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# FIBRATIONS OF ANALYTIC VARIETIES

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**The induced continuous, differentiable, or analytic fibering about any point of a continuous, differentiable, or analytic group $A$, by a subgroup $B$ is well known, as are generalizations to various spaces with operators. One may ask about analogous results for varieties per se. For instance, if $C$ is any arc in $E^2$ and $p \in C$, then there is always a homeomorphism $\varphi$ from a neighborhood $U$ of $p$ to $I \times I$ $(I = (0, 1))$, so that $\varphi(U \cap C) = I \times \{\frac{1}{2}\}$. But there are arcs in $E^3$ which are so wildly embedded that at no point of the arc is there an analogous fibering. This paper considers a general fibration problem for complex-analytic varieties, and extends a result on fibering hypersurfaces due to Hassler Whitney.**

Roughly, one can formulate a basic fibering question for complex-analytic varieties in this way: Try to decompose any variety $V$ into disjoint submanifolds, or "strata", so that $V$ has a fibration about each point $p$ of $V$, using a piece of the submanifold through $p$ as fiber (Conjecture 1.4). We require such a decomposition to be locally finite. This problem is easily solved if we ask only for continuous fibrations, but it is in general not solvable if one requires analytic fibrations (see Remark 1.6). A natural notion turns out to be "semi-analytic" fibration, in which analytic fibers vary continuously. For a hypersurface $W$, Whitney has found such fibrations about all points off a subvariety of codimension 2 in $W$. We extend his result to arbitrary varieties, and using some of the ideas of the proof, we also answer a question of his concerning the structure of any variety near a submanifold of codimension 1.

## 1. Preliminaries; statement of the fibering theorem.

Let $V$ be analytic of dimension $r$ in an open set $H$ of $\mathscr{C}^n$ (where $\mathscr{C}$ denotes the complex line).

DEFINITION 1.1. A set $M \subseteqq H$ is a *manifold* if about each point $p \in M$ there is a $\mathscr{C}^n$-open neighborhood $U$ such that $M \cap U$ is the set of common zeros in $U$ of a set of functions analytic in $U$, and $M$ is nonsingular. A *stratification of* $V$ is a splitting of $V$ into a disjoint union of a locally finite set of irreducible manifolds, called the strata, such that the boundary of each stratum is the union of a set of lower dimensional strata.

THEOREM 1.2. *There is a stratification of any variety* [3, p. 227].

Let $M$ be an $s$-dimensional stratum of $V$, and let $\Delta^r$ denote the $r$-fold product of an open disk in $\mathscr{C}$.

DEFINITION 1.3. *A $\mathscr{C}^n$-open neighborhood $U$ of $p \in M$ has a semi-analytic fibration* if there is a homeomorphism $\sigma$ from $\Delta^s \times \Delta^{n-s}$ to $U$ such that for each $q \in \Delta^{n-s}$, $\sigma(\Delta^s \times q)$ is biholomorphic to $M \cap U$; for $q \neq 0 \in \Delta^{n-s}$, $\sigma(\Delta^s \times q)$ lies entirely in $(V \backslash M) \cap U$ or $(H \backslash V) \cap U$, and $\sigma(\Delta^s \times 0) = M \cap U$.

CONJECTURE 1.4. Any analytic variety $V$ has a stratification such that each point has a neighborhood with a semi-analytic fibration [3, p. 230].

Our main fibering theorem is

THEOREM 1.5. *$V$ has a stratification so that about any point in an $(r-1)$-dimensional stratum, there is a neighborhood which can be semi-analytically fibered.*

REMARK 1.6. An example of Whitney (see [3, p. 239]) shows $\sigma$ cannot be made biholomorphic in general.

We collect here some other definitions and facts used in the sequel.

DEFINITION 1.7. The *tangent cone* $C(V, p)$ of $V$ at $p \in V$ is the set of all $q \in \mathscr{C}^n$ such that there are sequences $\{a_i\}(a_i \in \mathscr{C})$ and $q_i \to p$ $(q_i \in V)$ so $a_i(q_i - p) \to q$.

