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UNITARY DILATIONS FOR COMMUTING CONTRACTIONS

STEPHEN PARROTT

Let $S_1, S_2, \cdots, S_n$ be a set of commuting contraction operators on a Hilbert space H, let $U_1, U_2, \cdots, U_n$ be a set of commuting unitary operators on a Hilbert space K containing H, and let P be the projection from K to H. The set $U_1, \cdots, U_n$ is called a set of commuting unitary dilations for $S_1, \cdots, S_n$ provided that

$$PU_1^{m_1}U_2^{m_2}\cdots U_n^{m_n}x = S_1^{m_1}S_2^{m_2}\cdots S_n^{m_n}x$$

for all x in H and for all nonnegative integers $m_1, m_2, \cdots, m_n$. Sz.-Nagy proved that a single contraction has a unitary dilation, and Ando showed that any two commuting contractions possess a pair of commuting unitary dilations. This note presents several counterexamples which disprove the corresponding conjecture for three or more contractions.

In § 3, three commuting contractions, R, S, T are constructed which do not have commuting unitary dilations. The operators R and S each have norm one, while the operator T may be chosen to have any norm between zero and one. However, the proof yielding the counterexample fails completely if the operators R, S, T are replaced by $\lambda R, \lambda S, T$ with $0 < \lambda < 1$, and this raises another question.

It is known that a finite or infinite set of commuting contractions $S_1, S_2, \cdots$ which satisfies $\sum ||S_k||^2 \leqq 1$ possesses a set of commuting unitary dilations. Thus it appears that the "size" of a set of contractions may be relevant to the existence of commuting unitary dilations; and since two of the contractions in § 3 have norm one, it is conceivable that this example might be only a peculiar "boundary" phenomenon. In § 4 this notion is dispelled by a more complicated example showing that three commuting contractions, each of norm strictly less than one, can fail to have commuting unitary dilations. Although the example of § 4 is in most (but not all) respects more powerful than that of § 3, the latter is presented separately because of its simplicity.

Section 3 also observes that a recent result of Sz.-Nagy and Foias is equivalent to Ando's theorem. Section 5 shows that the counterexamples constructed in this paper to the unitary dilation conjecture cannot be used as counterexamples to another well-known conjecture concerning spectral sets.

2. Notation and preliminaries. If H is a subspace of a Hilbert space K, the orthogonal projection from K to H will be written as

P_H, and the restriction of an operator S to H will be written as $S|H$. A *contraction* operator S on a Hilbert space is a linear operator with $||S|| \leqq 1$; a *proper contraction* satisfies $||S|| < 1$.

We shall require a well-known result of Sz.-Nagy which states that the minimal unitary dilation of a contraction is unique up to unitary equivalence.

THEOREM (*Sz.-Nagy*). *Let S be a contraction operator on a Hilbert space H, and let U and U' be unitary dilations of S to Hilbert spaces K and K', respectively, containing H. Let K_0* (resp. K_0') *be the smallest subspace of K* (resp. K') *which contains H and reduces U* (resp. U'). *Then there is a unitary operator W from K_0 onto K_0' such that $W|H$ is the identity operator, and $W(U|K_0)W^{-1} = U'|K_0'$.*

The operator $U|K_0$ is called the *minimal unitary dilation* for the operator S.

3. A simple example. In this section we present a very simple example of three commuting contraction operators which do not possess commuting unitary dilations. Let H_0 be a Hilbert space of dimension at least two, and let $H = H_0 \oplus H_0$. Let V be any unitary operator on H_0 which is not a scalar multiple of the identity operator I, and let A be any contraction on H_0 which does *not* commute with V. Define operators R, S, T on H by the operator matrices:

$$R = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}.$$

Notice that R, S, T commute no matter how V and A are chosen; in fact, the product of any two of them is 0. We shall show that these operators cannot have commuting unitary dilations.

The proof is more natural when expressed in functional, rather than sequential, notation and for this reason we introduce the Hilbert space K_0 of all Fourier series $\sum_{n=-\infty}^{\infty} z^n x_n$, with the Fourier coefficients x_n in H_0, and $\sum ||x_n||^2 < \infty$. The inner product $\langle \cdot, \cdot \rangle$ on K_0 is defined, as usual, by $\langle \sum z^n x_n, \sum z^n y_n \rangle = \sum (x_n, y_n)$, where $(\cdot, \cdot)$ is the inner product on H_0. Such a Fourier series may be considered to define an honest square-integrable function from the unit circle into H_0 (see [8] for details), or it may be considered as merely a convenient way of keeping track of the components of a conventional infinite sequence of elements of H_0. For our purposes it makes no difference.

