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SOME MEASURE ALGEBRAS ON THE INTEGERS

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The author constructs some abstract algebras whose elements are subsets of the positive integers, and such that the measure of a set is its density. These algebras $\mathscr{A}$ are "abstract" in the sense that the countable join in the underlying lattice is not ordinary set union. However they are "concrete" in the sense that the elements of the algebra are sets, the notion of an integrable function is available and the normed vector space of integrable functions can be shown to be isometrically isomorphic to an ordinary L^1 space. If a function f is integrable, it is shown that its integral is given by

$$\lim_N \frac{1}{N}\sum_{j=1}^{N} f^*(j)$$

where f^* is a suitably chosen function differing from f only on a set of density 0.

This construction differs from others (several are described by Kubilius in his book on probabilistic methods in number theory), because usually countable additivity is sacrificed, whereas here the meaning of countable join has been altered.

The work was motivated by a desire to prove Theorems 3 and 4 which concern an application to sequences (mod 1). We also include some remarks concerning the possibility of constructing probabilistically independent measure algebras. Furthermore by means of the Cantor expansion of a number we construct an algebra which contains all periodic sequences.

2. Construction of $\mathscr{A}$. Let $(X, \mathscr{B}, \mu)$ be a probability space. Let $\mathscr{F} \subseteq \mathscr{B}$ be an algebra (not necessarily a σ-algebra) which generates $\mathscr{B}$, i.e., $\mathscr{B}$ is the smallest σ-algebra containing $\mathscr{F}$. We refer to the members of $\mathscr{B}$ as Borel sets. Suppose also that a sequence $\{z_n\}$ of elements from X is given satisfying

$$\mu(I) = \lim \frac{1}{N} \operatorname{card}\{j \mid z_j \in I; j = 1, 2, \cdots, N\}, \text{ for all } I \in \mathscr{F}.$$

This sequence will remain fixed throughout the discussion. A set $B \in \mathscr{B}$ is called *admissible* if there is a set of integers A such that

$$\mu(B \cap I) = \lim_N \frac{1}{N} \operatorname{card}\{j \mid z_j \in I; j \in A; j = 1, 2, \cdots, N\}, I \in \mathscr{F}.$$

A will be said to *go with B*. A set of integers A is said to *have density* if

$$\lim_N \frac{1}{N} \operatorname{card} \{j \mid j \in A; j = 1, 2, \cdots, N\}$$

exists. The value of the limit will be written dens (A). Moreover, if dens $(A_1 - A_2) = 0$ we will write $A_1 \subseteqq A_2$ (dens). We will write $A_1 = A_2$ (dens) if $A_1 \subseteqq A_2$ (dens) and $A_2 \subseteqq A_1$ (dens). We state now a series of lemmas.

LEMMA 1. *If A goes with B, then* $\mu(B) = \text{dens}(A)$.

Proof. $\mu(B) = \mu(B \cap X) = \text{dens}(A)$.

LEMMA 2. *If A goes with B and if $A = A_1$* (dens), *then A_1 goes with B.*

Proof. Obvious.

LEMMA 3. *If $I \in \mathscr{F}$, then I is admissible and the set $\{j \mid z_j \in I\}$ goes with I.*

Proof. Obvious.

LEMMA 4. *If B is admissible, then B^c is admissible.*

Proof. If A goes with B, A^c goes with B^c.

LEMMA 5. *If B_1 and B_2 are admissible with $B_1 \subseteqq B_2$ and if A_1 and A_2 go with B_1 and B_2 respectively, then $A_1 \subseteqq A_2$* (dens).

Proof.

$$\begin{aligned}
&\frac{1}{N} \operatorname{card} \{j \mid j \in A_1 - A_2, j = 1, \cdots, N\} \\
&\quad = \frac{1}{N} \operatorname{card} \{j \mid j \in A_1 - A_2; z_j \in I; j = 1, \cdots, N\} \\
&\qquad + \frac{1}{N} \operatorname{card} \{j \mid j \in A_1 - A_2; z_j \notin I; j = 1, \cdots, N\} \\
&\quad \leqq \frac{1}{N} \operatorname{card} \{j \mid j \in A_2^c; z_j \in I; j = 1, \cdots, N\} \\
&\qquad + \frac{1}{N} \operatorname{card} \{j \mid j \in A_1; z_j \notin I; j = 1, 2, \cdots, N\} .
\end{aligned}$$

