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FACIAL DECOMPOSITION OF LINEARLY COMPACT SIMPLEXES AND SEPARATION OF FUNCTIONS ON CONES

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Necessary and sufficient conditions for a linearly compact simplex K to be uniquely decomposable at a face are given. If P is a cone having the Riesz decomposition property and if $-f, g$ are subadditive homogeneous functions on P with $f \geqq g$ then it is shown that there is an additive homogeneous function h on P with $f \geqq h \geqq g$. If P is a lattice cone for the dual space of an ordered Banach space X and if $-f, g$ are also w^*-continuous then, under certain conditions, it is possible to choose $h \in X$; a consequence of this result is Andô's theorem, that an ordered Banach space has the Riesz decomposition property if its dual space is a lattice. A nonmeasure theoretic proof of Edwards' separation theorem for compact simplexes is also deduced from these results.

Let K be a linearly compact simplex in a real vector space E. Without loss of generality we will assume that K is contained in a hyperplane $e^{-1}(1)$ and that $E = \text{lin } K$, where lin K denotes the linear hull of K. Then it is well known that E is a vector lattice relative to the cone with base K, and that co $(K \cup -K)$ is the closed unit ball for a norm making E a pre-AL-space. In fact if K is compact for a locally convex Hausdorff topology on E then E is the Banach dual space of $A(K)$, the space of all affine continuous functions on K with supremum norm (cf. [5]). (We refer to [10] as a general reference for the lattice theory and terminology that is used.)

The set K is said to be *decomposable* at a face F if there exists a *complementary* face F' of K such that $F \cap F' = \phi$ while co $(F \cup F') = K$. If a complementary face to F exists then it is evident that it is uniquely determined; moreover, in this case, Alfsen [1] has shown that the decomposition is *unique* in the sense that each $k \in K$ has a unique decomposition $k = \lambda x + (1 - \lambda) y$ with $x \in F$, $y \in F'$ and $0 \leqq \lambda \leqq 1$. Alfsen has also given a necessary and sufficient condition for K to be decomposable at F; we give here other necessary and sufficient conditions which are perhaps more closely tied to the order and norm structure of E.

THEOREM 1. *Let K be a linearly compact simplex and F, F' disjoint faces of K. Then F and F' are complementary faces for a* *(necessarily unique) decomposition of K if and only if E is the order-direct sum of lin F and lin F'. Consequently, if E is complete in its norm then K is uniquely decomposable at F if and only if F is norm-closed.*

Proof. Since F is a face of K it is easily verified that lin F is a lattice ideal in E and that $(\text{lin } F) \cap K = F$. If E is the order-direct sum of lin F and lin F' then each $x \in K$ has a unique decomposition $x = y + z$ with $y, z \geqq 0$, $y \in \text{lin } F$, $z \in \text{lin } F'$; hence $K = \text{co}(F \cup F')$, and the decomposition is unique.

Suppose conversely that $K = \text{co}(F \cup F')$. Then, since $E = \text{lin } F + \text{lin } F'$, it will follow that E is the order-direct sum of lin F and lin F' if we prove that $\text{lin } F' = (\text{lin } F)^{\perp} = \{y \in E\colon |x| \wedge |y| = 0, \forall x \in \text{lin } F\}$ (cf. [10, p. 38]). Since $(\text{lin } F)^{\perp}$ is a lattice ideal the set $G = K \cap (\text{lin } F)^{\perp}$ is a face of K disjoint from F, and hence $G \subseteqq F'$. However if $x \in F' \backslash G$ then there exists a $y \in F$ such that $x \wedge y = z \neq 0$; but, since F and F' are faces of K and $x = z + (x - z)$, $y = z + (y - z)$, this implies that $z/\|z\| \in F \cap F'$ which is impossible. Therefore we have $(\text{lin } F)^{\perp} = \text{lin } G = \text{lin } F'$.