If $V$ has pure dimension $r$ at $p$, $C(V, p)$ is a homogeneous algebraic variety of pure dimension $r$. If $I(V, p)$ denotes the ring of germs of functions holomorphic in $\mathscr{C}^n$ at $p$ and vanishing on $V$, then $\mathscr{C}(V, p)$ is the variety of zeros in $\mathscr{C}^n$ of the set of all initial polynomials of each function in $I(V, p)$. (The initial polynomial of $f$ at $p$ is the polynomial of all terms of lowest order in $f$'s expansion at $p$.)

DEFINITION 1.8. Let varieties $V_1$, $V_2$ have pure dimension $r, s$ respectively in a $\mathscr{C}^n$-open neighborhood of a point $p$. *$V_1$ and $V_2$ intersect properly at $p$* if the codimension of $V_1 \cap V_2$ is the sum of the codimensions $n-r$ and $n-s$ of $V_1$ and $V_2$. *$V_1$ and $V_2$ intersect transversally at $p$* if $C(V_1, p)$ and $C(V_2, p)$ intersect properly at $p$.

THEOREM 1.9. *Suppose an $(n-r-1)$-dimensional linear variety $L_{n-r-1}$ intersects $V$ in an isolated point $p$. Then* [3, Lemma 9.7] *there is a $\mathscr{C}^n$-open neighborhood $U$ of $p$ so that the points in $U$ of* *the union of the parallel translates of* $L_{n-r-1}$ *through points of* $V \cap U$ *form an analytic variety in* $U$, *called the* cylinderization *of* $V$ *by* $L_{n-r-1}$ near $p$. *We denote it by* $V(L_{n-r-1}, p)$ *or simply by* $V(L_{n-r-1})$ *if the reference to* $p$ *is clear.*

THEOREM 1.10. *If* $W$ *is any subvariety of* $V$, *there is a stratification so* $W$ *lies outside the strata of dimension* $> \dim W$.

This is clear from the proof of (1.3).

NOTATION 1.11. $S(V)$ will denote the singular subvariety of $V$; $\mathscr{P}^n$, $n$-dimensional complex projective space; and $G_{n,r}$, the Grassmann manifold of all $r$-subspaces of $\mathscr{C}^n$. $(x_1, \cdots, x_n)$ will denote analytic co-ordinates about a point $p$ of an $(r-1)$-dimensional stratum $M$. $\bar{x}$ will stand for $(x_1, \cdots, x_{r-1}, 0, \cdots, 0)$, $\bar{\bar{x}}$ for $(0, \cdots, 0, x_r, \cdots, x_n)$, and by abuse of notation, $(\bar{x}, \bar{\bar{x}})$ for $(x_1, \cdots, x_n)$. Throughout the paper, any $\mathscr{C}^n$-open neighborhood $U$ about $p$ that we consider will be such that $M \cap U$ is an open subset of the $(x_1, \cdots, x_{r-1})$-plane $\mathscr{C}_{x_1, \ldots, x_{r-1}}$ (where $\mathscr{C}_{x_i}$ denotes the $x_i$-axis, $\mathscr{C}_{x_i x_j}$, the $(x_i, x_j)$-plane, etc.).

**2. Proof of the fibering theorem.** The strategy of the proof is this: We first prove the following theorem, which gives us the stratification used in our main fibration theorem.

THEOREM 2.1. *Any variety* $V$ *of dimension* $r$ *has a stratification so each* $(r-1)$*-stratum* $M$ *has the following three properties*:

(i) *The dimension of* $V$ *at any* $p \in M$ *is pure.*

(ii) *For any fixed* $p \in M$, *for each* $q \in V \backslash M$ *and each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that* $\text{distance}(q, p) < \delta \Rightarrow d(C(V, q), p(q)) < \varepsilon$. ($p(q)$ *is the analytic* $r$*-plane containing* $q$ *and the part of* $M$ *near* $p$, *and* $d$ *is any Hausdorff metric on* $G_{r,n}$.)

(iii) *For any fixed* $p \in M$, *for each* $q \in M$ *and each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that* $\text{distance}(q, p) < \delta \Rightarrow$ *for each* $x \in \overline{C(V, q)}$ (*the natural image of* $C(V, q)$ *in* $\mathscr{P}^{n-1}$), *there is a* $y \in \overline{C(V, p)}$ *within* $\varepsilon$ *of* $x$ (*relative to a metric in* $\mathscr{P}^{n-1}$).