Let U be the operator on K_0 defined by: $U(\sum z^n x_n) = \sum z^{n+1} x_n$. When expressed in sequential notation, U is just the familiar bilateral shift acting on a direct sum of copies of H_0. Further, if $H = H_0 \oplus H_0$ is identified in the obvious way with the subspace of K_0 consisting of all functions of the form $x_0 + zx_1$, with x_0, x_1 in H_0, then it is very easy to check that U is the minimal unitary dilation of the operator R defined above.

Now suppose R, S, T have commuting unitary dilations U_R, U_S, U_T acting on a Hilbert space K containing H, and let K_0' be the smallest subspace of K which contains H and reduces U_R. Then the theorem of Sz.-Nagy quoted in §2 shows that we may identify K_0' with the space K_0 of Fourier series in such a way that H is identified with $\{x_0 + zx_1 \mid x_0, x_1 \in H_0\}$ and $U_R \mid K_0 = U$.

Given x in H, let $\hat{x}$ be the Fourier series $\sum z^n x_n$ with $x_0 = x$ and $x_n = 0$ for $n \neq 0$. Since $P_H U_S \mid H = S$, $U_S\hat{x} = w + \sum z^n y_n$ with $y_0 = 0$, $y_1 = Vx$, and w orthogonal to K_0. Then, $\|x\|^2 = \|U_S\hat{x}\|^2 = \|Vx\|^2 + \|w\|^2 + \sum_{n\neq 1} \|y_n\|^2$, and since V is an isometry, we must have $y_n = 0$ for $n \neq 1$ and $w = 0$. Since U_S commutes with U_R, $U_S(z^n x) = U_S(U_R^n \hat{x}) = U_R^n U_S \hat{x} = z^{n+1}Vx$. Thus U_S maps K_0 onto itself (which implies that K_0 reduces U_S because U_S is unitary), and U_S is uniquely determined on K_0 by the equation above. In fact, $U_S \mid K_0$ may be considered as multiplication by the operator-valued function $z \to zV$.

Now let E be the projection on K_0, and let $\tilde{T} = EU_T \mid K_0$. Then, since K_0 reduces both U_R and U_S, E commutes with U_R and U_S, and hence $\tilde{T}$ commutes with both $U = U_R \mid K_0$ and $U_S \mid K_0$.

Let $\tilde{T}\hat{x} = \sum z^n L_n x$, where this defines L_n as operators from H_0 to H_0. Since $\tilde{T}$ commutes with U, it is easy to see that $\tilde{T}$ acts as multiplication by the operator-valued function $z \to \sum z^n L_n$, and it is obvious that this will commute with multiplication by $z \to zV$ if and only if each operator L_n commutes with V.

More explicitly, we compute: $\tilde{T}U_S\hat{x} = \tilde{T}(zVx) = \tilde{T}U(\widehat{Vx}) = U\tilde{T}(\widehat{Vx}) = \sum z^{n+1}L_n Vx$ and $U_S\tilde{T}\hat{x} = U_S(\sum z^n L_n x) = \sum z^{n+1} VL_n x$.

But finally we notice that $P_H\tilde{T} \mid H = P_H EU_T \mid H = P_H U_T \mid H = T$, and hence $L_0 = 0$ and $L_1 = A$. Thus if A is chosen to be any contraction which does not commute with V, we have a contradiction, and it is impossible to find commuting unitary dilations for R, S, T.

A. Lebow and R. Douglas have observed that a weaker example can be obtained more simply by taking the operator A to be unitary. Also, the reader may have noticed that the dilation condition $P_H U_R^k U_S^m U_T^n \mid H = R^k S^m T^n$ was not fully used. Actually, the counterexample is valid under the much weaker assumption that $P_H U_S \mid H = S$, $P_H U_T \mid H = T$, and $P_H U_R^n \mid H = R^n$ for $n \geqq 0$. Further, the assumption that U_T be unitary was not used at all, and there are several ways that the example can be strengthened at the expense of minor complications. However, I do not know of any simple modification which will produce three *proper* contractions without commuting unitary dilations.