If E is complete in its norm then it is an AL-space. If F is norm-closed then the continuity of the lattice operations in E shows that lin F is also norm-closed, and hence is a band. Therefore, by a theorem of Riesz (cf. [10, p. 39]), lin F has an order-direct complement in E, and so K is uniquely decomposable at F. If, conversely, K is uniquely decomposable at F then there exists a natural affine function f on K such that $F = f^{-1}(0)$, $F' = f^{-1}(1)$. The function f has an obvious extension to a continuous linear functional g on E and, since $F = K \cap g^{-1}(0)$, it follows that F is norm-closed.

If K is a compact simplex then E is certainly a Banach space, and so the following result is immediate.

COROLLARY. *If K is a compact simplex and F a face of K, then K is uniquely decomposable at F if and only if F is norm-closed.*

The corollary generalizes Alfsen's result that a compact simplex is uniquely decomposable at each closed (i.e., compact) face. When K is an arbitrary compact convex set Alfsen and Andersen [2] characterize the decomposable faces of K. However it is not true that every linearly compact simplex is decomposable at every norm-closed face, as the following example shows.

Example. Let K denote the continuous nonnegative functions f on

$$[0, 1] \text{ such that } \int_0^1 f(t)dt = 1, \text{ and let } F = \{f \in K \colon \int_0^{1/2} f(t)dt = 0\} .$$

Then K is a base for the lattice cone in $C\,[0, 1]$, and hence is a linearly compact simplex, and it is clear that F is a face of K. The norm in $C\,[0, 1]$ associated with K is the $L_1\,[0, 1]$-norm, and hence F is norm-closed. Suppose that there exists a face F' complementary to F in K. Then, since $f(1/2) = 0$ for all $f \in F$, there exists a $u \in F'$ such that $u(1/2) > 0$. However it is easy to decompose u nontrivially $u = \lambda g + (1-\lambda)h$ with $g \in F$, $h \in F'$ and $0 < \lambda < 1$. Since F' is a face of K it follows that $g \in F \cap F'$, which is a contradiction. Therefore K is not decomposable at the norm-closed face F.

It has been shown by Asimow [4] that the state space of a function algebra is decomposable at every extreme point, and so such a property does not characterize simplexes among compact convex sets; this property does however characterize simplexes among finite-dimensional compact convex sets as the following slightly more general result shows.

PROPOSITION. *If K is a compact convex set which is decomposable at each extreme point x, and such that each complementary face $\{x\}'$ is closed, then K is a finite-dimensional simplex.*

Proof. If the set K_e of extreme points of K is infinite then there exists an accumulation point $u \in K$. For each $x \in K_e$ the set K_e consists of x together with the extreme points of the closed set $\{x\}'$. Consequently $u \in \{x\}'$ for all $x \in K_e$. Therefore the intersection of the faces $\{x\}'$ forms a closed face F of K which is not empty, since $u \in F$. However if y is an extreme point of F then $y \in K_e$, and also $y \in \{y\}'$ which is impossible. Hence K_e is finite.

If K_e has m points and K has dimension n then, for each $x \in K_e$, it is clear that $\{x\}'$ has $m-1$ extreme points and has dimension $n-1$, and $\{x\}'$ has a similar decomposition property to K. Reducing in this way we see eventually that $m = n + 1$, that is K is an n-dimensional simplex.

If K is a compact simplex then the above result shows that not all faces $\{x\}'$ can be closed. For example, for the simplex $\{x \in l_1 \colon x \geqq 0, \; \|x\| \leqq 1\}$ all but one of the faces $\{x\}'$ are closed, while for the simplex of probability measures on $[0, 1]$ none of the faces $\{x\}'$ are closed.

2. We prove an analogue for linearly compact simplexes of Edwards' separation theorem [6], which characterizes compact simplexes; this is a corollary of the following result.

