PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 2, 1970

POWER SERIES SEMIGROUP RINGS

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A general method is given for constructing power series rings with exponents in a semigroup that need not be cancellative. Supports of power series need not be finite unions of inversely well ordered sets.

1. Semigroups.

The objective is to introduce the usual power series multiplication in subsets bigger than the semigroup ring.

NOTATION 1.1. For a semigroup Γ and any ring R consider functions $\alpha: \Gamma \to R$, or alternatively, $\alpha = \sum \alpha(s)s$ with $\alpha(s) \in R$. Define the *support* of α—written as $\operatorname{supp} \alpha$—to be the set

$$\operatorname{supp} = \alpha = \{s \in \Gamma \mid \alpha(s) \neq 0\} .$$

Let $\boldsymbol{P} = \boldsymbol{P}(\Gamma, R) = \boldsymbol{P}(\Gamma)$ denote all α such that $\operatorname{supp} \alpha$ is finite, i.e., the semigroup ring.

1.2. If in addition Γ is a partially ordered set (notation: *po*-set), let $W = W(\Gamma, R) = W(\Gamma)$ be the abelian group of all those α whose support $\operatorname{supp} \alpha$ is the join of a finite number of inversely well ordered sets.

DEFINITION 1.3. A semigroup Γ is a *block semigroup* if there exists a set $\{\Gamma(k) \mid k \in I\}$ of subsets of Γ, called *blocks*, such that $\Gamma = \bigcup \{\Gamma(k) \mid k \in I\}$, where each block is a *po*-set, and the following four conditions are satisfied.

(1) For any index n, there are only a finite number of ordered pairs (i, j) satisfying $\Gamma(i)\Gamma(j) \cap \Gamma(n) \neq \varnothing$. Furthermore, there exists some $\Gamma(k) \supseteqq \Gamma(n)$ such that the order on $\Gamma(n)$ is that induced from $\Gamma(k)$ and

$$\Gamma(i)\Gamma(j) \cap \Gamma(n) \neq \varnothing \Longrightarrow \Gamma(i)\Gamma(j) \subseteqq \Gamma(k) .$$

(2) For any $s, t \in \Gamma(i)$ and $\bar{s}, \bar{t} \in \Gamma(j)$, then

$$s \leqq t, \bar{s} \leqq \bar{t} \Longrightarrow s\bar{s} \leqq t\bar{t}$$

holds in the order on $\Gamma(k)$ for any k such that $\Gamma(i)\Gamma(j) \subseteqq \Gamma(k)$.

(3) If $s, t, u, \bar{s} \in \Gamma(i)$ for some i, then

$$s \leqq t \leqq \bar{s}, s \leqq u \leqq \bar{s} \Longrightarrow t < u \quad \text{or} \quad u \leqq t .$$

(4) Suppose $\{s(1) \geqq s(2) \geqq \cdots \geqq s(n) \geqq \cdots\} \subset \Gamma(i)$,

$$\{t(1) \geqq t(2) \geqq \cdots \geqq t(n) \geqq \cdots\} \subset \Gamma(j)$$

are nonincreasing sequences for some i and j with

$$s(1)t(1) = s(2)t(2) = \cdots = s(n)t(n) = \cdots .$$

Then both sequences are eventually constant.

A block semigroup will be said to be *coherent* if in addition to (1)-(4), also

(5) the union of the orders in the blocks is a partial order on Γ.

Frequently, Γ is a *po*-semigroup to begin with (i.e., $s < t$, $\bar{s} \leqq \bar{t} \Rightarrow s\bar{s} \leqq t\bar{t}$ and $\bar{s}s \leqq \bar{t}t$) and then the blocks are defined as certain subsets, where the order on the blocks is that inherited from Γ. Then the blocks induce a partial semigroup order on Γ that is smaller than the original one.

DEFINITION 1.4. A subset of a *po*-set is a *W-set* if it is a union of a finite number of inversely well ordered sets. If Γ is a block semigroup, then $W \subset \Gamma$ will be called a *partial W-set* if for any k the set $W \cap \Gamma(k)$ is a W-set in the *po*-set $\Gamma(k)$. Let $\boldsymbol{L} = \boldsymbol{L}(\Gamma, R) = \boldsymbol{L}(\Gamma)$ be the group of all α such that $\operatorname{supp} \alpha$ is a partial W-set.

The proof of the next lemma is motivated by [1; p. 76]. Note that condition (iii) below implies that $A \cup B$ and AB are contained inside a finite number of the $\Gamma(k)$.

