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ALGEBRAIC EQUIVALENCE OF LOCALLY NORMAL REPRESENTATIONS

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It will be shown that (i) the absolute value of every locally normal linear functional is again locally normal; (ii) two locally normal representations π_1 and π_2 of $\mathscr{A}$ generate isomorphic von Neumann algebras $\mathscr{M}(\pi_1)$ and $\mathscr{M}(\pi_2)$ if and only if there exists an automorphism σ of $\mathscr{A}$ such that $\pi_1 \circ \sigma$ and π_2 are quasi-equivalent, provided that either $\mathscr{M}(\pi_1)$ or $\mathscr{M}(\pi_2)$ is σ-finite.

This paper is motivated by a recent work [6] of R. Haag. R. V. Kadison and D. Kastler. As they mentioned, the recent progress in mathematical physics has made a precise analysis of representations of a C^*-algebra furnished with a net of von Neumann algebras a growing necessity.

In the first half of this paper, we shall show that the space of all locally normal linear functionals of a C^*-algebra with a net of von Neumann algebras is a closed invariant subspace of the conjugate space in the sense of [14], which will imply that the absolute value of a locally normal linear functional is locally normal too.

The last half of this paper will be devoted to extending a result of Powers [11] for UHF algebra to a C^*-algebra $\mathscr{A}$ with a proper sequential type I_∞ funnel. Namely it will be shown that two locally normal representations π_1 and π_2 of the C^*-algebra $\mathscr{A}$ generate isomorphic von Neumann algebras if and only if they are connected by an automorphism of $\mathscr{A}$. This is proven under the assumption that one of the generated von Neumann algebras is σ-finite.

2. The locally normal conjugate space of a C^*-algebra with a net of von Neumann algebras. Let $\mathscr{A}$ be a C^*-algebra. Suppose a system $\mathfrak{F} = (\mathscr{A}_\alpha\}$ of C^*-subalgebras of $\mathscr{A}$ indexed by a directed set $\{\alpha\}$ is given such that:

(i) $\mathscr{A}_\alpha$ is a von Neumann subalgebra of $\mathscr{A}_\beta$ if $\alpha \leqq \beta$;

(ii) $\bigcup_\alpha \mathscr{A}_\alpha$ is dense in $\mathscr{A}$ with respect to the norm topology.

The system $\mathfrak{F} = \{\mathscr{A}_\alpha\}$ is called a *net* (in $\mathscr{A}$) of von Neumann algebras and each $\mathscr{A}_\alpha$ is called *local subalgebra* of $\mathscr{A}$

DEFINITION 1. A continuous linear functional φ (resp. representation π) of $\mathscr{A}$ is said to be *locally normal* if φ (resp. π) is σ-weakly continuous on each local subalgebra $\mathscr{A}_\alpha$.

PROPOSITION 2. *Let V be the set of all locally normal linear functionals on a C^*-algebra $\mathscr{A}$ with a net $\mathfrak{F} = \{\mathscr{A}_\alpha\}$ of von Neumann algebras. Then V is a closed, invariant subspace of $\mathscr{A}^*$. Namely, if $\varphi \in \mathscr{A}^*$ is locally normal, then $a\varphi$ and φa, $a \in \mathscr{A}$, are both locally normal, where $a\varphi$ and φa defined by $a\varphi(x) = \varphi(xa)$ and $\varphi a(x) = \varphi(ax)$, $x \in \mathscr{A}$.*

Therefore, there exists a unique central projection z_0 of the universal enveloping von Neumann algebra $\tilde{\mathscr{A}}$ of $\mathscr{A}$, the second conjugate space of $\mathscr{A}$ as a Banach space, such that

$$z_0\mathscr{A}^* = V .$$

Proof. Let $\{\varphi_n\}$ be a sequence in V converging to $\varphi \in \mathscr{A}^*$ with respect to the norm topology. For each α, we have

$$||\varphi|_{\mathscr{A}_\alpha} - \varphi_n|_{\mathscr{A}_\alpha}|| \leqq ||\varphi - \varphi_n|| \to 0$$

as $n \to \infty$; hence $\{\varphi_n|_{\mathscr{A}_\alpha}\}$ converges to $\varphi|_{\mathscr{A}_\alpha}$. Since the predual $\mathscr{A}_{\alpha*}$ of each $\mathscr{A}_\alpha$ is complete, $\varphi|_{\mathscr{A}_\alpha}$ belongs to $\mathscr{A}_{\alpha*}$, so that φ is locally normal. Hence V is closed.