THEOREM 2. *Let P be a cone possessing the Riesz decomposition property, and let $-f$, g be subadditive homogeneous functionals on P with $f \geqq g$. Then there exists an additive homogeneous functional h on P such that $f \geqq h \geqq g$.*

Proof. If we define h on P by

$$h(x) = \inf \left\{ \sum_{i=1}^{n} f(x_i) \; : \; x = \sum_{i=1}^{n} x_i, \; x_i \in P \right\}$$

then it is clear that $f \geqq h \geqq g$, and hence h is finite-valued. Moreover, h is positive-homogeneous and subadditive. If $x = y + z$ with $x, y, z \in P$, and if $\varepsilon > 0$ choose $x_i \in P$ such that $x = \sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} f(x_i) \leqq h(x) + \varepsilon$. Then there exist $a_{ij} \in P$ such that $\sum_{i=1}^{n} a_{i1} = y$, $\sum_{i=1}^{n} a_{i2} = z$, $a_{i1} + a_{i2} = x_i$ for $i = 1, 2, \cdots, n$. We have

$$h(x) \geqq \sum_{i=1}^{n} f(x_i) - \varepsilon \geqq \sum_{i=1}^{n} f(a_{i1}) + \sum_{i=1}^{n} f(a_{i2}) - \varepsilon \geqq h(y) + h(z) - \varepsilon ,$$

so that h is additive and homogeneous.

In the corollary below K will denote a linearly compact subset of E, again contained in a hyperplane $e^{-1}(1)$ and such that lin $K = E$. By $A^b(K)$ we will denote the Banach space of all bounded real-valued affine functions on K with the supremum norm. If $\mathrm{co}(K \cup -K)$ is linearly bounded then its Minkowski functional is a norm in E and $A^b(K)$ is simply the Banach dual space of E for this norm. In the particular case when K is compact for some locally convex Hausdorff topology on E, $A^b(K)$ is the second dual space of $A(K)$.

COROLLARY. *The following statements are equivalent.*

(i) *K is a linearly compact simplex.*

(ii) *$\mathrm{co}(K \cup -K)$ is linearly compact and, if $-f$, g are bounded convex functions on K with $f \geqq g$, there exists an $h \in A^b(K)$ such that $f \geqq h \geqq g$.*

Proof. (i) $\rightarrow$ (ii). That co $(K \cup -K)$ is linearly compact was proved in [5, Th. 2]. If P is the cone generated by K as a base then P is a lattice-cone. If f and g are extended homogeneously to the rest of P then the existence of the required $h \in A^b(K)$ follows from the theorem.

(ii) $\rightarrow$ (i). If $u_1, u_2, v_1, v_2 \in A^b(K)$ and $u_1, u_2 \leqq v_1, v_2$ then, putting $g(x) = \max\, [u_1(x), u_2(x)]$, $f(x) = \min\, [v_1(x), v_2(x)]$ for all $x \in K$, we obtain a function $h \in A^b(K)$ such that $u_1, u_2 \leqq h \leqq v_1, v_2$. The w^*-compactness of order intervals in $A^b(K)$ now shows that $A^b(K)$ is a vector lattice, in fact an AM-space. Therefore E is an AL-space and, in particular, each $\eta \in E$ has a unique decomposition $\eta = \eta_1 - \eta_2$ with $\eta_i \geqq 0$ and $\|\eta\| = \|\eta_1\| + \|\eta_2\|$, namely for $\eta_1 = \eta^+$, $\eta_2 = \eta^-$. Since, by hypothesis, co $(K \cup -K)$ is the closed unit ball of E it follows that E is a sublattice of E. Therefore K is a linearly compact simplex.

It is perhaps surprising that the linear compactness condition on co $(K \cup -K)$ cannot be dropped, as the following two simple examples show.

