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SETS WHICH CAN BE MISSED BY SIDE APPROXIMATIONS TO SPHERES

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In this paper it is proved that a 2-sphere S in E^3 can be side approximated from Int S so as to miss a closed subset X of S if and only if $(S \cup \text{Int } S) - X$ is 1-ULC.

If $X = \{p\}$ is a singleton, then the 1-ULC property described in the first paragraph reduces to a property studied by McMillan [22] in connection with piercing points p of a crumpled cube (see (0.2) below). Our work was motivated by his paper although our methods bear little relationship to his.

Our result has as immediate corollaries a number of important, well-known theorems, four of which we now mention:

(0.1) ([2. Th. 2]). A 2-sphere S in E^3 is tame if $E^3 - S$ is 1-ULC.

(0.2) ([22, Th. 1]). If C is a crumpled cube in E^3 and $p \in \text{Bd } C$, then p is a piercing point of C if and only if $C - p$ is 1-ULC.

(0.3) ([19, Th. 3]). If S and S' are 2-spheres in E^3 and $X \subset U \subset S \cap S'$, where X is compact and U is open in S and S'. Then $(*, X, S')$ is satisfied if and only if $(*, X, S)$ is satisfied.

(0.4) ([8] and [10]). If S is a 2-sphere in E^3, X is a compact subset of S, and $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is compact and satisfies $(*, X_i, S)$, then $(*, X, S)$ is satisfied.

Properties similar to the 1-ULC property described in the first paragraph have been studied previously (e.g., properties (A, F, S) and (B, F, S) in [19]). However, the properties studied have, in general, had only restricted application to the question of determining which subsets of a 2-sphere can be missed by side approximations to that sphere and thus have been inapplicable in results such as (0.2), (0.3), and (0.4).

Precise definitions appear in § 1. Our main result, mentioned in the first paragraph, is proved in § 2. We also present in § 2 a slightly strengthened version of our main result for use in another paper (Lemma 2.5 and Theorem 2.6). The four corollaries mentioned above are proved in § 3. In § 3 we also remark on ways in which our main result can be used to sharpen other results which have appeared in the literature.

The part of this paper actually necessary for the proof of Theorem (0.1) appears in five short paragraphs. This proof is considerably shorter and conceptually easier to follow than the original proof. Our proof of Theorem (0.3) is likewise much shorter than the original. The

great simplifying factor is the use of Lister's form [18, Th. 3] of Bing's Side Approximation Theorem; Lister's theorem can be used to simplify the proof of many other results on the embeddings of surfaces in 3-manifolds (cf. [12]). Except for Lister's theorem, the tools we have chosen to use are essentially those which were used in the original proofs of Theorems (0.1)-(0.4). Lemma 2.5 and Theorem 2.6 are slightly more complex than the other results.

We had not originally planned to restrict ourselves in this paper to the applications of our main theorem which appear in § 3 (cf. [9, Ths. 2 and 3]), but we have found that our other applications fit nicely into more general settings which we shall study elsewhere (cf. [10] and [11]).

We thank the referee for useful comments and C. E. Burgess for discussions.

1. Property $(*, X, S)$, definitions, and notation. Suppose that X is a closed subset of a 2-sphere S in E^3. We say that S can be side approximated so as to miss X if the following property is satisfied:

Property $(*, X, S)$. ([15, p. 467]). If U is a component of $E^3 - S$ and ε is a positive number, then there are a polyhedral 2-sphere S', a finite collection $D_1, D_2, \cdots, D_k$ of mutually exclusive ε-disks lying in S', and a finite collection $E_1, E_2, \cdots, E_n$ of mutually exclusive ε-disks lying in S, such that

(1) There is an ε-homeomorphism from S onto S',
(2) $S' - \bigcup_{i=1}^{k} D_i \subset U$,
(3) $S \cap S' \subset \bigcup_{i=1}^{n} E_i$, and
(4) $X \cap (\bigcup_{i=1}^{n} E_i) = \varnothing$.

If $(*, X, S)$ is satisfied, then X is tame (*i*.e., X lies on a tame 2-sphere in E^3) [19, Th. 6]. If X has no degenerate components, then X is tame if and only if $(*, X, S)$ is satisfied [8, Ths. 1.1 and 5.1].

If the property defined above holds for one component U of $E^3 - S$ though perhaps not for the other, we say that $(*, X, U)$ is satisfied. If $(*, X, \operatorname{Int} S)$ is satisfied, then X lies on the surface of a 3-cell in E^3 [21, Th. 1]. Our goal is to characterize sets satisfying $(*, X, \operatorname{Int} S)$ as those compact sets X for which $(S \cup \operatorname{Int} S) - X$ is 1-ULC. A similar result will obviously hold for $U = \operatorname{Ext} S$.

REMARK. The definition that we have given of $(*, X, S)$ is the original formulation and not always the easiest to apply. If we wish to prove that $(*, X, \operatorname{Int} S)$ is satisfied, we shall use the definition we have stated. If we already know that $(*, X, \operatorname{Int} S)$ is satisfied and

wish to apply that fact, we shall use the following formulation due to Lister. Lister proves [18, Th. 2] that the two definitions are equivalent.