REMARK 2.2. Property (ii) expresses a kind of continuity of $C(V, q)$ as $q$ approaches a point $p$ in $M$ transversally, while property (iii) does the same for $q$ varying in $M$.

We next prove

THEOREM 2.3. *Let* $V$ *be stratified as in Theorem* 2.1. *For each* $p$ *in an* $(r-1)$*-stratum* $M$ (*where* $\dim_p V = r$), *and each* $(n-r)$*-plane*

*P transverse to V at p, there is a* $\mathscr{C}^n$*-open neighborhood U of p so every translate* $P + q$ ($q \in V \cap U$) *is transverse to V at q.*

Using the above theorem, we can now easily prove our main result, Theorem 1.5, as follows: If $p \in M$ and $\dim_p V = r$, then, assuming the part of $\mathscr{C}_{x_{r+1},\cdots,x_n}$ near $p$ forms an open set of $p$, (2.1) together with [3, p. 273, Zusatz II] shows that a $V$-open neighborhood of any point of $V \backslash M$ near $p$ may be represented in the form

$$x_i = f_i(x_1, \cdots, x_r) \ (i = r + 1, \cdots, n; \ f_i \text{ analytic}) . \tag{2.4}$$

With these functions we easily construct a fibration of a neighborhood of $p$ using Whitney's method [3, §§ 11, 12].

If $p \in M$ and $\dim_p V = r - 1$, then Theorem 1.5 is trivial.

*Proof of Theorem* 2.1.

(i) Suppose $r < n$. Write $V = V' \cup V''$ ($V'$, $V''$ varieties), $V'$ having pure dimension $r$, $\dim V'' < r$, $\dim (V' \cap V'') < r - 1$. Then we may stratify $V$ so $V' \cap V''$ is contained in the union of strata of dimension $< r - 1$ (1.10).

To prove (ii) and (iii) we show that in any stratification satisfying (i), those points of $M$ where the conditions of (ii) and (iii) do not hold are contained in an analytic subvariety of $V$ having dimension less than $r - 1$, and may therefore be put into lower dimensional strata (again by 1.10). Any $(r - 1)$-stratum of this new stratification will then satisfy (i), (ii), and (iii).

(ii) The points of $M$ satisfying (ii) are called *regular* points by Whitney; that we can choose a stratification satisfying (ii) is essentially done in [4, Th. 19.2].

(iii) We show the set of points where the condition in (iii) fails is contained in the set $A$, defined as follows: Denote the topological closure of the (countable) collection of $(r - 1)$-strata by $\bar{M}_1, \bar{M}_2, \cdots$. Let $A_i$ be the set of points of $\bar{M}_i$ having multiplicity in $V$ (Definition 2.5) greater than the minimum assumed on $M_i$. Then take $A$ to be $\bigcup_{i=1}^{\infty} A_i$. To prove (iii) we must show

(a) $A$ is an analytic subvariety of $V$ (of dimension $< r - 1$);

(b) The condition of (iii) holds in $M \backslash A$. The main part of the proof of (b) is establishing

($b_1$) For each $p \in M \backslash A$, there is a neighborhood $U_p$ such that $\mathscr{A} = \{(q, C(V, q)) : q \in M_i \backslash A\}$ is analytic in $(M \cap U_p) \times \mathscr{C}^n$; and

($b_2$) Each $C(V, q)$ ($q \in M \cap U_q$) is a union of $r$-planes containing $M \cap U_p$. We will see (b) is an easy consequence of these two facts.

*Proof of* (a). We recall the

DEFINITION 2.5. Let $p$ be any point of $V$, and suppose an $(n-r)$-dimensional linear variety $L_{n-r}$ intersects $V$ transversally at $p$. Then there is a $\mathscr{C}^n$-open neighborhood $U$ of $p$ and an integer $m(p)$ so that at each $q \in U$ off a proper analytic subvariety of $U$, $L_{n-r} + q$ intersects $V \cap U$ in exactly $m(p)$ distinct points. We call $m(p)$ the *multiplicity of $V$ at $p$*.