Examples. (i) Let E be the linear subspace of l_1 spanned by those elements with only finitely many nonzero coordinates, together with the two elements $\{2^{-n}\}$, $\{(-3)^{-n}\}$, and let $K = \{x \in E : x \geqq 0, \|x\|_1 \leqq 1\}$. If $S = \{x \in E : \|x\|_1 \leqq 1\}$ then it is obvious that for each $\varepsilon > 0$ we have co $(K \cup -K) \subsetneqq S \subsetneqq (1 + \varepsilon)$ co $(K \cup -K)$. If $x = \{(-3^{-n}\}$ then $x^+ \notin E$, so that $2x \in S$ but $2x \notin$ co $(K \cup -K)$. Therefore co $(K \cup -K)$ is not linearly closed; in other terminology E has a $(1+\varepsilon)$-generating cone for all $\varepsilon > 0$ but not a 1-generating cone. However, a straightforward verification shows that E has the Riesz decomposition property and hence, by Theorem 2, K has the separation property stated in part (ii) of the Corollary. However K is not a simplex.

(ii) Let K denote the polynomials p nonnegative on $[0, 1]$ and such that $\int_0^1 p(x)\, dx = 1$. It is clear that co $(K \cup -K)$ is not linearly compact because the polynomials do not constitute a sublattice of $L_1[0, 1]$. It is true, but less obvious, that lin K has the Riesz decomposition property (of. [7]). We are grateful to Professor W. A. J. Luxemburg for bringing this fact and reference to our notice.

By an *ordered Banach space* we shall mean a partially ordered Banach space which has a closed, normal, generating cone. If X is an ordered Banach space then so is X^* (cf. [8]). The following lemma now follows from a result of Kadison ([9, Lemma 4.3]).

LEMMA 1. *Let X be an ordered Banach space and let*

$$K = \{f \in X^* : f \geqq 0,\ \|f\| \leqq 1\}\ ,$$

equipped with the w^*-topology. Then X is order and topologically isomorphic to

$$A_0(K) \equiv \{f \in A(K) : f(0) = 0\}\ .$$

LEMMA 2. *Let C be a cone in a vector space V, let p be a function homogeneous on C and let f be a function affine on V such that $f(x) \leqq p(x)$ for all $x \in C$. Then the linear function $g = f - f(0)$ satisfies $g(x) \leqq p(x)$ for all $x \in C$.*

Proof. It is simple to check that g is linear on V. Suppose that there is a point $x \in C$ such that $g(x) > p(x)$. Then if $\varepsilon = -f(0)$ and $\delta = f(x) - p(x)$ we have $\varepsilon \geqq 0$ and $g(x) - p(x) = \delta + \varepsilon > 0$. Hence there exists an $r \geqq 1$ such that $r(\delta + \varepsilon) > \varepsilon$, and we have

$$f(x) = r^{-1}f(rx) + (1 - r^{-1})f(0) .$$

Therefore

$$f(rx) - p(rx) = r(f(x) - p(x)) + (r - 1)\varepsilon = r(\delta + \varepsilon) - \varepsilon > 0 ,$$

which gives a contradiction.

The following theorem is the main result of this section and is a topological version of Theorem 2.

THEOREM 3. *Let X be an ordered Banach space such that the dual cone P^* is a lattice cone in X^*, and let $-f$, g be w^*-continuous subadditive homogeneous functionals on P^* with $f \geqq g$. If either* (i) $f = u_1 \wedge u_2$, $g = v_1 \vee v_2$, *where* $u_1, u_2, v_1, v_2 \in X$, *or* (ii) *the dual cone in X^{**} possesses an interior point, then there exists an $h \in X$ such that $f \geqq h \geqq g$.*

Proof. If $K = \{x \in P^* : \|x\| \leqq 1\}$ then Lemma 1 shows that we can assume that $X = A_0(K)$, and it is sufficient to find an $h \in X$ such that $f(x) \geqq h(x) \geqq g(x)$ for all $x \in K$.