LEMMA 1.5. *If Γ is a block semigroup and if A and B are any partial W-sets, then:*

(i) $\{(a, b) \in A \times B \mid ab = s\}$ *is finite for each* $s \in \Gamma$;

(ii) AB *is a partial W-set;*

(iii) *If Γ is also coherent and A and B are W-sets, then so is AB.*

Proof. If $(i(p), j(p))$ with $p = 1, \cdots, m$ are the finite number of ordered pairs of indices for which $\Gamma(i(p))\Gamma(j(p)) \cap \Gamma(n) \neq \varnothing$ for some index n, and if $\Gamma(k) \supseteqq \Gamma(n)$ is as in 1.3 (1), then

$$AB \cap \Gamma(n) \subseteqq \bigcup \{[A \cap \Gamma(i(p))][B \cap \Gamma(j(p))] \mid p = 1, \cdots, m\} \subseteqq \Gamma(k) .$$

Thus if one of (i)-(iii) fails for A, B, then it already fails for some sets of the form $A \cap \Gamma(i(p))$, $B \cap \Gamma(j(p))$ contained in single blocks, i.e., we can assume $A \subseteqq \Gamma(i(p))$ and $B \subseteqq \Gamma(j(p))$ for some p. Furthermore, $A = A \cap \Gamma(i(p)) = A(1) \cup \cdots \cup A(r)$ and $B = B \cap \Gamma(j(p)) = B(1) \cup \cdots \cup B(s)$ where each $A(i)$ and $B(j)$ is a chain. Thus $AB =$

$\bigcup \{A(i)B(j) \mid i = 1, \cdots, r;\ j = 1, \cdots, s\}$, and by 1.3 (2) each $A(i)B(j)$ is a totally ordered subset of $\Gamma(k)$. Hence (iii) follows from (ii). Without any loss of generality, it suffices to prove (i) and (ii) for totally ordered sets of the form $A = A(i) \subseteqq \Gamma(i(p))$, $B = B(j) \subseteqq \Gamma(j(p))$. Hence by 1.3 (2) and 1.3 (3) also $AB = (AB) \cap \Gamma(k)$ is totally ordered. Conclusion (i) follows immediately from 1.3 (4); and it is interesting to observe that the latter cancellativity condition is not needed for the remainder of the lemma.

In order to prove (ii), the above shows that $(AB) \cap \Gamma(k)$ is a finite union of chains. Consequently, it suffices to verify that any arbitrary subset $t \subseteqq (AB) \cap \Gamma(k)$ contains a maximal element. For any subset $t \subseteqq AB$ whatever, define $b(t) \in B$ and $a(t) \in A$ by

$$b(t) = \max \{b \in B \mid \exists a \in A,\ ab \in t\}\ ,$$
$$a(t) = \max \{a \in A \mid ab(t) \in t\}\ .$$

Set $t(1) = \{ab \in t \mid ab > a(t)b(t)\}$. If $t(1) = \varnothing$, then $t \leqq a(t)b(t) = \max t$. Otherwise $ab \in t(1) \neq \varnothing$ implies that (1) $b < b(t)$, (2) $a > a(t)$, and (3) $a(t(1))b(t(1)) > a(t)b(t)$.

(1) First, $ab \in t$ implies $b \leqq b(t)$ by the definition of $b(t)$. If $b = b(t)$, then the definition of $a(t)$ and $ab(t) \in t$ imply $a \leqq a(t)$, which gives the contradiction that $ab \leqq a(t)b(t)$. Thus $b < b(t)$.

(2) If $a \leqq a(t)$, then again $b < b(t)$ implies $ab \leqq a(t)b(t)$. Thus $a > a(t)$. Repeat this process, i.e., set $t(0) = t$, define $t(i+1) = \{ab \in t(i) \mid ab > a(t(i))b(t(i))\}$ provided $t(i) \neq \varnothing$, and get

$$\begin{aligned} &t(0) \supset t(1) \supset \cdots && \supset t(n);\\ &a(t(0))b(t(0)) < a(t(1))b(t(1)) < \cdots && < a(t(n))b(t(n));\\ &a(t(0)) < a(t(1)) < \cdots && < a(t(n))\ . \end{aligned}$$

By the a.c.c. in A, for some n,

$$\varnothing = t(n+1) \equiv \{ab \in t \mid ab > a(t(n))b(t(n))\}\ ,$$

and hence $t \leqq a(t(n))b(t(n)) = \max t$.

The next theorem and corollary are consequences of the previous lemma.

THEOREM 1.6. *Suppose that Γ is a block semigroup and R is a ring. Then $L(\Gamma, R)$ is a ring.*

The group $W(\Gamma)$ can only be defined if Γ is a *po*-set.