A simple idea contained in the proof above sheds light on a recent result of Sz.-Nagy and Foias [9]. Let R and T be commuting contractions with $\|T\| = 1$, acting on a Hilbert space H, and let U_R, U_T be a pair of commuting unitary dilations acting on a Hilbert space K containing H. (The existence of U_R, U_T is guaranteed by Ando's theorem [1], [8, Chapter 1].) Let K_0 be the smallest subspace of K which contains H and reduces U_R, let E be the projection on K_0, and let $\tilde{T} = EU_T|K_0$. Then it is an elementary exercise to verify that $\tilde{T}$ commutes with $U_R|K_0$ (the minimal unitary dilation for R), $\|\tilde{T}\| = \|T\| = 1$, and the pair $U_R|K_0$, $\tilde{T}$ dilates the pair R, T. The first two statements are trivial, and the last will be evident to anyone familiar with the structure of unitary dilations. For the reader's convenience a proof of the last statement is sketched in the next paragraph.

By definition, the space K_0 is the closed linear span of the spaces $U_R^n H$, $-\infty < n < \infty$. Let K_0^+ be the closed linear span of the spaces $U_R^n H$, $n \geqq 0$, and let $M = K_0^+ \ominus H$. Then the following facts (a) and (b) are well-known and easy to verify:

(a) $K_0 \ominus K_0^+$ is the closed linear span of all vectors $(U_R^{*n} - R^{*n})x$, with $x \in H$ and $n \geqq 0$

(b) M is the closed linear span of all vectors $(U_R^n - R^n)x$, with $x \in H$ and $n \geqq 0$.

From (a) and (b) we deduce:

(c) $\tilde{T}K_0^+ \subset K_0^+$. This follows from (a) and a routine computation showing that for all $x, y \in H$ and $m, n \geqq 0$, $(\tilde{T}U_R^m x, (U_R^{*n} - R^{*n})y) = (EU_T U_R^m x, (U_R^{*n} - R^{*n})y) = (EU_T U_R^{m+n} x, y) - (EU_T U_R^m x, R^{*n} y) = (TR^{m+n}x, y) - (TR^m x, R^{*n}y) = 0$.

(d) $\tilde{T}M \subset M$. This follows from (b) and a similar computation showing that for all $x, y \in H$ and $m, n \geqq 0$, $(\tilde{T}(U_R^m - R^m)x, U_R^{*n}y) = 0$. Finally we compute, for $x, y \in H$ and $m, n > 0$,

$$\begin{aligned}(\tilde{T}^m U_R^n x, y) &= (\tilde{T}^m R^n x, y) + (\tilde{T}^m (U_R^n - R^n)x, y) \\ &= (\tilde{T}^m R^n x, y) = (T^m R^n x, y) .\end{aligned}$$

The vanishing of $(T^m(U_R^n - R^n)x, y)$ follows from (b) and (d), and the last equality follows from (c) and (d) (see [7], Lemma 0, for a complete proof.)

The relation $\|\tilde{T}\| = \|T\|$ depended on the assumption $\|T\| = 1$, but if we replace T by $T/\|T\|$ and apply similar reasoning, we obtain the following result, which is equivalent to the main result of [9] (see [3] for details).

THEOREM (*Sz.-Nagy and Foias*). *Let R and T be commuting contractions on a Hilbert space H, and let U be the minimal unitary dilation of R to a Hilbert space K. Then there exists an operator*

$\tilde{T}$ *on* K *such that*:

(i) *The pair* $U, \tilde{T}$ *dilates the pair* R, T.
(ii) $U\tilde{T} = \tilde{T}U$
(iii) $\|\tilde{T}\| = \|T\|$.

Therefore, the result of Sz.-Nagy and Foias may be viewed as a consequence of Ando's theorem on the existence of unitary dilations for a pair of commuting contractions. Conversely, it is easy to deduce Ando's theorem from the theorem above, and in effect, the two results are equivalent. However, the clever proof of Sz.-Nagy and Foias in [9] proceeds from first principles, and does not rely on Ando's theorem (as does the proof above.) Another proof, written in matricial notation and also independent of Ando's theorem, may be found in [3].

4. Proper contractions without commuting unitary dilations. In this section we give an example of three commuting *proper* contractions which fail to have commuting unitary dilations. Unfortunately, this example is not conveniently expressable in functional notation, and we are forced to use the more cumbersome sequential notation.

