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EXTREMAL ELEMENTS OF THE CONVEX CONE A_n OF FUNCTIONS

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Let A_1 be the set of nonnegative real functions f on $[0, 1]$ such that $\nabla_h^1 f(x) = f(x) - f(x+h) \leqq 0$, $h > 0$, for $[x, x+h] \subset [0, 1]$, and let A_n, $n > 1$, be the set of functions belonging to A_{n-1} such that $\nabla_h^n f(x) = \nabla_h^{n-1} f(x) - \nabla_h^{n-1} f(x+h) \leqq 0$ for

$$[x, x+nh] \subset [0, 1] .$$

Since the sum of two functions in A_n belongs to A_n and since a nonnegative real multiple of an A_n function is an A_n function, the set of A_n functions forms a convex cone. It is the purpose of this paper to give the extremal elements (i.e., the generators of extreme rays) of this cone, to prove that they form a closed set in a compact convex set that does not contain the origin but meets every ray of the cone, and to show that for the functions of the cone an integral representation in terms of extremal elements is possible. The intersection of the A_n cones is the class of functions alternating of order$^\infty$. Thus, the set of these functions, which will be denoted by A_∞, forms a convex cone also. The extremal elements for the convex cone A_∞ are given too.

Let f be a function in A_1 which assumes exactly one positive value in $[0, 1]$; that is, $f(x) = 0$, $x \in [0, \xi)$, $f(x) = c > 0$, $x \in [\xi, 1]$ where $0 \leqq \xi \leqq 1$, if $f = f_1 + f_2$, where f_1 and $f_2 \in A_1$, then $0 = \nabla_h^1 f(x) = \nabla_h^1 f_1(x) + \nabla_h^1 f_2(x)$ implies $\nabla_h^1 f_i(x) = 0$ for $i = 1, 2$ and $[x, x+h] \subset [\xi, 1]$. Therefore, $f_i(x) = 0$, $x \in [0, \xi)$, $f_i(x) = c_i \geqq 0$, $x \in [\xi, 1]$, $i = 1, 2$, where $c_1 + c_2 = c$. Hence, f is an extremal element of A_1. On the other hand, if f assumes at least two positive values in $[0, 1]$, then a nonproportional decomposition can be given by taking

$$f_1(x) = \min \{f(x), (1/2)[f(0) + f(1)]\}$$

and $f_2 = f - f_1$. Therefore, the extremal elements of A_1 are precisely the functions in A_1 which assume exactly one positive value in $[0, 1]$.

Since a function in A_n, $n > 1$, is nonnegative, nondecreasing, and concave on $[0, 1]$, it must be continuous on $(0, 1]$ (cf. [6], p. 148). It follows that the only extremal elements of A_1 which are in A_n are those functions f such that $f = c > 0$ on $(0, 1]$ while $f(0) = 0$ or $f(0) = c$ and these functions are again extremal in A_n. If $f \in A_n$, $n > 1$, f is not constant and $f(0) > 0$, then a nonproportional decomposition can be given by taking $f_1 = f(0)$ and $f_2 = f - f_1$. If $f \in A_n$, $f(0) = 0$, f is not constant on $(0, 1]$ and f is not continuous at 0 (that is,

$f(0+) > 0$), then take $f_1 = f(0+)$ on $(0, 1]$, $f_1(0) = 0$ and $f_2 = f - f_1$. In so doing, f_1 and $f_2 \in A_n$ and f_1 and f_2 are not proportional to f. Therefore, for $n > 1$, the only extremal elements of A_n such that $f(0) > 0$ are the positive constant functions, and the only extremal elements of A_n which are discontinuous at 0 are those functions f such that $f(0) = 0$ and $f = c > 0$ on $(0, 1]$. It will be shown that the remaining extremal elements of A_n, $n > 1$, are indefinite integrals of the extremal elements of a cone which is similar to A_1. This cone is given in the following definitions.