We next note

LEMMA 2.6. *The set of points of $V$ with multiplicity greater than a fixed integer forms a subvariety of $V$.*

(One proof will appear in a forthcoming book on analytic varieties by Whitney; a general ring-theoretic proof appears in [1, Th. 40.3].)

Now let $m_i$ be the smallest number such that there is a point of $M_i$ having multiplicity $m_i$ in $V$. Then by Lemma 2.6, the set $V_{m_i}$ of points of $V$ having multiplicity greater than $m_i$ is a subvariety of $V$; since, by [3, Lemma 8.2], $\bar{M}_i$ is also a subvariety of $V$,

$$A_i = \bar{M}_i \cap V_{m_i}$$

is a subvariety of $V$. And because $M_i$ is irreducible, $\dim A_i < \dim M_i$. Now by local finiteness of the $M_i$, we see that $\bigcup_{i=1}^{\infty} A_i$ is a subvariety of $V$ having dimension less than $r-1$.

REMARK 2.7. From here to the end of the proof of Theorem 2.3 a $\mathscr{C}^n$-open neighborhood about a typical point in $M \backslash A$ (which we will call 0) will be subjected to a finite succession of requirements. To keep notation simple, denote by $U$ a neighborhood small enough so all requirements at any stage are satisfied.

*Proof of* $(b_1)$. Let $0 \in \mathscr{C}^n$ be a typical point of $M \backslash A$. We show that for some $U$ $(0 \in U)$, $\mathscr{A} \cap ((M \cap U) \times \mathscr{C}^n)$ is analytic. (Assume $U \cap A = \varnothing$.) Since the proof is a bit long, we divide the proof into three parts:

First, we show there is an open neighborhood $N$ in the Grassmannian $G_{n,n-r-1}$ of all $(n-r-1)$-subspaces of $\mathscr{C}^n$, so that the cylindrizations $V(L^*_{n-r-1})$ of $V$ at 0 along each $L^*_{n-r-1}$ in $N$ are all analytic in some $\mathscr{C}^n$-open $U$, and such that the multiplicity of each point in $M \cap U$ is the same in any $V(L^*_{n-r-1})$ as it is in $V$. Second, using this fact we show $\mathscr{A}(L^*_{n-r-1}) = \{(q, C(V(L^*_{n-r-1}), q)) \mid q \in M \cap U\}$ is analytic in $((M \backslash A) \cap U) \times \mathscr{C}^n$ for any $L^*_{n-r-1}$ above.

Finally, using the analyticity of each $\mathscr{A}(L^*_{n-r-1})$, we prove $(b_1)$.

To establish the first assertion, let $L^*_{n-r-1}$ be a subspace of any $L_{n-r}$ in (2.5) so that for almost every $q \in U$, $\pi(U \cap V \cap (L_{n-r} + q))$

consists of $m$ points, where $m$ = multiplicity of 0 in $V$, and $\pi$ = projection of $L_{n-r}$ to $L^*_{n-r-1}$. From intersection theory it is clear that the conclusion holds for some $N$ about $L^*_{n-r-1}$, and possibly smaller $U$.

To prove the analyticity of each $\mathscr{A}(L^*_{n-r-1})$, we note there is a function $f$ holomorphic in $U$ and vanishing on $V$ and such that at any $q \in M \cap U$, $V(L^*_{n-r-1})$ has multiplicity equal to the order of $f$ at $q$ (essentially [1, (40.3)]). Therefore the order of $f$ is $m$ at each point of $M \cap U$, so the initial form is always the $m$-th-degree term of $f$. The analyticity of the coefficients of any term then implies each $\mathscr{A}(L^*_{n-r-1})$ is analytic in $((M\backslash A) \cap U) \times \mathscr{C}^n$.

We now prove $(b_1)$. We begin by showing

(2.8) $C(V(L^*_{n-r-1}), 0) = (C(V, 0)\,(L^*_{n-r-1})$ for any $L^*_{n-r-1}$ transverse to $C(V, 0)$.