We begin as in the preceding example. Let H_0 be a Hilbert space of dimension at least two, and let $H = H_0 \oplus H_0$. Choose noncommuting isometries V_2 and V_3 on H_0, and define operators S_1, S_2, S_3 on H by the operator matrices:

$$S_i = \begin{bmatrix} 0 & 0 \\ V_i & 0 \end{bmatrix},$$

where V_1 is the identity operator on H_0. We shall show that the operators λS_i, $i = 1, 2, 3$, do not have commuting unitary dilations when λ is sufficiently close to 1, $0 < \lambda < 1$. The idea is that commuting unitary dilations for λS_i would have to converge, as $\lambda \to 1$, to commuting unitary dilations for S_i, and this would contradict the result of §3.

The minimal unitary dilation $U(\lambda)$ for λS_1, $0 < \lambda < 1$, may be realized as follows on the space K_0 of all sequences $\{\cdots, (x_{-2}, x_{-1}), \boxed{(x_0, x_1)}, (x_2, x_3), \cdots\}$ of elements of $H = H_0 \oplus H_0$. (The zero' the component is boxed, and the space H is identified with the subspace of K_0 consisting of all sequences which vanish outside the box.)

$$\begin{aligned} U(\lambda)\{\cdots, (x_{-2}, x_{-1}), &\boxed{(x_0, x_1)}, (x_2, x_3), \cdots\} \\ = \{\cdots, (x_{-4}, x_{-3}), &\boxed{(x_{-2}, \sqrt{1-\lambda^2}x_{-1} + \lambda x_0)}, \\ &(\sqrt{1-\lambda^2}x_0 - \lambda x_{-1}, x_1), (x_2, x_3), \cdots\}. \end{aligned}$$

Note that $U(1)$ is a unitary dilation (but not the minimal unitary dilation) for S_1, and the operators $U(\lambda)$ converge uniformly to $U(1)$ as $\lambda \to 1$. Let M denote the smallest subspace of K_0 which contains H and reduces $U(1)$. Expressed concretely, M is the space of all sequences

$$\{\cdots, (x_{-4}, 0), (x_{-2}, 0), \boxed{(x_0, x_1)}, (0, x_3), (0, x_5), \cdots\} .$$

Now for each fixed λ, $0 < \lambda < 1$, we assume the existence of commuting unitary dilations $U_i(\lambda)$ for λS_i, $i = 1, 2, 3$, and there is no loss of generality in assuming that for all λ, $0 < \lambda < 1$, the operators $U_i(\lambda)$ act on a fixed Hilbert space K containing H. If $K_0(\lambda)$ is the smallest subspace of K which contains H and is invariant under $U_1(\lambda)$, then by the uniqueness theorem of Sz.-Nagy quoted in §2, we may assume that $K_0(\lambda) = K_0$ for all $0 < \lambda < 1$, and that $U_1(\lambda) | K_0$ is the operator $U(\lambda)$ defined above. (Given unitary operators $W(\lambda)$ which map $K_0(\lambda)$ onto K_0, fix all elements of H, and satisfy

$$W(\lambda)(U_1(\lambda) | K_0(\lambda)) W(\lambda)^{-1} = U(\lambda) ,$$

one can embed K_0 in a larger Hilbert space K' and choose arbitrary unitary operators $W'(\lambda)$ mapping K onto K' and satisfying $W'(\lambda) | K_0(\lambda) = W(\lambda)$. Then one can replace the operators $U_i(\lambda)$ by their unitary transforms $W'(\lambda) U_i(\lambda) W'(\lambda)^{-1}$. For fixed λ, these new operators are again commuting unitary dilations for λS_i, $i = 1, 2, 3$, and, by construction, $W'(\lambda) U_1(\lambda) W'(\lambda)^{-1} | K_0 = U(\lambda)$.)

The unit ball of operators is compact in the weak operator topology, and hence there exists a sequence $\{\lambda_n\}$, $\lambda_n \to 1$, such that the operators $P_M U_i(\lambda_n) P_M$ converge weakly, as $n \to \infty$, to operators Q_i, 1, 2, 3. Since $U_1(\lambda) | K_0 = U(\lambda)$, the uniform convergence of $U(\lambda)$ implies that $P_M U_1(\lambda) P_M$ converges uniformly to Q_1, and that $Q_1 | M = U(1) | M$. Further, it is routine to verify that $P_H Q_i | H = S_i$, $i = 1, 2, 3$.

