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ON S-UNITS ALMOST GENERATED BY S-UNITS OF SUBFIELDS

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Let K/k be a finite galois extension of number fields, S a finite set of primes of K, and Φ a set of intermediate fields. We assume that S and Φ are closed under the action of $G(K/k)$ and that S contains all the archimedean primes. This paper determines conditions under which the S-units of fields of Φ "almost generate" those of K (i.e., generate a subgroup of finite index).

Let U be the S-units of K and U' the subgroup generated by S-units of fields in Φ. For any subgroup, H, of G, let χ_H be the character of G induced by the trivial character on H and let M_H be the corresponding $\boldsymbol{C}[G]$-module.

THEOREM 1. *U/U' is finite if and only if every irreducible $\boldsymbol{C}[G]$-module, M, which occurs in some $M_{H(\mathfrak{P})}$, $\mathfrak{P} \in S$ also occurs in some $M_{J(F)}$, $F \in \Phi$. (Here $H(\mathfrak{P})$ denotes the splitting group of the prime $\mathfrak{P}$ and $J(F)$ the group of automorphisms fixing the elements of F).*

Proof. U/U' is finite if and only if $U \otimes \boldsymbol{C} = U' \otimes \boldsymbol{C}$. But we know the structure of $U \otimes \boldsymbol{C}$ (see, e.g., p. 10 of [1]). If θ is the sum, over all conjugacy classes of primes of S, of $\chi_{H(\mathfrak{P})}$, and $N = U \otimes \boldsymbol{C}$ then the character of N is $\theta - \chi_G$. Hence, except for components with character χ_G, the components of N are those and only those which occur in some $M_{H(\mathfrak{P})}$, $\mathfrak{P} \in S$.

Now U' is generated by those elements which are invariant under some $J(F)$, $F \in \Phi$. So U/U' is finite if and only if N is generated by such elements, which of course is the case if and only if each irreducible component is so generated. Such a component, N', is so generated if and only if it has a nontrivial element fixed by some $J(F)$. By Frobenius reciprocity this is equivalent to saying that N' occurs in some $M_{J(F)}$.

COROLLARY 1. *If for every $\mathfrak{P} \in S$ there is an $F \in \Phi$ such that $\mathfrak{P}$ does not split at all from F to K then U/U' is finite.*

Proof. In this case each $H(\mathfrak{P})$ contains some $J(F)$.

2. In this section we suppose that every irreducible character

of G occurs in some $M_{H(\mathfrak{P})}$, $\mathfrak{P} \in S$ (for example, if k has a complex prime or K has a real one.)

COROLLARY 2. *Let Φ be the set of all proper subextensions. Then U/U' is infinite if and only if G admits a fixed point free (complex) representation (i.e., one in which only the identity has eigenvalue* 1).

Proof. Clear.

REMARK. Groups admitting such a representation are fairly special, the only familiar ones being the cyclic groups, certain metacyclic groups, and SL(2, 5). A complete classification is given in [3].

COROLLARY 3. *Let G be abelian and let Φ consist of cyclic subextensions. Then U/U' is finite if and only if every maximal cyclic subextension belongs to Φ.*

Proof. Clear.

THEOREM 2. *The S-units if K of degree $\leqq m$ over k generate a subgroup of finite index in U if and only if every irreducible (complex) representation of G factors through a transitive permutation representation on at most m symbols.*

Proof. We let Φ be the set of all intermediate fields of degree m. Then the first condition is equivalent to the finiteness of U/U', which is equivalent to the occurrence of each irreducible representation of G in some $M_{J(F)}$, $F \in \Phi$.

Now the representation afforded by $M_{J(F)}$ factors through the action of G on cosets of $J(F)$ by translation, hence any component of it factors through permutations on $[G\colon J(F)] = [F\colon k] \leqq m$ elements. Conversely if a representation φ factors through a transitive representation on a set Ω, $|\Omega| \leqq m$, let $J \subset G$ be the stabilizer of a point of Ω. Then the action on Ω is equivalent to translation of the cosets of J, which gives rise to the character χ_J. Since φ, restricted to J, has a fixed point, φ occurs in χ_J by Frobenius reciprocity. Clearly the field F, corresponding to J, has degree $[G\colon J] \leqq m$.

REMARK. For $m = 2, 3, 4$ (but not higher) the above condition is easily seen to be equivalent to the assertion that every irreducible representation factors through S_m.

Since explicit algorithms are available for finding units (in fact fundamental units) in quadratic and cubic extensions of Q (see [2]) we mention the following example.

THEOREM 3. *The S-units of degrees* 2 *and* 3 *over* k *"almost generate" the S-units of* K *if and only if* G *is of one of the following forms*:

(1) G *abelian of exponent* 2 *or* 3

(2) G *has an abelian subgroup,* A *of exponent* 3 *such that* A *is of index* 2 *and* G/A *acts on* A *by inversion.*

Proof. It is easy to check that all irreducible representations of groups of the above forms factor through S_3. Conversely suppose all the irreducible representations of G factor through S_3. Then the same is true of quotients of G, and, by Frobenius reciprocity, of subgroups. In particular all elements are of orders 1, 2 or 3. This takes care of the abelian case.

If G is not abelian, let φ be any irreducible representation of G'. If ψ is an irreducible component of the induced representation of G then the restriction of ψ to G' contains φ by Frobenius reciprocity. Since ψ factors through S_3, φ factors through S_3'. Hence G' is abelian of exponent 3. Since G/G' is abelian it is abelian of exponent 2 or 3. If it is a 3-group, so is G, but then, since all irreducible representations of G factor through the 3-Sylow subgroup of S_3, G itself would be abelian. Hence G/G' is of exponent 2. The action of G/G' on G' gives an ordinary representation of G/G', which can be diagonalized. If G/G' had more than one generator some element of order 2 would commute with an element of order 3, giving an element of order 6 which is impossible. Hence $G/G' \cong Z/2Z$ and the action on G' is inversion.

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