}$. This implies that $\mathscr{S}(S) \subset M(S)^f$ and since the inclusion is continuous, $\mathscr{S}(S) \subset M(S)^{fo}$. Thus if $M(S)^{ff} = M(S)$, $\mathscr{S}(S)^f \supset M(S)$ so that $\mathscr{S}(S)^f = M(S)$ by 6.3.

(c) Let y'_{nj} be defined by

$$y'_{nj}(x) = u_n(j)x'_j(x) \qquad x \in X.$$

Let $\{m_n\}$ be any increasing sequence of integers such that $u_n(j) = 0$ for $j > m_n$. Then 7.2(d) implies that

$$\lim_n \left\|x - \sum_{j=1}^{m_n} y'_{nj}(x)x_j\right\| = 0$$

so that $\{x_n\}$ is an approximative basis of X.

(d) This is an immediate consequence of 7.2(d) with $F_n = F_{u_n}$ for each n.

THEOREM 7.4. *Let $\{x_j, x'_j\}$ and $\{y_j, y'_j\}$ be complete biorthogonal sequences in Banach spaces. If $\{x_j, x'_j\}$ is strongly series summable and $M(\{x_j\}) \subset M(\{y_j\})$ then $\{y_j, y'_j\}$ is a strongly series summable.*

Proof. Let S be the sequence space associated with $\{x_j, x'_j\}$ and T the sequence space associated with $\{y_j, y'_j\}$. If $M(S) \subset M(T)$ the inclusion is continuous since both are BK-spaces. Thus if (c) holds for $M(S)$ it also holds for $M(T)$ with the same sequence $\{u_n\}$.

I do not know an example of a series summable complete biorthogonal sequence which is not strongly series summable. If $\{x_j, x'_j\}$ is series summable and $\{e_j, E_j\}$ is norming in $\mathscr{S}(S)$ then $\mathscr{S}(S) = M(S)^{fo}$ so that $\{x_j, x'_j\}$ is strongly series summable.

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