DEFINITION 1. If g is a real function continuous almost everywhere on $(0, 1]$ and n is a positive integer, then g is said to satisfy property $P(n)$ if

$$\operatorname*{limit}_{\delta \to 0} \int_1^{\delta}\int_1^{t_{n-1}} \cdots \int_1^{t_2}\int_1^{t_1} g(t)dt\,dt_1 \cdots dt_{n-1}$$

exists and is finite.

DEFINITION 2. Let $K(n)$ denote the convex cone of nonnegative, nonincreasing real functions on $(0, 1]$ which satisfy property $P(n)$.

In the same manner that the extremal elements of A_1 were found, it can be shown that the extremal elements of $K(n)$ are precisely these functions which assume exactly one positive value in $(0,1]$. Preliminary to the determination of the extremal elements of A_n, it is shown in the following two lemmas how the A_n functions are related to the functions in $K(n-1)$, where $n > 1$.

LEMMA 1. *If $f \in A_n$, then $(-1)^n f_+^{(n-1)} \in K(n-1)$, where $n > 1$.*

Proof. The proof will be by induction on n. If $f \in A_2$, then f is nonnegative, nondecreasing and concave. It follows that f'_+ is nonnegative and nonincreasing on $(0, 1]$, where $f'_+(1) = f'_+(1-)$ [6]. Also,

$$f(x) = \int_0^x f'_+(t)dt + f(0+)$$

which implies that f'_+ satisfies property $P(1)$ [4].

Assume that $f \in A_n$ implies $(-1)^n f_+^{(n-1)} \in K(n-1)$ for $n \geqq 2$. If $f \in A_{n+1}$, then

$$\nabla_h^2 \nabla_h^{n-1} f(x) = \nabla_h^{n+1} f(x) \leqq 0$$

for $[x, x + (n+1)h] \subset [0, 1]$, which implies that

$$\nabla_h^2 \nabla_{\delta_1}^1 \nabla_{\delta_2}^1 \cdots \nabla_{\delta_{n-1}}^1 f(x) \leqq 0$$

for $[x, x+2h+\delta_1+\delta_2+\cdots+\delta_{n-1}] \subset [0, 1]$ [2]. It then follows that

$$(-1)^{n-1}\nabla_h^2 f_+^{(n-1)}(x) \leqq 0$$

for $[x, x+2h] \subset (0, 1]$, and hence, $(-1)^{n-1}f_+^{(n-1)}$ is concave on $(0, 1]$. Therefore, $f^{(n-1)} = f_+^{(n-1)}$, since $f_+^{(n-1)}$ is continuous. It follows that $f_+^{(n)}$ exists on $(0, 1]$, where $f_+^{(n)}(1) = f_+^{(n)}(1-)$, and $(-1)^{n+1}f_+^{(n)}$ is nonnegative and nonincreasing. It remains only to show that $f_+^{(n)}$ satisfies property $P(n)$. If $f \in A_{n+1}$, then $f \in A_n$, and

$$\begin{aligned} &\operatorname*{limit}_{\delta\to 0}\int_1^\delta\int_1^{t_{n-1}}\cdots\int_1^{t_2}\int_1^{t_1} f_+^{(n)}(t)dtdt_1\cdots dt_{n-1} \\ &= \operatorname*{limit}_{\delta\to 0}\int_1^\delta\int_1^{t_{n-1}}\cdots\int_1^{t_3}\int_1^{t_2} f^{(n-1)}(t_1)dt_1dt_2\cdots dt_{n-1} \\ &\quad - f^{(n-1)}(1)\int_1^0\int_1^{t_{n-1}}\cdots\int_1^{t_3}\int_1^{t_2} dt_1dt_2\cdots dt_{n-1} \end{aligned}$$

exists and is finite, since $f^{(n-1)}$ satisfies property $P(n-1)$ by the induction hypothesis.

DEFINITION 3. If g is a real function on $(0, 1]$ which satisfies property $P(n)$, then define the function $I(g, n;)$ by the equation

$$I(g, 1; x) = \int_0^x g(t)dt\ ,$$
$$I(g, n; x) = \int_0^x\int_1^{t_{n-1}}\cdots\int_1^{t_2}\int_1^{t_1} g(t)dtdt_1\cdots dt_{n-1}\ ,$$

$n = 2, 3, 4, \cdots$, for $x \in [0, 1]$.