Write $\mathscr{C}^n = L_{r+1} \times L^*_{n-r-1}$ for some $(r+1)$-space $L_{r+1}$; let $\rho$ be the natural projection onto the first factor. Since 0 is isolated in $L^*_{n-r-1} \cap C(V, 0)$,

$$C(\rho(V(L^*_{n-r-1}))) = \rho(C(V(L^*_{n-r-1}))) .$$

(This is just a restatement of [4, Lemma 9.7].) If $\mathscr{C}_{x_{r+2},\cdots,x_n} = L^*_{n-r-1}$, no germ in $I(V(L^*_{n-r-1}), 0)$ involves any of the variables $x_{r+2}, \cdots, x_n$. The characterization in (1.7) of tangent cones in terms of initial forms then gives (2.8).

Using this result, one easily checks that for each $q \in M \cap U$, there are subspaces

$$(L^*_{n-r-1})_1, \cdots, (L^*_{n-r-1})_{n_q}$$

such that the analytic set

$$\mathscr{A}_q = \bigcap_{i=1}^{n_q} ((L^*_{n-r-1})_i)$$

has fiber $C(V, q)$ above $q$. Furthermore, the fiber of $\mathscr{A}_q$ over any point $q'$ in $M \cap U$ contains the tangent cone at $q'$. (Each $V(L^*_{n-r-1})_i$ contains $V$, so

$$C(V(L^*_{n-r-1})_i, q') \supseteqq C(V, q') ;$$

hence

$$\bigcap_{i=1}^{n_q} C(V(L^*_{n-r-1})_i, q') \supseteqq C(V, q').)$$

It then follows that the analytic set

$$\bigcap_{L^*_{n-r-1} \in N} \mathscr{A}(L^*_{n-r-1}) \cap ((M \cap U) \times \mathscr{C}^n)$$

is just $\mathscr{A}_q \cap ((M \cap U) \times \mathscr{C}^n)$. We have thus proved $(b_1)$.

NOTATION 2.9. By local finiteness, for some $U$, $\mathscr{A} \cap ((M \cap U) \times \mathscr{C}^n)$ is defined by finitely many cylindrizations; we denote them by $f_1, \cdots, f_s$.

We next prove $(b_2)$. (b) will follow easily from $(b_1)$ and $(b_2)$.

*Proof of* $(b_2)$. Let $U$ be a common neighborhood in which all the cylindrizations above are defined. Now if the initial polynomial at $p \in M \cap U$ of any $f_k$ defining any cylindrization of $V$ involved any variables $x_1, \cdots, x_{r-1}$, the order of $f_k$ at $p$ could not be a constant $m$ on points of $M \cap U$. Therefore if $q \in \mathscr{A}(L^*_{n-r-1})$, then

$$(0, C_{\bar{x}}) + q \subseteqq \mathscr{A}(L^*_{n-r-1}) .$$

Hence a similar relation holds for $\mathscr{A} \cap (M \cap U) \times \mathscr{C}^n)$. This, together with the fact that $C(V, p)$ is homogeneous of pure dimension $r$, shows that $C(V, p) \cap \pi^{-1}(p)$ is the union of finitely many 1-subspaces of $\pi^{-1}(p)$ ($\pi$ being the projection $(\bar{x}, \bar{\bar{x}}) \rightarrow (\bar{x})$.) This in turn gives $(b_2)$.

*Proof of* (b). From $(b_2)$ we may clearly assume that the $x$ in (2.1, (iii)) is contained in $\pi^{-1}(0)$. (b) is then obvious when we observe that the variety in $(M \cap U) \times \mathscr{P}^{n-1}$ obtained from

$$\mathscr{A} \cap ((M \cap U) \times \pi^{-1}(M))$$

is of pure dimension 1 and its fiber above any $\bar{x} \in M_i \cap U$ has dimension 0.

We now assume $V$ is stratified as in Theorem 2.1. A proof of Theorem 2.3 can be given following Lemmas 2.10 and 2.11 which give information on tangent spaces near, but off, any $(r-1)$-stratum $M$.