Now we show that Q_1 commutes with Q_2 and Q_3. Writing $E = P_M$ and using the notation $[A, B] = AB - BA$, we first note:

$$(1) \qquad [Q_1, Q_i] = \text{weak} \lim_{n\to\infty} [EU_1(\lambda_n)E, EU_i(\lambda_n)E]$$

$$(2) \qquad \lim_{n\to\infty} ||(1 - E) U_1(\lambda_n) E|| = 0$$

$$(3) \qquad \lim_{n\to\infty} ||EU_1(\lambda_n)(1 - E)|| = 0 .$$

Equation (1) holds because $EU_1(\lambda_n)E$ converses *uniformly* to Q_1, and (2) and (3) follow immediately from the expression for $U(\lambda)$ above. Now the identity

$$(4) \qquad \begin{aligned} [EU_1(\lambda)E, EU_i(\lambda)E] = E[U_1(\lambda), U_i(\lambda)]E + EU_i(\lambda)\{(1 - E) U_1(\lambda)E\} \\ - \{EU_1(\lambda)(1 - E)\} U_i(\lambda)E \end{aligned}$$

together with (1), (2), (3), and $[U_1(\lambda), U_i(\lambda)] = 0$ imply that $[Q_1, Q_i] = 0$.

The contractions Q_i leave M invariant, $Q_i|M$ commutes with $Q_1|M = U(1)|M$, and $P_H Q_i|H = S_i$, 1, 2, 3. From this we conclude (the argument is identical to one in §3) that the action of Q_i on M is given by:

$$\begin{aligned} Q_i\{\cdots, (x_{-2}, 0), \boxed{(x_0, x_1)}, (0, x_3), \cdots\} \\ = \{\cdots, (V_i x_{-4}, 0), \boxed{(V_i x_{-2}, V_i x_0)}, (0, V_i x_1), \cdots\} . \end{aligned}$$

Of course, Q_2 and Q_3 do not commute because V_2 and V_3 do not commute. However, we will now show that the method of construction of Q_2 and Q_3 implies that they must commute. This is a contradiction, and therefore, the commuting unitary dilations $U_i(\lambda)$ cannot exist when λ is sufficiently close to 1, $0 < \lambda < 1$.

To see that Q_2 and Q_3 must commute, first note that their representation above shows that each is isometric on M. As is well-known, a sequence of contractions which converges weakly to an isometry also converges in the strong operator topology. Since the isometry $Q_i|M$ is a weak (hence strong) limit of the contractions $P_M U_i(\lambda_n) P_M|M$, elementary properties of the strong operator topology imply that for all x in M,

(a) $Q_2 Q_3 x = \lim_{n\to\infty} P_M U_2(\lambda_n) P_M U_3(\lambda_n) x$

(b) $\lim (1 - P_M) U_i(\lambda_n) x = 0$.

From this we obtain, for all x in M,

$$\begin{aligned} Q_2 Q_3 x &= \lim P_M U_2(\lambda_n) P_M U_3(\lambda_n) x \\ &= \lim P_M U_2(\lambda_n) U_3(\lambda_n) x + \lim P_M U_2(\lambda_n)(P_M - 1) U_3(\lambda_n) x \\ &= \lim P_M U_2(\lambda_n) U_3(\lambda_n) x = \lim P_M U_3(\lambda_n) U_2(\lambda_n) x = Q_3 Q_2 x . \end{aligned}$$

The proof is complete.

5. Remarks on an open problem. It has long been known that questions involving unitary dilations are often closely related to questions involving spectral sets. For instance, Von Neumann's theorem stating that the unit disc is a spectral set for every contraction is a simple consequence of the existence of a unitary dilation for a single contraction together with the easy fact that the disc is a spectral set for any unitary operator. Conversely, if a simply connected subset of the plane is a spectral set for an operator, then the operator has a normal dilation with spectrum in the boundary of the set [2, 4, 5]. (When the set in question is the unit disc, this says that Von Neumann's theorem implies that every contraction has a unitary dilation.)

One can try to generalize Von Neumann's theorem to apply to several commuting contractions as follows. Let D be the unit disc $\{z \| z| \leqq 1\}$ in the complex plane, and let D^n denote the n-fold direct product of D with itself. We shall say that D^n is a *spectral set* for n commuting contractions $S_1, \cdots, S_n$ if for each polynomial $p(z_1, \cdots, z_n)$ in n variables,

$$||p(S_1, \cdots, S_n)|| \leqq ||p||_\infty ,$$

where $||p||_\infty$ denotes the maximum of $|p(z_1, \cdots, z_n)|$ on D^n. The following conjecture is a natural generalization of Von Neumann's theorem.