LEMMA 2. *If* $g \in K(n-1)$, *then* $(-1)^n I(g, n-1;) \in A_n$, *where* $n > 1$.

Proof. The proof will be by induction on n. If $g \in K(1)$, then

$$I(g, 1; x) = \int_0^x g(t)dt \geqq 0\ ,$$

for $x \in [0, 1]$. If $[x, x+h] \subset [0, 1]$, then

$$\nabla_h^1 I(g, 1; x) = \int_{x+h}^x g(t)dt \leqq 0$$

since $g(t) \geqq 0$, where $x \leqq t \leqq x+h$. Since g is nonincreasing on $(0, 1]$, then $I(g, 1;)$ is concave on $[0, 1]$ and it follows that $\nabla_h^2 I(g, 1; x) \leqq 0$, for $h > 0$ and $[x, x+2h] \subset [0, 1]$ [4]. Hence, $I(g, 1;) \in A_2$ whenever $g \in K(1)$.

Assume that $(-1)^n I(g, n-1;) \in A_n$ for $g \in K(n-1)$ and $n > 1$. If $g \in K(n)$, then let

$$f(x) = \int_1^x g(t)dt ,$$

for $x \in (0, 1]$. Since $g \in K(n)$, it is easily seen that $-f \in K(n-1)$ and it follows from the induction hypothesis that

$$(-1)^{n+1}I(g, n;) = (-1)^n I(-f, n-1;) \in A_n .$$

By a repeated application of the mean value theorem for a Riemann integral, it can be shown that

$$\nabla_h^{n-1} I(g, n; x) = (-h)^{n-1} f(\xi)$$

for $[x, x+(n-1)h] \subset [0, 1]$, where $x < \xi < x + (n-1)h$. It follows that

$$\begin{aligned} &\nabla_h^{n+1}(-1)^{n+1} I(g, n; x) \\ &= (-1)^{n+1} \nabla_h^2 \nabla_h^{n-1} I(g, n; x) \\ &= (-1)^{2n} h^{n-1} \nabla_h^2 f(\xi) \leqq 0 \end{aligned}$$

for $[x, x+(n+1)h] \subset [0, 1]$, since f is concave on $(0, 1]$ [4]. This inequality, together with the fact that $(-1)^{n+1}I(g, n;) \in A_n$, implies that $(-1)^{n+1}I(g, n;) \in A_{n+1}$.

It is a consequence of Lemmas 1 and 2 that $f = I(f_+^{(n-1)}, n-1;)$ whenever $f \in A_n$, $n > 1$, $f(0+) = 0$ and $f^{(k)}(1) = 0$ for $1 \leqq k \leqq n-2$. If $f \in A_2$, then f is concave on $[0, 1]$ and

$$f(x) = \int_0^x f'_+(t)dt = I(f'_+, 1; x) .$$

If $f \in A_n$, $n > 2$, then $(-1)^{n-2} f^{(n-2)}$ is concave on $(0, 1]$ (cf. proof of Lemma 1). It follows that

$$f^{(n-2)}(x) = \int_1^x f_+^{(n-1)}(t)dt ,$$

which implies that $f = I(f_+^{(n-1)}, n-1;)$ [4].

PROPOSITION 1. *The function f defined by*

$$f(x) = m[\xi^{n-1} - (\xi - x)^{n-1}]$$

for $x \in [0, \xi]$ and $m\xi^{n-1}$ for $x \in [\xi, 1]$, where $0 < \xi \leqq 1$ and $m > 0$, is an extremal element of A_n, $n > 1$.

Proof. If f is such a function, then

$$f_+^{(n-1)}(x) = (-1)^n m(n-1)! ,$$

$x \in (0, \xi)$ and 0 for $x \in [\xi, 1]$, which implies that $(-1)^n f_+^{(n-1)}$ is an extremal element of $K(n-1)$. Since $f(0)=0$ and $f^{(k)}(1)=0$ for $1 \leqq k \leqq n-2$ (whenever $n>2$), then $f = I(f_+^{(n-1)}, n-1;)$ and it follows from Lemma 2 that $f \in A_n$.