Let G denote the w^*-closed convex hull of the graph of f in $K \times R$ and define $\hat{f}(x) = \sup\{u(x) : u \in A(K),\ u \leqq f\}$ for all $x \in K$. A straightforward calculation shows that $\hat{f}(x) \leqq \inf\{r : (x, r) \in G\}$ for each $x \in K$. If $\mu < \inf\{r : (x, r) \in G\}$ then by separating (x, μ) from G we obtain a $v \in A(K)$ such that $v \leqq f$ while $v(x) > \mu$; therefore $\hat{f}(x) = \inf\{r : (x, r) \in G\}$. Given $\varepsilon > 0$, for each $x \in K$ let N_x be a w^*-compact convex neighbourhood of x such that $|f(x) - f(y)| < \varepsilon$ for each $y \in N_x$, and let $K \subseteqq \bigcup_{i=1}^n N_{x_i}$. For each $x \in K$ we therefore have

$$(x, \hat{f}(x)) \subseteqq \mathrm{co} \bigcup_{i=1}^{n} \{N_{x_i} \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon]\} ,$$

and so we can write $x = \sum_{i=1}^n \lambda_i y_i$, $\hat{f}(x) = \sum_{i=1}^n \lambda_i r_i$ with $y_i \in N_{x_i}$ and $r_i \in [f(x_i) - \varepsilon, f(x_i) + \varepsilon]$ for each i. If we now define for each $x \in P^*$

$$\bar{f}(x) = \inf\left\{\sum_{i=1}^{n} f(x_i) : x_i \in P^*, \sum_{i=1}^{n} x_i = x\right\}$$

then, for each $x \in K$,

$$\bar{f}(x) \leqq \sum_{i=1}^{n} \lambda_i f(y_i) \leqq \sum_{i=1}^{n} \lambda_i f(x_i) + \varepsilon \leqq \sum_{i=1}^{n} \lambda_i r_i + 2\varepsilon = \hat{f}(x) + 2\varepsilon .$$

Therefore $\bar{f}(x) \leqq \hat{f}(x)$ for each $x \in K$. If $\alpha > 0$ and if $\hat{f}_\alpha(x) = \sup \{u(x) : u \in A(\alpha K),\ u \leqq f\}$ then the argument shows that $\bar{f}(x) \leqq \hat{f}_\alpha(x)$ for all $x \in \alpha K$; in particular $\hat{f}_\alpha(0) = 0 = \bar{f}(0)$.

If condition (i) holds then we have

$$\bar{f}(x) = \inf \{u_1(x_1) + u_2(x_2) : x = x_1 + x_2,\ x_i \in P^*\} .$$

Since P^* is a normal cone we can choose $\alpha > 0$ such that $\|x_1\| + \|x_2\| \leqq \alpha \|x\|$ whenever $x = x_1 + x_2$ with $x_i \in P^*$.

If condition (ii) holds and if ζ is an interior point of the dual cone in X^{**} then the order interval $[-\zeta, \zeta]$ is the unit ball for an equivalent norm in X^{**}, and hence X^* has an equivalent norm which is additive on P^*. Therefore there exists an $\alpha > 0$ such that $\sum_{i=1}^n \|x_i\| \leqq \alpha \|\sum_{i=1}^n x_i\|$ whenever $x_i \in P^*$.

Now let $x = \sum_{i=1}^n x_i \neq 0$, where $x_i \in P^*$, and with $n = 2$ if (i) holds. If $\lambda = \sum_{i=1}^n \|x_i\|$, and if $y_i = 0$ when $x_i = 0$, $y_i = \lambda x_i/\|x_i\|$ when $x_i \neq 0$, then $y_i \in \alpha K$ for each i. Since $\hat{f}_\alpha$ is convex on αK we have

$$\begin{aligned}\hat{f}_\alpha(x) &= \hat{f}_\alpha\left(\sum_{i=1}^n \frac{\|x_i\|}{\lambda} y_i\right) \leqq \sum_{i=1}^n \frac{\|x_i\|}{\lambda} \hat{f}_\alpha(y_i) \leqq \sum_{i=1}^n \frac{\|x_i\|}{\lambda} f(y_i) \\ &= \sum_{i=1}^n f(x_i) .\end{aligned}$$