Conjecture. The n-polydisc D^n is a spectral set for any n commuting contractions on a Hilbert space.

It is easy to employ the spectral theorem to show that this conjecture is true if the commuting contractions are normal operators. From this, it follows that the existence of commuting unitary dilations for n commuting contractions implies the conjecture for that n. Thus Ando's theorem implies that the conjecture is true for $n = 2$. For $n \geqq 3$, the conjecture is still open, and the results of this paper show that its proof (if, indeed, it is true) cannot rely on the existence of commuting unitary dilations for commuting contractions.

Also, the operators of §3 cannot be used to construct a counterexample to the conjecture, for D^3 is a spectral set for any three contractions defined on $H = H_0 \oplus H_0$ by operator matrices

$$S_i = \begin{bmatrix} 0 & 0 \\ Q_i & 0 \end{bmatrix}, \; i = 1, 2, 3, \; ||Q_i|| \leqq 1 .$$

This also shows that the spectral set problem and the commuting unitary dilation problem are *not* equivalent for $n \geqq 3$.

To see that D^3 is a spectral set for the operators S_1, S_2, S_3, consider an arbitrary polynomial p in three variables:

$$p(z_1, z_2, z_3) = a_0 + \sum_{i=1}^{3} a_i z_i + q(z_1, z_2, z_3) ,$$

where q is a polynomial containing only terms of second or higher degree. Then

$$p(S_1, S_2, S_3) = \begin{bmatrix} a_0 & 0 \\ \sum_{i=1}^{3} a_i Q_i & a_0 \end{bmatrix} .$$

The first step, left to the reader, is to verify that

$$\left\| \begin{bmatrix} a_0 & 0 \\ \sum_{i=1}^{3} a_i Q_i & a_0 \end{bmatrix} \right\| \leqq \left\| \begin{bmatrix} |a_0| & 0 \\ \sum_{i=1}^{3} |a_i| & |a_0| \end{bmatrix} \right\| .$$

Hence to show that D^3 is a spectral set for S_1, S_2, S_3, we need only show that the norm of the latter matrix is no larger than

$$\inf \left\| a_0 + \sum_{i=1}^{3} a_i z_i + q(z_1, z_2, z_3) \right\|_\infty ,$$

where the infemum is taken over all polynomials q containing only terms of degree two or higher.

Now a classical result of Caratheodory and Fejer states that

$$\inf \| b_0 + b_1 z + r(z) \|_\infty = \left\| \begin{bmatrix} b_0 & 0 \\ b_1 & b_0 \end{bmatrix} \right\|$$

where the infemum is taken over all polynomials $r(z)$ (in one variable) which contain only terms of degree two or higher. (For a modern proof, see Sarason's beautiful paper [6], where the result is derived as a consequence of a special case of the theorem of Sz.-Nagy and Foias discussed in §3.) Using this fact, we have:

$$\begin{aligned} \| p(S_1, S_2, S_3) \| &\leqq \left\| \begin{bmatrix} |a_0| & 0 \\ \sum_{i=1}^{3} |a_i| & |a_0| \end{bmatrix} \right\| \\ &= \inf \left\| |a_0| + \left(\sum_{i=1}^{3} |a_i| \right) z + r(z) \right\|_\infty \\ &\leqq \inf \left\| |a_0| + \sum_{i=1}^{3} |a_i| z_i + q(z_1, z_2, z_3) \right\|_\infty \\ &= \inf \left\| a_0 + \sum_{i=1}^{3} a_i z_i + q(z_1, z_2, z_3) \right\|_\infty , \end{aligned}$$

where, again, r and q range over all polynomials, in one and three variables, respectively, containing only terms of degree two or higher. The second inequality was obtained by setting all three variables equal in the polynomial $|a_0| + \sum_{i=1}^{3} |a_i| z_i + q(z_1, z_2, z_3)$, and the last equality was obtained by multiplying by $a_0/|a_0|$ and replacing z_i by $(\bar{a}_0/|a_0|) \cdot (a_i/|a_i|) z_i$.

Thus D^3 is a spectral set for three commuting contractions which admit no commuting unitary dilations, and it appears that the connection between spectral sets and unitary dilations may not be as close as has been assumed.

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University of Massachusetts at Boston