Property $(*, X, \operatorname{Int} S)$ (alternative definition). If $\varepsilon > 0$, then there are disjoint ε-disks $D_1, D_2, \cdots, D_n$ in $S - X$ and an ε-homeomorphism h from S onto a polyhedral sphere $h(S)$ in E^3 such that

(1) $h(S) - \bigcup_{i=1}^{n} \operatorname{Int} h(D_i) \subset \operatorname{Int} S$,

(2) $S - \bigcup_{i=1}^{n} \operatorname{Int} D_i \subset \operatorname{Ext} h(S)$, and

(3) $S \cap h(S) = \bigcup_{i=1}^{n} [\operatorname{Int} D_i \cap \operatorname{Int} h(D_i)]$.

REMARK. Property $(*, \varnothing, \operatorname{Int} S)$ is always satisfied ([4, Th. 16'] (original formulation) and [18, Th. 2] (alternative definition)).

We shall use the symbol ρ for the Euclidean metric. If A and A' are subsets of E^3 and ε is a positive number, then we define

(1) $\operatorname{Diam} A = \sup \rho(x, y)$ $(x, y \in A)$,

(2) $\rho(A, A') = \inf \rho(a, a')$ $(a \in A, a' \in A')$, and

(3) $B(A, A'; \varepsilon) = \{a' \in A' \mid \rho(a', A) < \varepsilon\}$. (If $A' = E^3$, we write $B(A, A'; \varepsilon) = B(A; \varepsilon)$.)

The set A is an ε-set if $\operatorname{Diam} A < \varepsilon$. A map $f: A \to E^3$ is called an ε-map if f moves no point as far as ε. Let Δ and I denote a standard disk or 2-simplex and a standard interval or 1-simplex, respectively. We say that A is 1-LC (0-LC) in A' at $x \in E^3$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that any map $f: \operatorname{Bd} \Delta \to B(x, A; \delta)$ $(f: \operatorname{Bd} I \to B(x, A; \delta))$ has a continuous extension $f^*: \Delta \to B(x, A'; \varepsilon)$ $(f^*: I \to B(x, A'; \varepsilon))$. Speaking loosely, A is 1-LC in A' at $x \in E^3$ if loops near x in A can be shrunk to a point near x in A'. We say that A is 1-ULC (0-ULC) in A' if A is 1-LC (0-LC) in A' at each point x of $\operatorname{Cl} A$ and if for each $\varepsilon > 0$ the corresponding δ may be chosen independently of $x \in \operatorname{Cl} A$. If A is 1-ULC in A, then we simply say that A is 1-ULC. We say that A is *weakly* 1-ULC in A' if (speaking loosely again) small *unknotted* simple closed curves in A bound small singular disks in A'. (A simple closed curve in E^3 is said to be unknotted if it bounds a disk in E^3.) We leave it to the reader to formalize the definition of "weakly 1-ULC".

A crumpled cube C is the union of a 2-sphere in E^3 and its interior in E^3. The 2-sphere is called the boundary of C, denoted $\operatorname{Bd} C$; and $C - \operatorname{Bd} C$ is called the interior of C, denoted $\operatorname{Int} C$. Any homeomorphic image of C is called a crumpled cube. A point $p \in \operatorname{Bd} C$ is said to be a piercing point of C if there is an embedding $h: C \to E^3$ such that $h(\operatorname{Bd} C)$ can be pierced by a tame arc at $h(p)$.

Most of our terminology not defined explicitly in this section is fairly standard and can be found in most of the references listed.

See, for example, [2], [7], and [19].

2. A 1-ULC characterization of sets satisfying $(*, X, \operatorname{Int} S)$. Our characterization will appear in two parts which we prove separately (Ths. 2.1 and 2.4).

THEOREM 2.1. *If S is a 2-sphere in E^3, X is a closed subset of S, and $(*, X, \operatorname{Int} S)$ is satisfied, then $(S \cup \operatorname{Int} S) - X$ is* 1-ULC.

Proof. Let ε be a positive number. There is a δ in the range $0 < \delta < \varepsilon/6$ having the following property: if S' is a 2-sphere in E^3 such that there is a δ-homeomorphism from S onto S', then any δ-set on S' lies on an $\varepsilon/6$-disk on S'. Let $f: \operatorname{Bd} \Delta \to (S \cup \operatorname{Int} S) - X$ be a map into a δ-set in $(S \cup \operatorname{Int} S) - X$. In order to show that $(S \cup \operatorname{Int} S) - X$ is 1-ULC it suffices to show that f is nullhomotopic ($f \sim 0$) in an ε-subset of $(S \cup \operatorname{Int} S) - X$.