Take an arbitrary element $\varphi \in V$. Let a be an element of $\mathscr{A}_\alpha$. For each β, there exists an index γ such that $\alpha \leqq \gamma, \beta \leqq \gamma$. Since $\varphi|_{\mathscr{A}_\gamma}$ is normal and a is in $\mathscr{A}_\gamma$, $a\varphi|_{\mathscr{A}_\gamma}$ is normal, so that

$$a\varphi|_{\mathscr{A}_\beta} = (a\varphi|_{\mathscr{A}_\gamma})|_{\mathscr{A}_\beta}$$

is normal. Hence $a\varphi$ belongs to V. Therefore, if a belongs to $\bigcup \mathscr{A}_\alpha$, then $a\varphi$ is locally normal. If a is an arbitrary element of $\mathscr{A}$, then there exists a sequence $\{a_n\}$ in $\bigcup \mathscr{A}_\alpha$ such that

$$\lim_{n\to\infty} ||a - a_n|| = 0 ;$$

hence

$$\lim_{n\to\infty} ||a\varphi - a_n\varphi|| \leqq \lim_{n\to\infty} ||a - a_n|| \, ||\varphi|| = 0 .$$

Therefore $a\varphi$ belongs to V since V is closed. By symmetry, φa is also in V. Hence V is invariant.

The last half of our assertion follows from the fact that V is invariant as a subspace of $\tilde{\mathscr{A}}_*$ by [14]. This completes that proof.

As an immediate consequence of the above result, we get

COROLLARY 3. *In the same situation as Proposition 1, if $\varphi \in \mathscr{A}^*$ is locally normal, then the absolute value $|\varphi|$ of φ, in the sense of the polar decomposition, is locally normal too. In particular, if $\varphi \in \mathscr{A}^*$ is locally normal and self-adjoint, then the positive part φ^+*

and the negative part φ^- *of* φ *are both locally normal.*

This generalizes a result [6; Proposition 6] of Haag, Kadison and Kastler.

PROPOSITION 4. *In the same situation as before,* V *is weak* sequencially complete. That is, if* φ *is a weak* limit of a sequence* $\{\varphi_n\}$ *of locally normal linear functionals, then* φ *is locally normal too.*

This follows directly from the weak sequential completeness of the predual $\mathscr{A}_{\alpha *}$ of each $\mathscr{A}_\alpha$, see for example [12].

3. Algebraic equivalence of locally normal representations. First of all, we recall the definition of algebraic equivalence of two representations given by Powers [11]:

DEFINITION 5. Let $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ be two representations of a C^*-algebra $\mathscr{A}$. If the von Neumann algebras $\mathscr{M}(\pi_1)$ and $\mathscr{M}(\pi_2)$ generated by $\pi_1(\mathscr{A})$ and $\pi_2(\mathscr{A})$ respectively are isomorphic, then π_1 and π_2 are said to be *algebraically equivalent.*

The following is a slight modification of a definition given by Haag, Kadison and Kastler [6].

DEFINITION 6. A sequential type I_∞ funnel $\{\mathscr{A}_n\}$ of a C^*-algebra $\mathscr{A}$ is said to be *proper* if each relative commutant $\mathscr{A}_n' \cap \mathscr{A}_{n+1}$ of $\mathscr{A}_n$ in $\mathscr{A}_{n+1}$ is of type I_∞.

The following lemma is a modification of Glimm and Kadison's result [3].

LEMMA 7. *Let* $\mathscr{M}$ *be a von Neumann algebra generated by an increasing sequence* $\{\mathscr{A}_n\}$ *of* C^**-algebras, each of which contains the identity* 1 *of* $\mathscr{M}$. *Let* $\mathscr{U}(\mathscr{A}_n)$ *denote the group of all unitary operators of* $\mathscr{A}_n$. *Then the union* $\bigcup_{n=1}^\infty \mathscr{U}(\mathscr{A}_n)$ *is strongly dense in the group* $\mathscr{U}(\mathscr{M})$ *of unitary operators of* $\mathscr{M}$.