Let 0 be a typical point of $M$, and suppose $\mathscr{C}_{x_{r+1},\ldots,x_n}$ is transverse to $C(V, 0)$. Let dist( , ) be the usual distance on $\mathscr{C}_{\bar{x}}$ and let $\pi$ be as above.

LEMMA 2.10. *For* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if*

$$(\bar{x}, \bar{\bar{x}}) \in V, |\bar{x}| < \delta, |\bar{x}'| < \delta$$

*and* $0 < |\bar{\bar{x}}| < \delta$, *then*

$$\operatorname{dist}\Big(\Big(\frac{\bar{\bar{x}}}{|\bar{\bar{x}}|}\Big), C(V, \bar{x}') \cap \pi^{-1}(\bar{x}')\Big) < \varepsilon .$$

*Proof*. We may choose $\delta > 0$ so $C(V, \bar{x}')$ intersects $\pi^{-1}(\bar{x}')$ properly for each $\bar{x}'$, $|\bar{x}'| < \delta$ (using $(b_2)$). The family $C(V, \bar{x}') \cap \pi^{-1}(\bar{x}')$ is then easily seen to be continuous at all $\bar{x}'$ close to 0. We may therefore assume $\bar{x}' = 0$.

Let $g_{i,\bar{x}}$ denote the initial polynomial of $f_i$ expanded about $\bar{x}$. Let $S$ be the unit sphere in $\pi^{-1}(0)$, center 0. For $\varepsilon > 0$, the set $A$ of all points in $S$ at a distance $\geqq \varepsilon$ from $C(V, 0) \cap \pi^{-1}(0)$ is compact; let $\alpha > 0$ be the minimum of

$$|g_{1,0}| + \cdots + |g_{s,0}|$$

on $A$. Then

$$\left|g_{1,0}\left(\frac{\bar{\bar{x}}}{|\bar{\bar{x}}|}\right)\right| + \cdots + \left|g_{n-r,0}\left(\frac{\bar{\bar{x}}}{|\bar{\bar{x}}|}\right)\right| < \alpha$$

implies

$$\operatorname{dist}\left(\left(\frac{\bar{\bar{x}}}{|\bar{\bar{x}}|}\right), C(V, 0) \cap \pi^{-1}(0)\right) < \varepsilon .$$

One can easily find such a $\delta$ so this holds; if $(\bar{x}, \bar{\bar{x}}) \in V$, then

$$g_{i,0}\left(\frac{\bar{\bar{x}}}{|\bar{\bar{x}}|}\right) = \left[g_{i,0}\left(\frac{\bar{\bar{x}}}{|\bar{\bar{x}}|}\right) - g_{i,x}\left(\bar{x}, \frac{\bar{\bar{x}}}{|\bar{\bar{x}}|}\right)\right] + \left[|\bar{\bar{x}}|^{-m_i} f_i(\bar{x}, \bar{\bar{x}}) - |\bar{\bar{x}}|^{-m_i} g_{i,x}(\bar{x}, \bar{\bar{x}})\right]$$

($m_i = \deg(g_i)$). The first difference is small for $(\bar{x}, \bar{\bar{x}})$ near 0 by the continuity of $g_{i,\bar{x}}$; the other difference is too, since $f_i(\bar{x}, \bar{\bar{x}}) - g_{i,x}(\bar{x}, \bar{\bar{x}})$ is holomorphic of order at least $m_i + 1$ at all $\bar{x}$ sufficiently near 0.

Now let $P(q)$ be as in (2.4), and let $d( \, , \, )$ be a metric in $G_{r,n}$.

LEMMA 2.11. *There is a $\mathscr{C}^n$-open neighborhood $V$ about* 0 *such that if $q \in (V \backslash M) \cap U$, then*

(1) *$d(C(V, q), P(q))$ is small; and*

(2) *$d(P(q), C(V, 0))$ is small.*

*Proof*. (1) is obvious, since 0 is regular. (2) follows easily from Lemma 2.10 since $C(V, 0)$ and $P(q)$ are just the cylindrizations in $\mathscr{C}^n$ along $\mathscr{C}_{\bar{x}}$ of $C(V, 0) \cap \pi^{-1}(0)$, and the analytic line through (0, 0) and $q$, respectively.