If g and $h \in A_n$ such that $f = g + h$, then $(-1)^n g_+^{(n-1)}$ and

$$(-1)^n h_+^{(n-1)} \in K(n-1)$$

and $f_+^{(n-1)} = g_+^{(n-1)} + h_+^{(n-1)}$. Since $(-1)^n f_+^{(n-1)}$ is extremal in $K(n-1)$, there are constants $\lambda_i \geqq 0$, $i = 1, 2$, such that $g_+^{(n-1)} = \lambda_1 f_+^{(n-1)}$ and $h_+^{(n-1)} = \lambda_2 f_+^{(n-1)}$. Since $f(0) = 0$ and $f^{(k)}(1) = 0$ for $1 \leqq k \leqq n-2$, it follows that $g(0) = g^{(k)}(1) = 0$ and $h(0) = h^{(k)}(1) = 0$ for $1 \leqq k \leqq n-2$. Hence

$$\begin{aligned} g &= I(g_+^{(n-1)}, n-1;) = I(\lambda_1 f_+^{(n-1)}, n-1;) \\ &= \lambda_1 I(f_+^{(n-1)}, n-1;) = \lambda_1 f\,, \end{aligned}$$

and similarly, $h = \lambda_2 f$. Thus, if $f(x) = m[\xi^{n-1} - (\xi - x)^{n-1}]$, $x \in [0, \xi]$ and $m\xi^{n-1}$ for $x \in [\xi, 1]$, where $0 < \xi \leqq 1$ and $m > 0$, then f is extremal in A_n, $n > 1$. Denote this latter function by $e(m, \xi, n-1;)$.

If $f \in A_2$ such that $f(0+) = f(0) = 0$, $f \neq 0$ and $f \neq e(m, \xi, 1;)$, for $m > 0$ and $0 < \xi \leqq 1$, then f'_+ is not extremal in $K(1)$, since f'_+ assumes at least two positive values in $(0, 1]$. It follows that there are functions g_1 and $g_2 \in K(1)$ such that $f'_+ = g_1 + g_2$ and g_1 and g_2 are not proportional to f'_+. Since $f(0) = 0$, then $f = I(f'_+, 1;)$ and it follows that

$$f = I(f'_+, 1;) = I(g_1 + g_2, 1;) = I(g_1, 1;) + I(g_2, 1;)\ .$$

Thus, if $f_i = I(g_i, 1;)$, then $f_i \in A_2$, $i = 1, 2$, and $f = f_1 + f_2$. This gives a nonproportional decomposition of f. Therefore, the extremal elements of A_2 are the positive constant functions, the functions which are a positive constant on $(0, 1]$ and zero at 0 and the functions $e(m, \xi, 1;)$, where $m > 0$ and $0 < \xi \leqq 1$. The remaining extremal elements of A_n, $n > 2$, are given in the next proposition.

PROPOSITION 2. *If $m > 0$, the function $e(m, 1, k;)$ is an extremal element of A_n for $n > 2$ and $1 \leqq k \leqq n-2$.*

Proof. Since A_n is a subcone of A_{k+1} and $e(m, 1, k;)$ is an extremal element of A_{k+1}, it is sufficient to show that $e(m, 1, k;) \in A_n$. If $f = e(m, 1, k;)$, then $f = I(f^{(k)}, k;)$ where

$$f^{(k)}(x) = (-1)^{k+1} m(k!)$$

for $0 < x \leqq 1$. Since $f^{(k)}$ is constant on $(0, 1]$, it follows from a repeated application of the mean value theorem for a Riemann integral that

$$\nabla_h^{k+1} f(x) = \nabla_h^1 \nabla_h^k f(x) = (-h)^k \nabla_h^1 f^{(k)}(\xi) = 0$$

for $h > 0$, $[x, x + (k + 1)h] \subset [0, 1]$, and thus, $\nabla_h^p f(x) = 0$ for $h > 0$, $[x, x + ph] \subset [0, 1]$ and $p \geqq k + 1$. Hence, $f \in A_n$ for every n, which implies that f is extremal in A_p, for $p \geqq k + 1$.