1.7. COROLLARY 1 TO THE THEOREM. *If Γ is coherent then $W(\Gamma, R)$ is also a ring.*

1.8. COROLLARY 2 TO THE THEOREM. *Suppose Γ is a block semigroup satisfying the following.*

(a) *For $s, t \in \Gamma(i)$; $\bar{s}, \bar{t} \in \Gamma(j)$*

$$s < t,\ \bar{s} \leqq \bar{t} \Longrightarrow s\bar{s} < t\bar{t} \quad \text{and} \quad \bar{s}s < \bar{t}t$$

holds in any block $\Gamma(k)$ whenever $s\bar{s}, t\bar{t} \in \Gamma(k)$ or $\bar{s}s, \bar{t}t \in \Gamma(k)$.

(b) *Each block is totally ordered.*

(c) *For any n, if $i(p)$, $j(p)$ with $p = 1, \cdots, m$ are the finite number of ordered pairs of indices for which*

$$\Gamma(i(p))\Gamma(j(p)) \cap \Gamma(n) \neq \varnothing \text{ , then the } m\text{-subsets}$$
$$\Gamma(i(p))\Gamma(j(p)) \text{ are pairwise disjoint.}$$

Then $\boldsymbol{L}(\Gamma, R)$ is an integral domain.

2. Applications and examples.

NOTATION 2.1. From now on R will be a fixed totally ordered ring and Γ a coherent block semigroup. For any subset $S \subseteqq \Gamma(k)$ of a block $\Gamma(k)$, $\max S$ denotes the set of maximal elements of S. If $\alpha \in \boldsymbol{L}(\Gamma, R) = \boldsymbol{L}(\Gamma) = \boldsymbol{L}$, an element $s \in \max\,[\operatorname{supp} \alpha \cap \Gamma(k)]$ will be called a *maximal component* of α. Note that if each $\Gamma(k)$ is totally ordered, that then there is only one such s for each block.

DEFINITION 2.2. For $\alpha \in \boldsymbol{L}$, define $0 \leqq \alpha$ provided $0 \leqq \alpha(s) \in R$ for each maximal component s of α.

PROPOSITION 2.3. *In the above order, $\boldsymbol{L}(\Gamma, R)$ and $\boldsymbol{W}(\Gamma, R)$ are partially ordered rings.*

In order to obtain examples of partially ordered rings, some coherent block semigroups are described.

EXAMPLE 2.4. Consider the noncancellative semigroup

$$\Gamma = \{2^{-k} \mid k = 0, 1, 2, \cdots\} \subset [0, 1]$$

with $st = s \wedge t = \min(s, t)$ for $s, t \in \Gamma$. Take Γ in the natural order as a single block.

2.5. Let $\boldsymbol{Z} = \{0, \pm 1, \pm 2, \cdots\}$ and $\boldsymbol{N} = \{0, 1, 2, \cdots\}$ with the usual order while $\bar{\boldsymbol{N}}$ denotes the discrete order. Let $\Gamma = \bar{\boldsymbol{N}} \times \bar{\boldsymbol{N}} \times \boldsymbol{Z}$ under componentwise addition where $(i, j, k) > (p, q, r)$ if and only if $i = p$, $j = q$, but $k > r$. Let the blocks be $\Gamma(i, j) = \{i\} \times \{j\} \times \boldsymbol{Z}$. Then $\boldsymbol{W}(\Gamma) \subset \boldsymbol{L}(\Gamma)$ properly.

2.6. Take $G = \bar{N} \times (N \times Z)$ with $(i, j, k) > (p, q, r)$ if either $i = p$, $j > q$; or if $i = p$, $j = q$, but $k > r$. The blocks are

$$G(i) = \{(i, j, k) \mid j, k \in Z\} ,$$

and again $W(G) \neq L(G)$.

2.7. As a set, take $H = N \times N \times Z$ but where $(i, j, k) > (p, q, r)$ provided one of the following four cases holds:

(i) $i > p$;

(ii) $i = p \geqq 1$, $j > q$;

(iii) $i = p \geqq 1$, $j = q$, $k > r$;

(iv) $i = p = 0$, $j = q$, $k > r$.

Note that $(N\backslash\{0\}) \times N \times Z$ is totally ordered and dominates $\{0\} \times \bar{N} \times Z$. The blocks are $H(k) = \{(p, q, r) \mid p, q \in N, r \in Z; p \leqq k\}$. Here $L(H) \neq W(H)$, and $W(H)$ is a lattice ordered ring.

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Received April 3, 1969, and in revised form December 13, 1969.

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