Because $\operatorname{Int} S$ is 0-ULC [24, p. 66] and S is an absolute neighborhood retract, there is a map $F: \operatorname{Bd} \Delta \times I \to (S \cup \operatorname{Int} S) - X$ such that $F_0 = f$, $F_1: \operatorname{Bd} \Delta \to \operatorname{Int} S$, and $\operatorname{Diam} F(\operatorname{Bd} \Delta \times I) < \delta$ (where $F_t: \operatorname{Bd} \Delta \to E^3$ is defined by $F_t(x) = F(x, t)$). The map F_1 can be extended to a map $G: \Delta \to E^3$ such that $\operatorname{Diam} G(\Delta) < \delta$.

Since $(*, X, \operatorname{Int} S)$ is satisfied, there are disjoint δ-disks $D_1, D_2, \cdots, D_n$ in $S - X$ and a δ-homeomorphism h from S onto a polyhedral 2-sphere $h(S)$ in E^3 such that (1), (2), and (3) of the alternative definition for $(*, X, \operatorname{Int} S)$ are satisfied and such that $h(S)$ separates $G(\operatorname{Bd} \Delta) = F_1(\operatorname{Bd} \Delta)$ from X.

The intersection of $G(\Delta)$ with $h(S)$ is a δ-set. Hence by our choice of δ, there is an $\varepsilon/6$-disk E on $h(S)$ which contains $G(\Delta) \cap h(S)$. By the Tietze extension theorem (see [7, Lemma 1] for details), there is a map $G_1: \Delta \to E^3$ which agrees with G on the component K of

$$\Delta - G^{-1}[G(\Delta) \cap h(S)]$$

which contains $\operatorname{Bd} \Delta$ and takes $\Delta - K$ into E. Then

$$G_1(\Delta) \cap S \subset \bigcup_{i=1}^{n} D_i .$$

Again by the Tietze theorem there is a map $G_2: \Delta \to E^3$ which agrees with G_1 on the component K' of $\Delta - G_1^{-1}[G_1(\Delta) \cap \bigcup_{i=1}^{n} D_i]$ which contains $\operatorname{Bd} \Delta$ and takes $\Delta - K'$ into $\bigcup_{i=1}^{n} D_i$.

Clearly $f \sim 0$ in $F(\operatorname{Bd} \Delta \times I) \cup G_2(\Delta)$. Thus it suffices to show that $F(\operatorname{Bd} \Delta \times I) \cup G_2(\Delta) \subset (S \subset \operatorname{Int} S) - X$ and that

$$\operatorname{Diam} [F(\operatorname{Bd} \Delta \times I) \cup G_2(\Delta)] < \varepsilon .$$

That these conditions are satisfied is easily checked. This completes the proof of Theorem 2.1.

The second half of our characterization requires two lemmas. The first, Lemma 2.2, is a special case of [22, Lemma 2], which in turn is a special case of [14, Th. 2]. We indicate a proof for Lemma 2.2 since it is a simple special case of the more difficult general results of [14]. Lemma 2.3, a consequence of Lemma 2.2, could be generalized considerably by appeal to these more general results. This is interesting when one notes that Theorem (0.4), a conjecture by Loveland, is simply the translation of Lemma 2.3, via Theorems 2.1 and 2.4, into a statement about side approximations of 2-spheres.

LEMMA 2.2. *Let B denote a closed subset of a disk D, let S be a 2-sphere in E^3, and let X be a subset of S such that $(S \cup \operatorname{Int} S) - X$ is* 1-ULC. *Let $\varepsilon > 0$ and a map $f: D \to S \cup \operatorname{Int} S$ be given. Then there is a map $f^*: D \to S \cup \operatorname{Int} S$ such that*

$$f^*(D - B) \subset (S \cup \operatorname{Int} S) - X,$$

$$f^* | B = f | B, \text{ and}$$

$$\rho(f^*(x), f(x)) < \varepsilon \text{ for each } x \in D.$$

Indication of proof. One triangulates $D - B$ so that the mesh of the triangulation approaches zero very rapidly near B. Then using the facts that $\operatorname{Cl}((S \cup \operatorname{Int} S) - X) = (S \cup \operatorname{Int} S)$, $(S \cup \operatorname{Int} S) - X$ is 0-ULC [24, p. 66], and $(S \cup \operatorname{Int} S) - X$ is 1-ULC, one moves the image of the 0-, 1-, and 2-skeletons of the triangulation away from X. Note that we do not require that X be closed.

LEMMA 2.3. *Let S be a 2-sphere in E^3, and let $\{X_i\}_{i=1}^{\infty}$ be a family of closed subsets of S such that, for each i, $(S \cup \operatorname{Int} S) - X_i$ is* 1-ULC. *Then $(S \cup \operatorname{Int} S) - \bigcup_{i=1}^{\infty} X_i$ is* 1-ULC.