It will follow, as a consequence of the next three lemmas, that no other functions in A_n are extremal elements of A_n, $n > 2$.

LEMMA 3. *Let $f \in A_n$, $n > 2$, such that $f(0+) = f(0) = 0$ and $f \neq e(m, 1, k;)$, where $m > 0$ and $1 \leqq k \leqq n - 2$. If there is an integer k such that $1 \leqq k \leqq n - 2$ and $f^{(k)}(1) \neq 0$, then f is not an extremal element of A_n.*

Proof. Let k denote the smallest integer such that $f^{(k)}(1) \neq 0$ Then $f \in A_n \subset A_{k+2}$ implies that $(-1)^k f_+^{(k+1)} \in K(k + 1)$, and it follows from Lemma 2 that $I(f_+^{(k+1)}, k + 1;) \in A_{k+2}$. Since $f(0) = 0$ and $f^{(p)}(1) = 0$ for $1 \leqq p < k$, then $I(f_+^{(k+1)}, k + 1;) = I(f^{(k)}, k;) - f^{(k)}(1) I(1, k;) = f - e(m, 1, k;)$, where $m = (-1)^{k-1}[1/(k!)] f^{(k)}(1) > 0$. Since

$$\nabla_h^p e(m, 1, k; x) = 0$$

for $h > 0$, $[x, x + ph] \subset [0, 1]$ and $p \geqq k + 1$ and $f \in A_n$, it follows that

$$\nabla_h^p I(f_+^{(k+1)}, k + 1; x) = \nabla_h^p f(x) \leqq 0 ,$$

for $[x, x + ph] \subset [0, 1]$, $k + 1 \leqq p \leqq n$. Hence, $f - e(m, 1, k;) \in A_n$, where $m = (-1)^{k-1}[1/(k!)] f^{(k)}(1)$, and a nonproportional decomposition of f can be given by taking $f_1 = e(m, 1, k;)$ and $f_2 = f - f_1$. Thus, f is not extremal.

LEMMA 4. *Let $f \in A_n$, $n > 2$, such that $f \neq 0$, $f(0+) = f(0) = 0$ and $f \neq e(m, 1, k;)$, where $m > 0$ and $1 \leqq k \leqq n - 2$. If $f_+^{(n-1)} = 0$ on $(0, 1]$, then f is not an extremal element of A_n.*

Proof. If $f_+^{(n-1)} = 0$, then there is a positive integer $k \leqq n - 2$ such that $f^{(k)} \neq 0$ and $f^{(k)}$ is constant on $(0, 1]$. Thus, $f^{(k)}(1) \neq 0$ and it follows from Lemma 3 that f is not extremal.

It follows from Lemmas 3 and 4 that if f is an extremal element of A_n, $n > 2$, such that $f(0+) = f(0) = 0$ and either $f_+^{(n-1)} = 0$ or $f^{(k)}(1) \neq 0$ for some k, $1 \leqq k \leqq n - 2$, then $f = e(m, 1, k;)$, where $m > 0$ and $1 \leqq k \leqq n - 2$.

LEMMA 5. *Let $f \in A_n$, $n > 2$, such that $f(0+) = f(0) = 0$, $f_+^{(n-1)} \neq 0$ and $f^{(k)}(1) = 0$ for $1 \leqq k \leqq n - 2$. If f is an extremal element of A_n, then $f = e(m, \xi, n - 1;)$, where $m > 0$ and $0 < \xi \leqq 1$.*