*Proof of Theorem* 2.3. Suppose $U$ has diameter $\delta$; denote it by $U(\delta)$. Then Lemma 2.11 shows

(2.12) for any $\varepsilon' > 0$, there is a $\delta' > 0$ so that for

$$q \in (V \backslash M) \cap U(\delta'), \quad d(C(V, q), C(V, 0)) < \varepsilon' .$$

Let $\bar{W}$ be the image in $\mathscr{P}^{n-1}$ of a homogeneous variety $W \subseteqq \mathscr{C}^n$, and $d^*$, a metric in $\mathscr{P}^{n-1}$; then

(2.13) For $\varepsilon > 0$, there is a $\delta > 0$ such that if $q \in V \cap U(\delta)$, then for each $p \in \overline{C(V, q)}$, there is a point of $\overline{C(V, 0)}$ within $\varepsilon$ of $p$.

When $p \notin M$, this follows easily from (2.12) and the compactness of $\overline{C(V, q)}$ and $\overline{C(V, 0)}$; when $p \in M$, this is just (2.1, (ii)).

Now let $P$ be any $(n - r)$-plane of $\mathscr{C}^n$ transverse to $V$ at 0. $P + q$ is transverse to $V$ at $q \in M \cap U$ if and only if $\bar{P}$ and $\overline{C(V, q)}$ are disjoint. Now let $d_0$ be the infimum of $d^*(x, y)$ over $x \in \bar{P}$, $y \in \overline{C(V, 0)}$; then $d_0 > 0$. Choose $\varepsilon$ in (2.13) to be $\frac{1}{2} d_0$; any corresponding $U(\delta)$ then serves as the required $U$ in the statement of Theorem 2.3.

## 3. Proof of the fibering theorem.

Using Theorem 2.3, one may now prove our main result, Theorem 1.5. Let 0 be a typical point of any $(r - 1)$-stratum $M$ $(M \cap A = \varnothing)$, and suppose $\mathscr{C}_{x_{r+1}, \cdots, x_n}$ is transverse to $V$ at 0. Then (e.g., from [2, $p$. 273, Zusatz II]) there is a $\mathscr{C}^n$-open $U$ about 0 with the following property:

(3.1) There is a neighborhood $N(q)$ open in $U \cap \mathscr{C}_{\bar{x}, x_r}$, about any $q \in U \cap \mathscr{C}_{\bar{x}, x_r} \backslash M$, such that the points in $(V \backslash M) \cap U$ above $N(q)$ are given by holomorphic functions $x_i = \varphi_{ij}(x, \cdots, x_r)$ $(i = r + 1, \cdots, n;$ $j = 1, \cdots, m$ for some fixed $m$).

Now when $x_r \neq 0$, define $\psi_{ij}(0, \cdots, 0, x_r, x_i)$ to be

$$\left[\sum_{k=1}^{m} \frac{|x_i - \varphi_{ij}(0, \cdots, 0, x_r)|}{|x_i - \varphi_{ik}(0, \cdots, 0, x_r)|}\right]^{-1}$$

when for each $k = 1, \cdots, m$, $x_i \neq \varphi_{ij}(0, \cdots, 0, x_r)$; let

$$\psi_{ij}(0, \cdots, 0, x_r, \varphi_{ik}(0, \cdots, 0, x_r)) = \delta_{jk} .$$

It easily follows from [3, §§ 11, 12] that for $i = r + 1, \cdots, n$,

$$h_i(x_1, \cdots, x_r, x_j) = \sum_{j=1}^{m} \{\psi_{ij}(0, \cdots, 0, x_r, x_i) \cdot [\varphi_{ij}(x_1, \cdots, x_r) - \varphi_{ij}(0, \cdots, 0, x_r)]\}$$

when $x_r \neq 0$, and $h_i(x_1, \cdots, x_{r-1}, 0, x_j) = x_j$ define a semi-analytic fibration in a $\mathscr{C}^n$-open neighborhood of 0. (An individual fiber is obtained by setting $x_r$ and $x_j$ equal to constants.)