Proof. Take an arbitrary unitary operator $u \in \mathscr{U}(M)$. There exists a self-adjoint operator $h \in \mathscr{M}$ such that $u = \exp(2\pi i h)$ and $\|h\| \leqq 1$. Since $\bigcup_{n=1}^\infty \mathscr{A}_n$ is a strongly dense *-subalgebra of $\mathscr{M}$, there exists, by Kaplansky's density theorem [2: Th. 3, p. 43], a net $\{h_j\}_{j \in J}$ of self-adjoint elements in $\bigcup_{n=1}^\infty \mathscr{A}_n$ such that $\{h_j\}_{j \in J}$ converges strongly to h and $\|h_j\| \leqq 1$. Put $u_j = \exp(2\pi i h_j)$, $j \in J$. Since each h_j

belongs to some $\mathscr{A}_n$, u_j belongs to $\bigcup_{n=1}^{\infty}\mathscr{U}(\mathscr{A})$. By the strong continuity of the functional calculus on the bounded set of self-adjoint elements (see [10]), the net $\{u_j\}$ converges strongly to u. This completes the proof.

LEMMA 8. *Let $\mathscr{M}$ be a σ-finite (countably decomposable) von Neumann algebra. Suppose $\mathscr{A}$ and $\mathscr{B}$ are type I_∞ subfactors of $\mathscr{M}$ with properly infinite relative commutants $\mathscr{A}'\cap\mathscr{M}$ and $\mathscr{B}'\cap\mathscr{M}$. Then there exists a unitary operator u in $\mathscr{M}$ such that $u\mathscr{B}u^{-1}=\mathscr{A}$.*

Proof. Let $\{u_{i,j}: i, j = 1, 2, \cdots\}$ and $\{v_{i,j}: i, j = 1, 2, \cdots\}$ be matrix units of $\mathscr{A}$ and $\mathscr{B}$ respectively. Put $e = u_{1,1}$ and $f = v_{1,1}$. Then e and f are minimal projections in $\mathscr{A}$ and $\mathscr{B}$ respectively. Since $\mathscr{A}'\cap\mathscr{M}$ is properly infinite, $\mathscr{A}'\cap\mathscr{M}$ contains an infinite sequence $\{p_n\}$ of equivalent orthogonal projections with $\sum p_n = 1$. For each index n, let u_n be a partial isometry in $\mathscr{A}'\cap\mathscr{M}$ such that $u_n^*u_n = p_1$ and $u_nu_n^* = p_n$. Then we have

$$(u_ne)^*(u_ne) = eu_n^*u_ne = ep_1\ ;$$
$$(u_ne)(u_ne)^* = u_neu_n^* = ep_n\ .$$

Hence $\{ep_n\}$ is an infinite sequence of equivalent orthogonal projections with $\sum_{n=1}^{\infty} ep_n = e$, which means that e is a properly infinite projection in $\mathscr{M}$. Similarly f is a properly infinite projection in $\mathscr{M}$. Since e and f both have central support 1, they are equivalent in $\mathscr{M}$, that is, there exists a partial isometry $w\in\mathscr{M}$ with $w^*w = e$ and $ww^* = f$ because of the σ-finiteness of $\mathscr{M}$. Put

$$u = \sum_{i=1}^{\infty} v_{i,1}wu_{1,i}\ .$$

Then we have

$$\begin{aligned} u^*u &= \sum_{i,j=1}^{\infty} u_{i,1}w^*v_{1,i}v_{j,1}wu_{1,j} \\ &= \sum_{i=1}^{\infty} u_{i,1}w^*wu_{1,i} \\ &= \sum_{i=1}^{\infty} u_{i,1}u_{1,1}u_{1,i} = \sum_{i=1}^{\infty} u_{i,i} \\ &= 1\ ; \end{aligned}$$

similarly

$$uu^* = 1\ .$$

Hence u is a unitary operator in $\mathscr{M}$. By a straightforward calculation, we have

$$uu_{i,j}u^* = v_{i,j}, \qquad i, j = 1, 2, \cdots\ ;$$

hence we have

$$u\mathscr{A}u^* = \mathscr{B}.$$

This completes the proof.