In case (ii) this inequality gives $\hat{f}_\alpha(x) \leqq \bar{f}(x)$ for each $x \in K$, while in case (i) we have $\hat{f}_\alpha(x) \leqq u_1(x_1) + u_2(x_2)$ which again gives $\hat{f}_\alpha(x) \leqq \bar{f}(x)$; in either case therefore we have proved that $\bar{f}(x) = \hat{f}_\alpha(x)$ for each $x \in K$. If we define $\|f\| = \sup \{|f(x)| : x \in K\}$, and $\|\bar{f}\|$ similarly, then we have $|\bar{f}(x)| \leqq \sum_{i=1}^n \|f\|\,\|x_i\| \leqq \alpha \|f\|\,\|x\|$ for each $x \in K$, so that $\|\bar{f}\| \leqq \alpha \|f\|$.

By Theorem 2 $\bar{f}$ is additive on P^* and the above argument shows that $\bar{f}$ is w^*-l. s. c. on βK for each $\beta > 0$. The set $\{x \in P^* : \bar{f}(x) \leqq r\}$ is convex and its intersection with each multiple of the unit ball of X^* is w^*-closed; hence $\bar{f}$ is w^*-l. s. c. on P^*. If we write $\bar{g}$ for -($\overline{-g}$) then $\bar{g}$ is w^*-u. s. c. on P^* and is additive, homogeneous and satisfies $g \leqq \bar{g} \leqq \bar{f} \leqq f$. If $\varepsilon > 0$ and $r > \alpha$ then, by separating the sets $\{(x, t) \in P^* \times R ; t > \bar{f}(x)\}$ and $\{(y, s-\varepsilon/r) \in K \times R : s \leqq \bar{g}(y)\}$ and applying Lemma 2, we obtain a $w_\varepsilon \in X$ such that $w_\varepsilon \leqq f$ and $w_\varepsilon(x) > g(x) - \varepsilon/r$ for all $x \in K$. Hence if $z_\varepsilon = (g - w_\varepsilon) \vee 0$, z_ε is homogeneous, subadditive and w*-continuous on P^* with $\|z_\varepsilon\| < \varepsilon/r$. The above argument shows that $\bar{z}_\varepsilon$ is w^*-u. s. c. on P^* and that $\|\bar{z}_\varepsilon\| \leqq \alpha \|z_\varepsilon\| < \varepsilon$. Since the set $K \times \{\varepsilon/r\}$ is disjoint from the w^*-closed cone $\{(x, t) : x \in P^*,\ t \leqq \bar{z}_\varepsilon(x)\}$ the separation theorem gives a $p_\varepsilon \in X$ such that $p_\varepsilon \geqq \bar{z}_\varepsilon \geqq g - w_\varepsilon$, 0 and $\|p_\varepsilon\| \leqq \varepsilon$.

Using the procedure of the preceding paragraph choose f_1, $g_1 \in X$

such that $f_1 \leqq f$, $g_1 \geqq 0$, $g \leqq f_1 + g_1$ and $\|g_1\| < 1/2$, in particular we have $f \wedge (f_1 + g_1) \geqq g \vee f_1$. By induction there exist sequences $\{f_n\}$ and $\{g_n\}$ in X such that (a) $g_n \geqq 0$, $\|g_n\| < 2^{-n}$, (b) $g \vee f_n \leqq f_{n+1} + g_{n+1}$, (c) $f_{n+1} \leqq f \wedge (f_n + g_n)$. Properties (b) and (c) give $-g_{n+1} \leqq f_{n+1} - f_n \leqq g_n$ so that $\|f_{n+1} - f_n\| < 2^{-n}$. Therefore $\{f_n\}$ converges to $h \in X$ such that $h \leqq f$ by (c), and $h \geqq g$ by (b).