Proof. Since $f(0) = f^{(k)}(1) = 0$ for $1 \leqq k \leqq n - 2$, then

$$f = I(f_+^{(n-1)}, n - 1;)$$

and it follows from Lemma 1 that $(-1)^n f_+^{(n-1)} \in K(n - 1)$. If g_1 and $g_2 \in K(n - 1)$ such that $(-1)^n f_+^{(n-1)} = g_1 + g_2$, then

$$\begin{aligned} f = I(f_+^{(n-1)}, n - 1;) &= (-1)^n I(g_1 + g_2, n - 1;) \\ &= (-1)^n I(g_1, n - 1;) + (-1)^n I(g_2, n - 1;) . \end{aligned}$$

Then $f_i = (-1)^n I(g_i, n - 1;)$, $i = 1, 2$, implies that f_1 and $f_2 \in A_n$ and $f = f_1 + f_2$. Since f is extremal in A_n, there are numbers $\lambda_i \geqq 0$ such that $f_i = \lambda_i f$, $i = 1, 2$, which implies that $g_i = \lambda_i(-1)^n f_+^{(n-1)}$, $i = 1, 2$, and $(-1)^n f_+^{(n-1)}$ is therefore extremal in $K(n - 1)$. Thus,

$$(-1)^n f_+^{(n-1)}(x) = c > 0, x \in (0, \xi)$$

and 0 for $x \in [\xi, 1]$, which implies that

$$f = I(f_+^{(n-1)}, n - 1;) = e(m, \xi, n - 1;) ,$$

where $m = c/(n - 1)!$.

Therefore, the extremal elements of A_n, $n > 2$, are the positive constant functions, the functions which are a positive constant on $(0, 1]$ and zero at 0, the functions $e(m, 1, k;)$, where $m > 0, 1 \leqq k \leqq n - 2$, and the functions $e(m, \xi, n - 1;)$, where $m > 0$ and $0 < \xi \leqq 1$.

Since A_∞ is a subcone of A_n, it follows that the function $e(m, 1, n;)$, $m > 0$, is an extremal element of A_∞ for every positive integer n. It is shown in the following proposition that A_∞ has no other extremal elements which are continuous and zero at 0.

PROPOSITION 3. *If $f \in A_\infty$ such that $f(0+) = f(0) = 0$ and $f \neq e(m, 1, k;)$, where $m > 0$ and k is a positive integer, then f is not an extremal element of A_∞.*

Proof. Since $f \in A_\infty$ is a function of class C^∞ on $(0, 1]$, it follows from a theorem of Bernstein, Theorem 13-31 in [1], that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x - 1)^n$$

for $0 < x < 1$ by noting that the function g defined by

$$g(x) = f(1) - f(1 - x)$$

satisfies the hypothesis of the theorem. If there is a positive integer k such that $f^{(k)}(1) \neq 0$, then assume, without loss of generality, that

k is the least such integer. Then $f \in A_\infty \subset A_{k+2}$ implies that

$$(-1)^k f^{(k+1)} \in K(k+1)$$

from which it follows that $I(f^{(k+1)}, k+1;) \in A_{k+2}$. Hence,

$$I(f^{(k+1)}, k+1;) = I(f^{(k)}, k;) - f^{(k)}(1)I(1, k;) = f - e(m, 1, k;) ,$$

where $m = (-1)^{k-1}[1/(k!)]f^{(k)}(1) > 0$. If $f_1 = e(m, 1, k;)$ and $f_2 = f - f_1$, then $f_1 \in A_\infty$ since $f_1 \in A_n$ for every n and $f_2 \in A_\infty$ since $f_2 \in A_{k+2}$ and

$$\nabla_h^n f_2(x) = \nabla_h^n[f(x) - e(m, 1, k; x)] = \nabla_h^n f(x) \leqq 0 ,$$

for $h > 0$, $[x, x + nh] \subset [0, 1]$ and $n \geqq k+3$. Since f_1 is not proportional to f, this gives a nonproportional decomposition of f, and f is therefore not extremal. On the other hand, if $f^{(k)}(0) = 0$ for each positive integer k, then $f(x) = f(1)$ for $0 < x \leqq 1$, and $f(0+) = f(0) = 0$ implies that $f = 0$.