LEMMA 9. *Let $\mathscr{M}$ be a σ-finite von Neumann algebra generated by a C^*-algebra $\mathscr{A}$ with a proper sequential type I_∞ funnel $\{\mathscr{A}_n\}$, where we assume each $\mathscr{A}_n$ to be a von Neumann subalgebra of $\mathscr{M}$. Suppose $\mathscr{B}$ is a type I_∞ factor contained in $\mathscr{M}$ with properly infinite relative commutant $\mathscr{B}' \cap \mathscr{M}$. For any σ-strong* neighborhood*[1] *U of the identity 1 in $\mathscr{M}$, there exists n and a unitary operator u in U such that*

$$u\mathscr{B}u^{-1} \subset \mathscr{A}_n.$$

Proof. By Lemma 8, there exists a unitary operator $v \in \mathscr{M}$ such that $v\mathscr{B}v^{-1} = \mathscr{A}_1$. Since v^{-1} is in $\mathscr{M}$, it follows from Lemma 7 that there exists a unitary operator $w \in \mathscr{A}_n$ such that $w \in Uv^{-1}$. Put $u = wv$. Then u belongs to U and

$$u\mathscr{B}u^{-1} = wv\mathscr{B}v^{-1}w^{-1} = w\mathscr{A}_1w^{-1} \subset \mathscr{A}_n.$$

This completes the proof.

LEMMA 10. *Suppose $\mathscr{M}$, $\mathscr{A}$ and $\{\mathscr{A}_n\}$ are as in Lemma 9. Suppose $\mathscr{B}$ and $\mathscr{B}_1$ are both type I_∞ subfactors of M such that $\mathscr{B} \subset \mathscr{B}_1$ and the relative commutant $\mathscr{B}_1' \cap \mathscr{M}$ is properly infinite. Suppose u is a unitary operator in $\mathscr{M}$ such that*

$$u\mathscr{B}u^{-1} \subset \mathscr{A}_{n_0}.$$

For any σ-strong neighborhood U of 1 in $\mathscr{M}$, there exist a unitary operator u_1 and index $n_1 > n_0$ such that*

(i) $u_1\mathscr{B}u_1^{-1} \subset \mathscr{A}_{n_1}$;

(ii) $u_1xu_1^{-1} = uxu^{-1}$ *for every* $x \in \mathscr{B}$;

(iii) $u_1u^{-1} \in U$.

Proof. Put $\mathscr{C} = u\mathscr{B}u^{-1}$ and $\mathscr{C}_1 = u\mathscr{B}_1u^{-1}$. Then $\mathscr{C}$ and $\mathscr{C}_1$ are both type I_∞ subfactors of $\mathscr{M}$ with properly infinite relative commutant in $\mathscr{M}$. Put $\mathscr{N} = \mathscr{C}' \cap \mathscr{M}$. Since $\mathscr{C}$ is a type I subfactor of $\mathscr{M}$, $\mathscr{M}$ is decomposed into the tensor product:

[1] The σ-strong* topology in a von Neumann algebra $\mathscr{M}$ is defined as the locally convex topology induced by the family of seminorms: $x \in \mathscr{M} \to p_\omega(x) = \omega(x^*x + xx^*)^{1/2}$, where ω runs over all normal states of $\mathscr{M}$. The σ-strong* topology agree with the strong operator topology on the unitary group of $\mathscr{M}$, but their uniform structures are different.

$$\mathscr{M} \cong \mathscr{C} \otimes (\mathscr{C}' \cap \mathscr{M}) .$$

If $\mathscr{A}_n \supset \mathscr{C}$, then $\mathscr{A}_n$ is also decomposed with respect to this tensor product:

$$\mathscr{A}_n \cong \mathscr{C} \otimes (\mathscr{C}' \cap \mathscr{A}_n) .$$

Since $\bigcup \mathscr{A}_n$ generates $\mathscr{M}$, $\bigcup_{n=1}^{\infty} (\mathscr{C}' \cap \mathscr{A}_n)$ generates $\mathscr{C}' \cap \mathscr{M} = \mathscr{N}$. Let $\mathscr{D}$ be the uniform closure of $\bigcup_{n=1}^{\infty} (\mathscr{C}' \cap \mathscr{A}_n)$. Then $\mathscr{D}$ has a sequential type I funnel $\{\mathscr{C}' \cap \mathscr{A}_n\}$. Since

$$\mathscr{C} = u\mathscr{B}u^{-1} \subset \mathscr{A}_{n_0}$$

by assumption, $\mathscr{C}' \cap \mathscr{A}_n$ is properly infinite for $n > n_0$ because

$$\mathscr{C}' \cap \mathscr{A}_n \supset \mathscr{A}_n \cap \mathscr{A}_{n_0}' .$$