Proof. Suppose $\varepsilon > 0$ and $x \in S \cup \operatorname{Int} S$ given, and let Δ denote a disk. Since $(S \cup \operatorname{Int} S) - X_1$ is 1-ULC, there is a $\delta > 0$ such that each map $f: \operatorname{Bd} \Delta \to B(x, (S \cup \operatorname{Int} S) - X_1; \delta)$ extends to a map

$$f^*: \Delta \to B(x, (S \cup \operatorname{Int} S) - X_1; \varepsilon) .$$

Let $f: \operatorname{Bd} \Delta \to B(x, (S \cup \operatorname{Int} S) - \bigcup_{i=1}^{\infty} X_i; \delta)$ be given, and let

$$f_1: \Delta \to B(x, (S \cup \operatorname{Int} S) - X_1; \varepsilon)$$

be an extension of f. We now define positive numbers $\varepsilon_1, \varepsilon_2, \cdots$ and maps $f_2, f_3, \cdots$ inductively as follows:

Let

$$0 < \varepsilon_1 < 1/2\, \rho[f_1(\Delta), X_1 \cup ((S \cup \operatorname{Int} S) - B(x, (S \cup \operatorname{Int} S); \varepsilon))].$$

Let $f_2\colon \Delta \to (S \cup \operatorname{Int} S) - X_2$ be such that $f_2 \mid \operatorname{Bd} \Delta = f_1 \mid \operatorname{Bd} \Delta$ and

$$\rho(f_1(p), f_2(p)) < \varepsilon_1$$

for each $p \in \Delta$. Such a map exists by Lemma 2.2.

For $i > 1$, let ε_i be in the range $0 < \varepsilon_i < 1/2 \min \{\varepsilon_{i-1}, \rho[f_i(\Delta), X_i]\}$. Let $f_{i+1}\colon \Delta \to (S \cup \operatorname{Int} S) - X_{i+1}$ be such that $f_{i+1} \mid \operatorname{Bd} \Delta = f_i \mid \operatorname{Bd} \Delta$ and $\rho(f_i(p), f_{i+1}(p)) < \varepsilon_i$ for each $p \in \Delta$. Again, at each stage such a map exists by Lemma 2.2.

Because $S \cup \operatorname{Int} S$ is a complete space, it follows that $f^* = \lim f_i$ exists, is continuous, extends f, and takes Δ into

$$B\Big(x, (S \cup \operatorname{Int} S) - \bigcup_{i=1}^{\infty} X_i; \varepsilon\Big).$$

Since x is arbitrary and δ may be chosen independently of x, $(S \cup \operatorname{Int} S) - \bigcup_{i=1}^{\infty} X_i$ is 1-ULC.

THEOREM 2.4. *If S is a 2-sphere in E^3, X is a closed subset of S, and $(S \cup \operatorname{Int} S) - X$ is* 1-ULC, *then* $(*, X, \operatorname{Int} S)$ *is satisfied.*

Proof. We first establish the theorem for the special case $X = S$ and then show how the proof can be altered to yield the general result. There are four steps in the special case.

1. There are disjoint disks $D_1, D_2, \cdots, D_n$ in S and an ε-homeomorphism h from S onto a polyhedral sphere $h(S)$ in E^3 such that (1), (2), and (3) of the alternative definition of $(*, \varnothing, \operatorname{Int} S)$ are satisfied and such that $\operatorname{Diam} [D_i \cup h(D_i)] < \varepsilon$ for each i. This is because $(*, \varnothing, \operatorname{Int} S)$ is always satisfied.

2. By the Tietze Extension Theorem, there are maps $f_1, f_2, \cdots, f_n$ such that for each i, $f\colon D_i \to h(D_i) \cup \operatorname{Int} D_i$, f_i agrees with h on the component K_i of $D_i - h^{-1}[h(D_i) \cap D_i]$ which contains $\operatorname{Bd} D_i$, and f_i takes $D_i - K_i$ into $\operatorname{Int} D_i$.

3. By Lemma 2.2, since $\operatorname{Int} S$ is 1-ULC, if $\delta > 0$, then there is a map $g_i\colon D_i \to f_i(K_i) \cup B(D_i, \operatorname{Int} S; \delta)$ which agrees with f_i on

$$K_i - f_i^{-1}[B(D_i, f_i(D); \delta)].$$

Thus we may assume that $g_1(D_1), g_2(D_2), \cdots, g_n(D_n)$ are disjoint singular disks in $\operatorname{Int} S$ such that for each i, (1) $g_i(D_i)$ has no singularities near $g_i(\operatorname{Bd} D_i)$, (2) $\operatorname{Diam} [g_i(D_i) \cup D_i] < \varepsilon$, and (3) $g_i(D_i) \cap h[S - \bigcup_{i=1}^{n} \operatorname{Int} D_i] = g_i(\operatorname{Bd} D_i)$.

4. By Dehn's Lemma [23], each singular disk $g_i(D_i)$ may be replaced by a polyhedral disk E_i in $\operatorname{Int} S$ such that

$$h\Big[S - \bigcup_{i=1}^{n} \operatorname{Int} D_i\Big] \cup \bigcup_{i=1}^{n} E_i$$

is a polyhedral 2-sphere S' in $\operatorname{Int} S$ and for each i, $\operatorname{Diam}[E_i \cup D_i] < \varepsilon$. If h' is any homeomorphism from S onto S' which extends

$$h \mid S - \bigcup_{i=1}^{n} \operatorname{Int} D_i ,$$

then h' is an ε-homeomorphism from S into $\operatorname{Int} S$. Since ε is arbitrary, this shows that $(*, S, \operatorname{Int} S)$ is satisfied.