## 4. A theorem on points of $V$ near a submanifold.

Using some of the above ideas (particularly Theorem 2.3) we can answer another question raised by Whitney concerning the structure of $V$ near points of a submanifold $M$ of codimension 1 in $V$. He showed the sheets of $V$ attach smoothly to $M$ (Definition 4.1) off a closed nowhere dense subset of $M$. We prove this nowhere dense may be taken to be an analytic subvariety of $M$.

DEFINITION 4.1. Let 0 be a typical point of $M$ above. Then $V$ *attaches smoothly to* $M$ *near* 0 if there is a $\mathscr{C}^n$-open $U$ of 0 such that:

(a) Representation (3.1) holds;

(b) If $V_i$ is any irreducible component of $V \cap U$, then there is a holomorphic vector field $v^i(\bar{x}) = (0, \cdots, 0, 1, v^i_{r+1}(\bar{x}), \cdots, v^i_n(\bar{x}))$ such that $C(V_i, \bar{x}) = \mathscr{C}_{\bar{x}} \times L(v_i(\bar{x}))$ ($L(v_i(\bar{x}))$ is the 1-subspace of $\mathscr{C}_x$ through $v^i(\bar{x})$);

(c) 0 is a regular point; and given $\varepsilon > 0$, there is a $\delta > 0$ so that if $y \in V_i \cap (\bar{x} + \mathscr{C}_{\bar{\bar{x}}})$ and $0 < |y - \bar{x}| < \delta$, then

$$\operatorname{dist}\left(\frac{y - \bar{x}}{|y - \bar{x}|}, \frac{v^i(\bar{x})}{|v^i(\bar{x})|}\right) < \varepsilon \, .$$

THEOREM 4.2. *Let $M$ be any submanifold of codimension* 1 *in* $V$. *Then there is a proper subvariety $W$ of $M$ so $V$ attaches smoothly to $V$ near point of $M \backslash W$.*

*Proof.* (a) Let $A'$ be the variety of § 2 having dimension $< r - 1$, containing $A$ and the nonregular points; we saw in § 3 that representation (3.1) holds at any point of $M \backslash A$.

(b) To find a proper subvariety of $M$ off which (b) holds, let $\mathscr{B} = \mathscr{A} \cap [M \cap U) \times \mathscr{C}_{\bar{\bar{x}}}]$ (with co-ordinates $\bar{x}, \bar{\bar{x}}$ as before). Assume $\bar{x}, \bar{\bar{x}}$ are such that for each $p \in M \cap U$, the part in $(\mathscr{C}_{\bar{\bar{x}}} + p) \cap U$ of the hyperplane $K$ defined by $x_r = 1$, intersects each of the 1-subspaces of the fiber above $p$ (see (2.1, (iii, $b_2$))). Using standard arguments, one may then show there is a proper subvariety $D$ of $M$ so that if $0 \in M \backslash (D \cup A')$, the part of $\mathscr{B}$ above a small neighborhood in $\mathscr{C}_{\bar{x}, x_r} \cap K$ about the point $\bar{x} = 0$, $x_r = 1$ is representable by holomorphic functions $h_{r+1}(\bar{x}), \cdots, h_n(\bar{x})$. $D$ will be a "discriminant"—the union of

$$M \cap \operatorname{Clos}\,[S(\mathscr{B}) \backslash M \times (0)]$$

($S(\mathscr{B})$ = singular variety of $\mathscr{B}$) and the set of points $q$ of $M$ where $\mathscr{B}$ fails to intersect $(q, \mathscr{C}_{\bar{\bar{x}}})$ transversally. That there is an analytic subvariety of $V$ coinciding with this last set on $V \backslash S(\mathscr{B})$ may be seen by noting that the closures of the set of tangent cones of any variety $V \subseteqq H \subseteqq \mathscr{C}^n$ is analytic in $V \times \mathscr{C}^n$, the fiber above any simple point $p \in V$ being just the tangent space to $V$ at $p$ [3, Th. 5.1]. Hence $D$ is intrinsically defined. Hence (b) holds at each point of $M \backslash (D \cup A')$.

(c) Any point of $M \backslash A'$ is regular; further, one may verify that the proof of the last part of (c) given in [4, § 14] at any point off the dense set considered there, may be used at any point of $M \backslash (D \cup A')$.

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