The results to this point are summarized in the following theorem.

THEOREM. *The extremal elements of A_1 are the functions which assume exactly one positive value in* $[0, 1]$. *The positive constant functions and the functions which are a positive constant on* $(0, 1]$ *and zero at* 0 *are extremal elements of A_n, $n > 1$, and are therefore extremal in A_∞. The functions $e(m, \xi, n-1; x) = m[\xi^{n-1} - (\xi - x)^{n-1}]$, $x \in [0, \xi]$ and $m\xi^{n-1}$ for $x \in [\xi, 1]$, where $m > 0$ and $0 < \xi \leqq 1$, are extremal elements of A_n, $n \geqq 2$. The only other extremal elements of A_n, $n \geqq 3$, are those functions $e(m, 1, k;)$, $1 \leqq k \leqq n-2$. The extremal elements of A_∞ which are continuous and zero at* 0 *are the functions $e(m, 1, k;)$, $k \geqq 1$.*

The set of functions $A_n - A_n$, $n \geqq 1$, forms the smallest linear space containing the convex cone A_n. With the topology of simple convergence, $A_n - A_n$ is a Hausdorff locally convex space. Let C_n be the set of functions $f \in A_n$ such that $f(1) = 1$. Then C_n is a convex set which meets every ray of A_n once and only once but does not contain the origin, that is the zero function. It then follows that f is an extreme point of C_n if, and only if, f is an extremal element of A_n which lies in C_n. A proof similar to that found on page 992 of [5] can be used here to show that C_n is compact. It follows from the next proposition that the set of extreme points of C_n is compact.

PROPOSITION 4. *The set of extreme points of C_n is closed in C_n, $n \geqq 1$.*

Proof. Since the topology of simple convergence is equivalent to the topology of pointwise convergence, it will suffice to show that if $\{f_i\}$ is a net of functions in ext C_n which converges pointwise to a function f, then $f \in \operatorname{ext} C_n$, $n \geqq 1$, where $\operatorname{ext} C_n$ denotes the set of extreme points of C_n. The proof for $n = 1$ is obvious. Since all except a finite number of the functions in ext C_n, $n > 1$, are of the form $e((1/\xi)^{n-1}, \xi, n-1;)$, where $0 < \xi \leqq 1$, it can be assumed without loss of generality that $f_i = e((1/\xi_i)^{n-1}, \xi_i, n-1;)$, for each i.

If the net $\{\xi_i\}$ of real numbers converges to 0, then it is easily seen that

$$\operatorname*{limit}_i f_i(x) = 1$$

for $x \in (0, 1]$. Since the topology is Hausdorff, it follows that $f(0) = 0$ and $f(x) = 1$, $x \in (0, 1]$, which implies that $f \in \operatorname{ext} C_n$.

On the other hand, if $\{\xi_i\}$ does not converge to 0, then there is a positive real number ξ_0 and a subnet $\{\xi_j\}$ of $\{\xi_i\}$ such that $\{\xi_j\}$ converges to ξ_0. If $0 \leqq x < \xi_0$, then

$$\operatorname*{limit}_j f_j(x) = \frac{1}{\xi_0^{n-1}}[\xi_0^{n-1} - (\xi_0 - x)^{n-1}] ;$$

whereas

$$\operatorname*{limit}_j f_j(x) = 1$$

if $\xi_0 \leqq x \leqq 1$. Therefore, since the topology is Hausdorff,

$$f = e((1/\xi_0)^{n-1}, \xi_0, n-1;) ,$$

and it follows that $f \in \operatorname{ext} C_n$.

Since ext C_n and C_n are both compact subsets of the locally convex space $A_n - A_n$, $n \geqq 1$, it follows from Theorem 39.4 of Choquet [3] that for any function $f_0 \in C_n$ there exists a probability measure μ_0 on ext C_n such that

$$f_0(x) = \int f(x) d\mu_0 ,$$

for $x \in [0, 1]$. Since C_n meets every ray of A_n and does not contain the origin, it follows that each function of A_n is a scalar multiple of such a representation.

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