Moreover, we have, for $n > n_0$,

$$\begin{aligned}(\mathscr{C}' \cap \mathscr{A}_n)' \cap (\mathscr{C}' \cap \mathscr{A}_{n+1}) &\supset \mathscr{C}' \cap \mathscr{A}_n' \cap \mathscr{A}_{n+1} \\ &\supset \mathscr{A}_{n_0}' \cap \mathscr{A}_n' \cap \mathscr{A}_{n+1} = \mathscr{A}_n' \cap \mathscr{A}_{n+1} ,\end{aligned}$$

hence $(\mathscr{C}' \cap \mathscr{A}_n)' \cap (\mathscr{C}' \cap \mathscr{A}_{n+1})$ is properly infinite. Hence the type I_∞ funnel $\{\mathscr{C}' \cap \mathscr{A}_n\}_{n>n_0}$ of $\mathscr{D}$ is proper. Put $\mathscr{D}_1 = \mathscr{C}' \cap \mathscr{C}_1$. Then $\mathscr{D}_1$ is a type I_∞ subfactor of $\mathscr{N}$. Since

$$\begin{aligned}\mathscr{D}_1' \cap \mathscr{N} &= (\mathscr{C}' \cap \mathscr{C}_1)' \cap (\mathscr{C}' \cap \mathscr{M}) \\ &\supseteqq \mathscr{C}_1' \cap \mathscr{M} = u(\mathscr{B}_1' \cap \mathscr{M})u^{-1} ,\end{aligned}$$

$\mathscr{D}_1$ has a properly infinite relative commutant $\mathscr{D}_1' \cap \mathscr{N}$. By Lemma 9, there exists a unitary operator $v \in U \cap \mathscr{N}$ and an index n_1 such that $v\mathscr{D}_1 v^{-1} \subset (\mathscr{A}_{n_1} \cap \mathscr{C}') \subset \mathscr{A}_{n_1}$. Put $u_1 = vu$. Then u_1 is in $\mathscr{M}$ and $u_1 u^{-1}$ is in U. For each $x \in \mathscr{B}$, uxu^{-1} is in $\mathscr{C}$; hence it commutes with v, so that

$$u_1 x u_1^{-1} = v(uxu^{-1})v^{-1} = uxu^{-1} .$$

Since

$$\begin{aligned}u_1(\mathscr{B}' \cap \mathscr{B}_1)u_1^{-1} &= vu(\mathscr{B}' \cap \mathscr{B}_1)u^{-1}v \\ &= v(\mathscr{C}' \cap \mathscr{C}_1)v^{-1} \\ &= v\mathscr{D}v^{-1} \subset \mathscr{A}_{n_1}\end{aligned}$$

we have

$$\begin{aligned}u_1\mathscr{B}_1u_1^{-1} &= u_1(\mathscr{B} \cup (\mathscr{B}' \cap \mathscr{B}_1))''u_1^{-1} \\ &= (u_1\mathscr{B}u_1^{-1} \cup u_1(\mathscr{B}' \cap \mathscr{B}_1)u_1^{-1})'' \\ &= (\mathscr{C} \cup v\mathscr{D}v^{-1})'' \subset (\mathscr{A}_{n_0} \cup \mathscr{A}_{n_1})'' = \mathscr{A}_{n_1} .\end{aligned}$$

This completes the proof.

LEMMA 11. *Suppose* $\mathscr{M}$, $\mathscr{A}$ *and* $\{\mathscr{A}_n\}$ *are as in Lemma* 9. *Suppose* $\mathscr{B}$ *is another* C^**-subalgebra of* $\mathscr{M}$ *with a proper sequential type* I_∞ *funnel* $\{\mathscr{B}_n\}$, *which is* σ*-weakly dense in* $\mathscr{M}$. *Then there exists a unitary operator* $u \in \mathscr{M}$ *such that*

$$u\left(\bigcup_{n=1}^{\infty} \mathscr{A}_n\right)u^{-1} = \bigcup_{n=1}^{\infty} \mathscr{B}_n ;$$

hence

$$u\mathscr{A}u^{-1} = \mathscr{B} .$$

Proof. By the σ-finiteness of $\mathscr{M}$, there exists a faithful normal state φ of $\mathscr{M}$. Define a distance function d on $\mathscr{M}$ by:

$$d(x, y) = \{\varphi((x-y)^*(x-y)) + \varphi((x-y)(x-y)^*)\}\}^{1/2} , \qquad x, y \in \mathscr{M} .$$

Then the topology induced by the metric d coincides with the σ-strong* topology on the bounded part of $\mathscr{M}$. Furthermore, the group $\mathscr{U}$ of all unitary operators of $\mathscr{M}$ is complete with respect to this metric d.