In the general case, we make the following changes:

3′. Let $\{X_i\}_{i=1}^{\infty}$ be a family of tame finite graphs on S such that $S - \bigcup_{i=1}^{\infty} X_i$ is totally disconnected. That such a family exists follows from [3, Th. 1]. If A is an arc in any X_i, then $(*, A, \operatorname{Int} S)$ is satisfied by [15, Th. 2]. Thus by Theorem 2.1, $(S \cup \operatorname{Int} S) - A$ is 1-ULC; and by Lemma 2.3, $(S \cup \operatorname{Int} S) - (\bigcup_{i=1}^{\infty} X_i \cup X)$ is 1-ULC since $\bigcup_{i=1}^{\infty} X_i \cup X$ is a countable union of closed sets A for which $(S \cup \operatorname{Int} S) - A$ is 1-ULC. Hence by Lemma 2.2, if $\delta > 0$, there is a map $g_i\colon D_i \to f_i(K_i) \cup B(D_i, (S \cup \operatorname{Int} S) - (\bigcup_{i=1}^{\infty} X_i \cup X); \delta)$ which agrees with f_i on $K_i - f_i^{-1}[B(D_i, f_i(D_i); \delta)]$. Thus we may assume that

$$g_1(D_1), g_2(D_2), \cdots, g_n(D_n)$$

are disjoint singular disks in $(S \cup \operatorname{Int} S) - X$ which satisfy (1), (2), and (3) of Step 3 above and such that for each i, $g_i(D_i) \cap S$ is a 0-dimensional subset of $\operatorname{Int} D_i$.

4′. The set $g_i(D_i) \cap S$, being a closed 0-dimensional subset of $\operatorname{Int} D_i - X$, can be covered by a finite collection $E_{i1}, E_{i2}, \cdots, E_{in_i}$ of disjoint disks in $\operatorname{Int} D_i - X$. Again by Dehn's Lemma, each singular disk $g_i(D_i)$ may be replaced by a polyhedral disk E_i such that

$$h\Big[S - \bigcup_{i=1}^{n} \operatorname{Int} D_i\Big] \cup \bigcup_{i=1}^{n} E_i$$

is a polyhedral 2-sphere S' in E^3 and for each i, $E_i \cap S \subset \bigcup_{j=1}^{n_i} E_{ij}$ and $\operatorname{Diam}[E_i \cup D_i] < \varepsilon$. But since ε is arbitrary, S' and the disks $\{E_i\}$ and $\{E_{ij}\}$ are precisely what one needs in order to show that $(*, X, \operatorname{Int} S)$ is satisfied, where in this case we apply the original definition.

We now show that the hypotheses of Theorem 2.4 can be weakened. Our goal is Theorem 2.6. In the proof of Theorem 2.6 we need to know that a polyhedral disk in E^3 whose boundary lies in the interior of a particular 2-sphere in E^3 (possibly a wild 2-sphere in E^3) can, in a particularly nice way, almost be cut off inside the 2-sphere. This fact, stated precisely in Lemma 2.5, will be apparent to those thoroughly familiar with the methods of [5]. However, following the suggestion of the referee, we give a proof which does not rely on the methods of

[5] and depends only on Bing's Side Approximation Theorem [4]. The brief outline with which the proof of Lemma 2.5 begins is due to the referee and had also been suggested to the author by C. E. Burgess.

LEMMA 2.5. *Suppose that*
S is a 2-sphere in E^3;
U is a component of $E^3 - S$;
D is a polyhedral disk in E^3 with $\operatorname{Bd} D \subset U$;
R is a disk in S with $D \cap S \subset R$;
$0 < \varepsilon < \rho(\operatorname{Bd} D, S)$; and
K is the the component of $D - S$ which contains $\operatorname{Bd} D$.
Then there is a map $h: D \to (D \cap U) \cup B(R, \operatorname{Cl} U; \delta)$ such that

$$h|K - B(R; \varepsilon) = \text{identity}$$
$$h(D) \cap S \text{ is 0-dimensional; and}$$
$$h|D - h^{-1}(S) \text{ is a homeomorphism.}$$

Proof. The plan is the following: Let S_1 be a side approximation to S from U sufficiently close to S. Cut $D_0 = D$ off on S_1 to get a new polyhedral disk D_1. Again let S_2 be a side approximation to S from U sufficiently close to S (much closer than S_1). Cut D_1 off on S_2 to obtain a new polyhedral disk D_2. Repeat this procedure. If proper care is taken in the process, then $h(D)$ may be taken to be $\lim D_i$. In the next few paragraphs we explain in turn how each of the conclusions of the lemma can be obtained by this procedure.