COROLLARY 1 (*Andô* [3]). *If X is an ordered Banach space such that X^* is a lattice for the dual ordering then X has the Riesz decomposition property.*

COROLLARY 2 (*Edwards* [6]). *If K is a compact simplex and if $-f$, g are u. s. c. convex functions on K with $f \geqq g$ then there exists an $h \in A(K)$ such that $f \geqq h \geqq g$.*

Proof. By truncating if necessary we may assume that f and g are bounded, say $|f(x)|$, $|g(x)| \leqq \lambda$ for all $x \in K$. First suppose that the strict inequality $f > g$ holds ; then the set $G = \{(x,t) : \lambda \leqq t \leqq g(x)\}$ is compact in $K \times R$ and is a subset of the convex set $H = \{(y,s) : s < f(y)\}$ which is relatively open in $K \times R$. Therefore, taking the convex hull of a finite covering of G by compact convex neighbourhoods in H, we see that H contains the closed convex hull of G. Hence for each $x \in K$ there is an $f_x \in A(K)$ and a neighbourhood U_x of x such that $g < f_x$ while $f_x(y) < f(y)$ for all $y \in U_x$. If $K \leqq \bigcup_{i=1}^n U_{x_i}$ and if $f' = f_{x_1} \wedge \cdots \wedge f_{x_n}$ then f' is continuous and concave on K with $g < f' < f$. Similarly we can construct a continuous convex function g' on K such that $g < g' < f' < f$. The functions $-f'$, g' have natural extensions to w*-continuous subadditive homogeneous functions on the positive cone P^* of $A(K)^*$ such that $g' \leqq f'$, and so Theorem 3 gives an $h' \in A(K)$ such that $g < g' \leqq h' \leqq f' \leqq f$.

In the general case $f \geqq g$ there exists an $h_1 \in A(K)$ such that $f + 1 > h_1 > g - 1$. By considering the functions $(f \wedge h_1) + 1/2$ and $(g \vee h_1) - 1/2$ we similarly obtain an $h_2 \in A(K)$ such that $f + 1/2 > h_2 > g - 1/2$ while $\|h_2 - h_1\| < 1/2$. Proceeding in this way we obtain a sequence $\{h_n\}$ which converges in $A(K)$ to h such that $g \leqq h \leqq f$.

REFERENCES

1. E. M. Alfsen, *On the decomposition of a Choquet simplex into a direct convex sum of complementary faces*, Math. Scand. **17** (1965), 169-176.

2. E. M. Alfsen and T. B. Andersen, *Split faces of compact convex sets*, Aarhus Universitet, Preprint Series, 1968/69, No. 32.

3. T. Andô, *On fundamental properties of a Banach space with a cone*, Pacific J. Math. **12** (1962), 1163-1169.

4. L. Asimow, *Directed Banach spaces of affine functions*, Trans. Amer. Math. Soc. **143** (1969), 117-132.

5. D. A. Edwards, *On the homeomorphic affine embedding of a locally compact cone into a Banach dual space endowed with the vague topology*, Proc. London Math. Soc. (3) **14** (1964), 399-414.

6. ———, *Séparation des fonctions réelles définies sur un simplexe de Choquet*, C. R. Acad. Sci. Paris **261** (1965), 2798-2800.

7. L. Fuchs, *Riesz vector spaces and Riesz algebras*, Queens Papers in Pure and Applied Math. No. 1, 1966.

8. J. Grosberg and M. Krein, *Sur la décomposition des fonctionelles en composantes positives*, C. R. (Doklady) de l'Acad. Sci. de l'URSS **25** (1939), 723-726.

9. R. V. Kadison, *Transformations of states in operator theory and dynamics*, Topology, suppl. **2** (1965), 177-198.

10. A. L. Peressini, *Ordered topological vector spaces*, Harper and Row, New York, 1967.

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