By induction, we construct increasing sequences $\{\mathscr{C}_i\}$, $\{\mathscr{D}_i\}$ of type I_∞ subfactors of $\mathscr{M}$, a sequence $\{u_i: i = 1, 2, \cdots\}$ of unitary operators in $\mathscr{M}$ and increasing sequences $\{n_i\}$ and $\{m_i\}$ of integers with the properties:

(i)
$$\mathscr{A}_{n_i+1} \subset \mathscr{C}_{i+1} \subset \mathscr{A}_{n_{i+1}} ;$$
$$\mathscr{B}_{m_i+1} \subset \mathscr{D}_i \subset \mathscr{B}_{m_{i+1}}$$

for $i = 1, 2, \cdots, k$;

(ii)
$$u_{2i-1}\mathscr{B}_{m_i+1}u_{2i-1}^{-1} = \mathscr{C}_i ,$$
$$uu_{2i}^{-1}\mathscr{A}_{n_i+1}u_{2i} = \mathscr{D}_i$$

for $i = 1, 2, \cdots, k$;

(iii)
$$u_{2i}^{-1}xu_{2i} = u_{2i-1}^{-1}xu_{2i-1} \text{ for } x \in \mathscr{C}_i ,$$
$$u_{2i+1}xu_{2i+1}^{-1} = u_{2i}xu_{2i}^{-1} \text{ for } x \in \mathscr{D}_i$$

for $i = 1, 2, \cdots, k$;

(iv)
$$d(u_i, u_{i+1}) < 1/2^i$$

for $i = 1, 2, \cdots, 2k$.

For $k = 1$, we choose $m_1 = 0$, $n_1 = 1$ and $\mathscr{C}_1 = \mathscr{A}_1$. Then by Lemma 8, there exists a unitary operator u_1 such that

$$u_1\mathscr{B}_1u_1^{-1} = u_1\mathscr{B}_{m_1+1}u_1^{-1} = \mathscr{C}_1 = \mathscr{A}_1 \subseteqq \mathscr{A}_{n_1} .$$

Consider the triplet $(\mathscr{C}_1, \mathscr{A}_2, u_1^{-1})$ as $(\mathscr{B}, \mathscr{B}_1, u)$ in Lemma 10. Then we can find a unitary operator $v \in \mathscr{M}$ and index m_2 such that

$$v\mathscr{A}_{n_1+1}v^{-1} = v\mathscr{A}_2 v^{-1} \subset \mathscr{B}_{m_2} ;$$

$$vxv^{-1} = u_1^{-1}xu_1 \text{ for every } x \in \mathscr{C}_1 ;$$

$$d(u_1^{-1}, v) < \tfrac{1}{2} .$$

Put $u_2 = v$ and $\mathscr{D}_1 = v\mathscr{A}_2 v^{-1}$.

Suppose $\{n_1, \cdots, n_k\}$, $\{m_1, \cdots, m_k\}$, $\{\mathscr{C}_1, \cdots, \mathscr{C}_k\}$, $\{\mathscr{D}_1, \cdots, \mathscr{D}_k\}$ and $\{u_1, \cdots, u_{2k}\}$ have been chosen so that condition (i), (ii), (iii) and (iv) are satisfied. Applying Lemma 10 to $\{\mathscr{D}_k, \mathscr{B}_{m_k+1}, u_{2k}\}$, we can find an index n_{k+1} and a unitary operator u_{2k+1} such that

$$u_{2k+1}\mathscr{B}_{m_k+1}u_{2k+1}^{-1} \subset \mathscr{A}_{n_{k+1}} ;$$

$$u_{2k+1}xu_{2k+1}^{-1} = u_{2k}xu_{2k}^{-1} \text{ for } x \in \mathscr{D}_k ;$$