Associating a continuous function h with $\lim D_i$. Cutting D_{i-1} off on S_i can be realized as a process of removing from D_{i-1} finitely many disjoint disks ${}^i\Delta_1, \cdots, {}^i\Delta_r$ in $\operatorname{Int} D_{i-1}$ and replacing them by disjoint disks ${}^i\Delta'_1, \cdots, {}^i\Delta'_r$ near S_i. Thus there is a natural, though nonunique, way of defining a homeomorphism $h_i: D_{i-1} \to D_i$: h_i differs from the identity only in that it takes a disk ${}^i\Delta_j$ removed in the cutting process to its replacement ${}^i\Delta'_j$ near S_i. A function

$$h = \lim h_i \cdots h_2 h_1: D \to \lim D_i$$

will therefore exist and be continuous provided the size of the disks removed and the size of their replacements converge to 0 rapidly enough with i. Suppose that to this end we want the disks $\{{}^{i+1}\Delta_j\}$ and $\{{}^{i+1}\Delta'_j\}$ to have diameter less than α. This can be accomplished by using caution in the adjustments which change D_{i-1} to D_i. The procedure is tedious but straightforward. The reader may wish to omit a reading of the details which follow:

Choose β, $0 < \beta < \alpha$, such that β-sets in a β-approximation S' to S lie in α-disks in S'.

Choose γ, $0 < \gamma < \beta/3$, such that γ-sets in a γ-approximation S'' to S lie in $\beta/3$-disks in S''.

Choose δ, $0 < \delta < \gamma$, such that δ-sets in a δ-approximation D'_{i-1} to D_{i-1} lie in γ-disks in D'_{i-1}.

Choose a δ-side approximation S_i to S from U with associated δ-disks $E_1, \cdots, E_n$ on S_i and $F_1, \cdots, F_m$ on S containing the intersections of S_i with S.

Adjust D_{i-1} slightly near S_i (by a δ-adjustment) so that the adjusted D_{i-1} misses $\cup E_j$ and is in general position with respect to S_i. Call the adjusted D_{i-1} by the same name as the original.

Cut D_{i-1} off on S_i to form D_i. This cutting removes disjoint disks $\{{}^i\Delta_j\}$ from $\operatorname{Int} D_{i-1}$ and replaces them by disks $\{{}^i\Delta'_j\}$ near S_i.

We now show that $D_i \cap S$ can be covered by a finite collection of disjoint β-disks in $\operatorname{Int} D_i$. Since $[D_{i-1}\text{ (adjusted)}] \cap (\cup E_j) = \varnothing$, the disk-with-holes $K_i = \operatorname{cl}(D_{i-1} - \cup {}^i\Delta_j)$ intersects S only in $\operatorname{Int} K_i$ and if care has been taken, also only in $\cup F_j$. From this latter fact, each component of $K_i \cap S$ has diameter less than δ. Hence by our choice of δ, $K_i \cap S$ can be covered by a finite collection of disjoint γ-disks in D_{i-1}, each of which has boundary in $\operatorname{Int} K_i$. The image of each such disk under $h_i\colon D_{i-1} \to D_i$ has diameter less than

$$\gamma + 2\beta/3 < \beta\ .$$

A disk ${}^i\Delta'_j$ is either contained in one of these β-disks or intersects none of them. In the latter case, note that ${}^i\Delta'_j$ is very close homeomorphically to some disk H in S_i such that $\operatorname{Bd} H \cap (\cup E_k) = \varnothing$. If ${}^i\Delta'_j$ is close enough to H, then ${}^i\Delta'_j \cap S$ is covered in a natural way by β-disks in $\operatorname{Int} {}^i\Delta'_j$ associated with the δ-disks $\{E_k \mid E_k \subset \operatorname{Int} H\}$. We thus obtain finally a finite collection of disjoint β-disks in $\operatorname{Int} D_i$ covering $D_i \cap S$.

If S_{i+1} is a β-approximation to S that intersects D_i only in the finite union of these disjoint β-disks, then the disks $\{{}^{i+1}\Delta_j\}$ will be of diameter less than $\beta(<\alpha)$ and the disks $\{{}^{i+1}\Delta'_j\}$ of diameter less than α by our choice of β.

These considerations show that the approximations S_i and disks D_i may be chosen so that $h = \lim h_i \cdots h_2 h_1$ automatically exists and is continuous.

Making $h \mid K - B(R;\varepsilon)$ *the identity.* One simply needs to keep all the spheres S_i away from some continuum in K which contains $K - B(R;\varepsilon)$.