$$d(u_{2k}, u_{2k+1}) < 1/2^{2k+1} .$$

Put $\mathscr{C}_{k+1} = u_{2k+1}\mathscr{B}_{m_k+1}u_{2k+1}^{-1}$. Since

$$u_{2k}\mathscr{D}_k u_{2k}^{-1} = \mathscr{A}_{n_k+1} ,$$

$\mathscr{C}_{k+1}$ contains $\mathscr{A}_{n_k+1}$. Now again applying Lemma 10 to the triplet $\{\mathscr{C}_{k+1}, \mathscr{A}_{n_{k+1}+1}, u_{2k+1}^{-1}\}$, we can choose an index m_{k+1} and a unitary operator $u_{2(k+1)}$ in $\mathscr{M}$ such that

$$u_{2(k+1)}^{-1}\mathscr{A}_{n_{k+1}+1}u_{2(k+1)} \subset \mathscr{B}_{m_{k+1}} ;$$

$$u_{2(k+1)}^{-1}xu_{2(k+1)} = u_{2k+1}^{-1}xu_{2k+1} \text{ for } x \in \mathscr{C}_{k+1} ;$$

$$d(u_{2k+1}, u_{2(k+1)}) < 1/2^{2(k+1)} .$$

Put $\mathscr{D}_{k+1} = u_{2(k+1)}^{-1}\mathscr{A}_{n_{k+1}+1}u_{2(k+1)}$.

Hence the existence of sequences $\{m_i\}$, $\{n_i\}$, $\{\mathscr{C}_i\}$, $\{\mathscr{D}_i\}$ and $\{u_i\}$ has been established. From condition (i) it follows that

$$\bigcup_{i=1}^{\infty} \mathscr{A}_i = \bigcup_{i=1}^{\infty} \mathscr{C}_i \text{ and } \bigcup_{i=1}^{\infty} \mathscr{B}_i = \bigcup_{i=1}^{\infty} \mathscr{D}_i .$$

From condition (iv), $\{u_k\}$ is a Cauchy sequence of unitary operators with respect to the metric d. Hence u_k converges σ-strongly* to a unitary operator u of $\mathscr{M}$. By condition (iii), for every $x \in \mathscr{B}_{m_i+1}$, we have

$$u_k v u_k^{-1} = u_{2i-1}xu_{2i-1}^{-1}$$

for each $k \geqq 2i$. Hence we have

$$u\mathscr{B}_{m_i+1}u^{-1} = \mathscr{C}_i .$$

Thus we have

$$u\Big(\bigcup_{k=1}^{\infty}\mathscr{B}_k\Big)u^{-1} = u\Big(\bigcup_{j=1}^{\infty}\mathscr{B}_{m_j+1}\Big)u^{-1}$$
$$= \bigcup_{i=1}^{\infty}\mathscr{C}_i = \bigcup_{i=1}^{\infty}\mathscr{A}_i\,.$$

This completes the proof.

As an immediate consequence of Lemma 11, we have the following extension of a corresponding result of Powers for UHF-algebras in [11].

THEOREM 12. *Suppose $\mathscr{A}$ is a C^*-algebra with a proper sequencial type I_∞ funnel $\{\mathscr{A}_n\}$. Suppose $\{\pi_1, \mathscr{H}_1\}$ and $\{\pi_2, \mathscr{H}_2\}$ are two locally normall representations of $\mathscr{A}$. Suppose either the von Neumann algebra $\mathscr{M}(\pi_1)$ generated by $\pi_1(\mathscr{A})$ or the one $\mathscr{M}(\pi_2)$ generated by $\pi_2(\mathscr{A})$ is σ-finite. Then the representations π_1 an π_2 are algebraically equivalent if and only if there exists an automorphism σ of $\mathscr{A}$ such that the representations π_1 and $\pi_2\cdot\sigma$ are quasi-equivalent. If this is the case, then σ may be chosen such that $\sigma(\bigcup_{n=1}^{\infty}\mathscr{A}_n) = \bigcup_{n=1}^{\infty}\mathscr{A}_n$.*

Proof. If there exists an automorphism σ of $\mathscr{A}$ such that π_1 and $\pi_2\circ\sigma$ are quasi-equivalent, then there exists an isomorphism ρ of $\mathscr{M}(\pi_1)$ onto the von Neumann algebra $\mathscr{M}(\pi_2\circ\sigma)$ generated by $\pi_2\circ\sigma(\mathscr{A})$ such that $\rho\circ\pi_1 = \pi_2\circ\sigma$. But it is clear that $\mathscr{M}(\pi_2\circ\sigma) = \mathscr{M}(\pi_2)$. Hence ρ implements the algebraic equivalence of π_1 and π_2.

Suppose $\mathscr{M}(\pi_1)$ is σ-finite. Since $\mathscr{A}$ is simple (see [6: Proposition 10]), π_1 is an isomorphism of $\mathscr{A}$ into $\mathscr{M}(\pi_1)$. Hence we may identity $\mathscr{A}$ with the subalgebra $\pi_1(\mathscr{A})$ of $\mathscr{M}(\pi_1)$ which generates $\mathscr{M}(\pi_1)$. Suppose ρ is an isomorphism of $\mathscr{M}(\pi_2)$ onto $\mathscr{M}(\pi_1)$. Put $\mathscr{B} = \rho\circ\pi_2(\mathscr{A})$ and $\mathscr{B}_n = \rho\circ\pi_2(\mathscr{A}_n)$. Then $\mathscr{M}(\pi_1)$, $\mathscr{A}$, $\{\mathscr{A}_n\}$ and $\mathscr{B}$, $\{\mathscr{B}_n\}$ satisfy all the assumptions of Lemma 11. Hence there exists a unitary operator u in $\mathscr{M}(\pi_1)$ such that

$$u\mathscr{A}u^{-1} = \mathscr{B},\ u\Big(\bigcup_{n=1}^{\infty}\mathscr{A}_n\Big)u^{-1} = \bigcup_{n=1}^{\infty}\mathscr{B}_n\,.$$

Define a map σ of $\mathscr{A}$ into $\mathscr{A}$ by

$$\sigma(x) = u^{-1}\rho\circ\pi_2(x)u\,, \qquad x\in\mathscr{A}\,.$$

Then we have, for $x\in\mathscr{A}$,

$$\pi_1\circ\sigma(x) = \sigma(x) = u(\rho\circ\pi_2(x))u^{-1}\,;$$

hence $\pi_1\circ\sigma$ is unitary equivalent to $\rho\circ\pi_2$ and $\rho\circ\pi_2$ is quasi-equivalent to π_2 by definition, so that $\pi_1\circ\sigma$ and π_2 are quasi-equivalent. This completes the proof.

COROLLARY 13. *If $\mathscr{A}$ is a C^*-algebra with a proper sequential type I_∞ funnel, then the group of automorphisms of $\mathscr{A}$ acts transitively on the set of all locally normal pure states.*

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REFERENCES

1. J. Dixmier, *Les C*-Algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.
2. ———, *Les algebres d'opérateurs dans l'espace Hilbertien*, 2é edition, Gauthier-Villars, Paris, 1968.
3. J. Glimm and R. V. Kadison, *Unitary operators in C*-algebras*, Pacific J. Math. **10** (1960), 547-556.
4. A. Guichardet, *Algebres d'observables associes aux relations de commutation* (to appear)
5. ———, *Tensor products of C**-algebras, Part II Infinite tensor product, Lecture notes, Aahus Univ., No. 12, 1969.
6. R. Haag, R. V. Kadison, and D. Kastler, *Nets of C*-algebras and classification of states* (to appear)
7. R. Haag and D. Kastler, *An algebraic approach to field theory*, J. Math. Phys. **5** (1964), 848-861.
8. N. M. Hugenholtz, M. Takesaki, and M. Winnink, *Local normality of* the KMS-*states in quantum statistical mechanics* (in preparation)
9. N. M. Hugenholtz and J. D. Wieringa, *On locally normal states in quantum statistical mechanics*, Comm. Math. Phys. **11** (1969), 183-197.
10. I. Kaplansky, *A theorem on rings of operators*, Pacific J. Math. **1** (1951), 227-232.
11. R. T. Powers, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Ann. of Math. **86** (1967), 138-171.
12. S. Sakai, *The theory of W*-algebras*, mimeographed notes, Yale Univ., 1962.
13. Z. Takeda, *Inductive limit and infinite direct product of operator* algebras, Tôhoku Math. J. **7** (1955), 67-86.
14. M. Takesaki, *On the conjugate space of operator algebra*, Tôhoku Math. J. **10** (1958), 194-203.

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