Keeping $h(D)$ *in* $(D \cap U) \cup B(R, \operatorname{Cl} U;\varepsilon)$. Note that

$$h(D) = \lim D_i \subset \lim (S_i \cup U(\mathrm{S}_i)) = \operatorname{Cl} U$$

(where $U(\mathrm{S}_i)$ is the complementary domain of S_i which "almost coin-

cides" with U). Then if one keeps all of the disks $\{{}^i\Delta'_j\}$ in $B(R, \varepsilon/2)$, which one may clearly do, it follows immediately that

$$h(D) \subset (D \cap U) \cup B(R; \varepsilon) .$$

Making $h(D) \cap S$ 0-dimensional. As we showed in the first part of this proof, by being careful at stage i we can require at stage $i+1$ that $D_i \cap S$ be covered by a collection of disjoint α-disks in S. One may then require that all of the disks $\{{}^k\Delta'_j\}$ $(k \geqq i+1)$ lie in an arbitrary neighborhood of the union of these α-disks. It follows then that $h(D) \cap S$ also lies very close to the union of these α-disks, hence has no component of diameter much larger than α. If our α's converge to 0, then $h(D) \cap S$ will be 0-dimensional.

Making $h \mid D - h^{-1}(S)$ a homeomorphism. This is automatic since each h_i is a homeomorphism and each point mapped into U has a neighborhood moved by only finitely many of the maps h_i.

THEOREM 2.6. *If S is a 2-sphere in E^3, X is a closed subset of S, and* $\operatorname{Int} S$ *is weakly* 1-ULC *in* $(S \cup \operatorname{Int} S) - X$, *then* $(S \cup \operatorname{Int} S) - X$ *is* 1-ULC.

Proof. By Theorem 2.1, it suffices to show that $(*, X, \operatorname{Int} S)$ is satisfied. The proof is like that of the general case of Theorem 2.4 except that Lemma 2.5 allows us to alter the second step so that our weaker hypothesis is applicable in the third step. The first and fourth steps remain unchanged.

2″. There are subdisks $E_1, E_2, \cdots, E_n$ of $D_1, D_2, \cdots, D_n$ respectively such that for each i, $h(D_i) \cap D_i \subset E_i \subset \operatorname{Int} D_i$. Choose $\alpha > 0$ such that $\alpha < 1/3\rho[E_i, (S - D_i) \cup h(S - D_i)]$ and such that

$$\operatorname{Diam}[h(D_i) \cup B(D_i; \alpha)] < \varepsilon$$

for each i. By Lemma 2.5, there are maps $f_1, f_2, \cdots, f_n$ such that for each i, f_i: $D_i \to h(D_i) \cup B(E_i, S \cup \operatorname{Int} S; \alpha)$, f_i agrees with h on the component of $D_i - h^{-1}[B(D_i, S \cup \operatorname{Int} S; \alpha)]$ which contains $\operatorname{Bd} D_i$, $f_i(D_i) \cap S$ is a 0-dimensional subset of $\operatorname{Int} D_i$, and $f_i \mid D_i - f_i^{-1}(S)$ is a homeomorphism. Define f: $S \to E^3$ piecewise by $f \mid S - \bigcup_{i=1}^n D_i = h \mid S - \bigcup_{i=1}^n D_i$ and $f \mid D_i = f_i$.

3″. Choose β such that

$$0 < \beta < 1/3\rho(f(D_i) \cap D_i, (S - D_i) \cup f(S - D_i))$$

and such that $\operatorname{Diam}[f(D_i) \cup B(D_i; \beta)] < \varepsilon$ for each i. Choose γ, using

the hypothesis that $\operatorname{Int} S$ is weakly 1-ULC in $(S \cup \operatorname{Int} S) - X$, such that each unknotted γ-simple closed curve in $\operatorname{Int} S$ bounds a singular $\beta/2$-disk in $(S \cup \operatorname{Int} S) - X$. We require that $0 < \gamma < \beta/2$. Let K_i denote the component of $D_i - f^{-1}(S)$ which contains $\operatorname{Bd} D_i$. Since $f(D_i) \cap S$ is 0-dimensional, there is a finite collection $J_1, J_2, \cdots, J_k$ of disjoint simple closed curves in K_i such that $\bigcup_{j=1}^{k} J_j$ separates $\operatorname{Bd} D_i$ from $f^{-1}(S)$ in D_i and such that each $f(J_j)$ is a γ-loop in a γ-neighborhood of the set $f(D_i) \cap D_i$. We assume that no proper subcollection of $J_1, J_2, \cdots, J_k$ separates $\operatorname{Bd} D_i$ from $f^{-1}(S)$ in D_i. Because $f | D_i - f^{-1}(S)$ is a homeomorphism and $K_i \subset D_i - f^{-1}(S)$, it follows from Dehn's Lemma [23] that each $f(J_j)$ is an unknotted γ-simple closed curve in $\operatorname{Int} S$. Hence by the way γ was chosen, there is a map $g_i: D_i \to (S \cup \operatorname{Int} S) - X$ such that g_i agrees with f on that component of $K_i - \bigcup_{j=1}^{k} J_j$ which contains $\operatorname{Bd} D_i$ and such that g_i takes the disk in D_i bounded by J_j into a $\beta/2$-subset of $(S \cup \operatorname{Int} S) - X$. Define $g: S \to E^3$ piecewise by $g | S - \bigcup_{i=1}^{n} D_i = f | S - \bigcup_{i=1}^{n} D_i$ and $g | D_i = g_i$.

Now proceed as in Step 3′ to choose $\{X_i\}_{i=1}^{\infty}$. Apply Lemma 2.3 only to the set $(S \cup \operatorname{Int} S) - \bigcup_{i=1}^{\infty} X_i$ to find this set to be 1-ULC. Then relative to some fixed $\delta > 0$, choose $h_i: D_i \to (S \cup \operatorname{Int} S) - \bigcup_{i=1}^{\infty} X_i$ as g_i was chosen in Step 3′. Require further that $\rho(g(x), h_i(x)) < \delta$ for each $x \in D_i$. Now for $0 < \delta < \rho(g(D_i), X)$ and for δ chosen otherwise sufficiently small, it follows as in Step 3′ that

$$h_1(D_1), h_2(D_2), \cdots, h_n(D_n)$$

are disjoint singular disks in $(S \cup \operatorname{Int} S) - X$ which satisfy (1), (2) and (3) of Step 3 (special case) and such that for each i, $h_i(D_i) \cap S$ is a 0-dimensional subset of $\operatorname{Int} D_i$.

The proof can now be completed as in Step 4′.

3. Proofs of Theorems (0.1)-(0.4).

Proof of Theorem (0.1). This is essentially the content of the special case of Theorem 2.4 which we proved first. Since $E^3 - S$ is 1-ULC, Theorem 2.4 and its counterpart for sets satisfying $(*, X, \operatorname{Ext} S)$ imply that $(*, S, S)$ is satisfied. That is, S can be homeomorphically approximated from each of its complementary domains. Thus S is tame [1, Th. 2.2].

Proof of Theorem (0.2). Suppose the point p is a piercing point of C, and $h: C \to E^3$ is an embedding so that $\operatorname{Bd} h(C)$ can be pierced

by a tame arc at $h(p)$. Then by (15, see Th. 11], $(*, h(p), \operatorname{Int} h(C))$ is satisfied. By [18, Th. 7] (or from Theorems 2.1 and 2.4 of this paper) $(*, p, \operatorname{Int} C)$ is satisfied. Thus by Theorem 2.1, $C - p$ is 1-ULC.

Conversely, if $C - p$ is 1-ULC, let $h: C \to E^3$ be an embedding such that $\operatorname{Bd} h(C)$ is tame from $E^3 - h(C)$ ([16], [17], or [12]). Then $h(C) - h(p)$ is 1-ULC, and clearly $(*, h(p), E^3 - h(C))$ is satisfied. Hence by Theorem 2.4, $(*, h(p), h(\operatorname{Bd} C))$ is satisfied. Therefore by [15, see Th. 11], p is a piercing point of C.

Proof of Theorem (0.3). Since the statement of the theorem is symmetric in S and S', we assume that $(*, X, S)$ is satisfied and prove under this assumption that $(*, X, S')$ is satisfied. By the compactness of S', the uniform nature of E^3, and Theorems 2.1 and 2.4, it is sufficient to show that the sets $(S' \cup \operatorname{Int} S') - X$ and $(S' \cup \operatorname{Ext} S') - X$ are 1-LC at each point $x \in S'$. If $x \in S' - X$, both sets are clearly 1-LC at x. If $x \in X$, then since S and S' share an open set containing X, both spheres are locally the same at x. Thus clearly $(S' \cup \operatorname{Int} S') - X$ and $(S' \cup \operatorname{Ext} S') - X$ are 1-LC at x if and only if $(S \cup \operatorname{Int} S) - X$ and $(S \cup \operatorname{Ext} S) - X$ are 1-LC at x. This completes the proof.

Proof of Theorem (0.4). The set $(S \cup \operatorname{Int} S) - X$ is 1-ULC by Theorem 2.1 and Lemma 2.3. Thus since X is closed $(*, X, \operatorname{Int} S)$ is satisfied by Theorem 2.4. Similarly $(*, X, \operatorname{Ext} S)$ is satisfied.

REMARK. [6, Th. 2.1] can be shown to be a special case of Theorem (0.4). Theorems 8, 9, and 10 of [20] are stated for closed subsets G of a 2-sphere S in E^3 such that the diameters of the components of G have a positive lower bound. It follows from Theorem 2.4 that the restriction that the diameters of G have a positive lower bound is unnecessary. That is, slight modifications of Loveland's proofs yield the following theorem:

THEOREM. *If G is a compact subset of a 2-sphere S in E^3 and if any of the following conditions is satisfied, then G lies on a tame 2-sphere in E^3.*

(i) *The set G can be locally spanned in each complementary domain of S on tame simple closed curves.*

(ii) *The set G can be locally spanned in each component of $E^3 - S$ missing Cantor sets.*

(iii) *The sphere S can be locally spanned at each point of G from each complementary domain of S.*

Similarly, a version of Eaton's theorem on strongly locally spherical spheres [13, Th. 3] can be stated and proved using Theorem 2.4 and modifications of Eaton's proof.

THEOREM. *If G is a compact subset of a 2-sphere S in E^3 such that S is strongly locally spherical at each point of G, then G lies on a tame 2-sphere.*

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