Hence

$$\lim_N \frac{1}{N} \operatorname{card} \{j \mid j \in A_1 - A_2; j = 1, \cdots, N\} \\ \leqq \mu(B_2^c \cap I) + \mu(B_1 \cap I^c), I \in \mathscr{F}.$$

Now since $\mathscr{F}$ generates $\mathscr{B}$, $\mu(I \triangle (B_2 - B_1))$ can be made as small as we wish so we see that

$$\lim_N \frac{1}{N} \operatorname{card} \{j \mid j \in A_1 - A_2; j = 1, \cdots, N\} = 0 .$$

LEMMA 6. *If A_1 and A_2 go with B, then $A_1 = A_2$* (dens).

Proof. This follows from Lemma 5.

LEMMA 7. *If B_1 and B_2 are admissible with $B_1 \cap B_2 = \varnothing$ then we can find sets A_1 and A_2 going with B_1 and B_2, respectively, such that $A_1 \cap A_2 = \varnothing$.*

Proof. Suppose A_1 and A_3 go with B_1 and B_2. Then A_3^c goes with B_2^c, and since $B_1 \subseteqq B_2^c$, $A_1 \subseteqq A_3^c$ (dens). Hence $A_3 - A_1 = A_3$ (dens), so by Lemma 2 we may set $A_2 = A_3 - A_1$.

LEMMA 8. *If $B_1 \subseteqq B_2$ are both admissible then so is $B_2 - B_1$.*

Proof. Let A_1 and A_2 go with B_1 and B_2, respectively. By the preceding lemmas we can assume that $A_1 \subseteqq A_2$. Then for any $I \in \mathscr{F}$,

$$\begin{aligned} \mu((B_2 - B_1) \cap I) &= \mu(B_2 \cap I) - \mu(B_1 \cap I) \\ &= \lim_N \frac{1}{N} \operatorname{card} \{j \mid j \in A_2; z_j \in I; j = 1, \cdots, N\} \\ &\quad - \lim_N \frac{1}{N} \operatorname{card} \{j \mid j \in A_1; z_j \in I; j = 1, \cdots, N\} \\ &= \lim_N \frac{1}{N} \operatorname{card} \{j \mid j \in A_2 - A_1; z_j \in I; j = 1, \cdots, N\} . \end{aligned}$$

LEMMA 9. *If $B_1, B_2, \cdots,$ is a sequence of mutually disjoint admissible sets, then $\cup B_i$ is admissible.*

Proof. Let $A_1, A_2, \cdots,$ be a sequence of mutually disjoint sets going with $B_1, B_2, \cdots,$ respectively. Since

$$\lim_N \frac{1}{N} \operatorname{card} \{j \mid j \in A_i; j = 1, \cdots, N\} = \mu(B_i) ,$$

we may assume, by removing a finite number of elements from A_i if necessary, that for all i

$$\frac{1}{N}\operatorname{card}\{j \mid j \in A_i; j = 1, \cdots, N\} < \mu(B_i) + 2^{-i} \text{ for all } N .$$

Let $A = \cup A_i$ and let $I \in \mathscr{F}$. Then

$$\frac{1}{N}\operatorname{card}\{j \mid z_j \in A; j = 1, \cdots, N\}$$
$$= \sum_{i=1}^{\infty} \frac{1}{N}\operatorname{card}\{j \mid z_j \in I; j \in A_i; j = 1, \cdots, N\} .$$

Now passage to the limit within the summation is justified since, by (1), the series is dominated by the series $\sum (\mu(B_i) + 2^{-i})$. Hence we get

$$\lim_N \frac{1}{N}\operatorname{card}\{j \mid z_j \in I; j \in A; j = 1, \cdots, N\}$$
$$= \sum \mu(B_i \cap I) = \mu(B \cap I) .$$

LEMMA 10. *The admissible sets form a σ-algebra.*

Proof. This is a simple consequence of Lemmas 8 and 9.

THEOREM 1. *Every Borel set is admissible.*

Proof. This is obvious from Lemma 10 since the sets of $\mathscr{F}$ are admissible.

We denote by $\mathscr{A}$ the collection of all sets of integers which go with some $B \in \mathscr{B}$. Then it is clear from the preceding discussion that $\mathscr{A}$, modulo sets of density 0, is isomorphic as a lattice to $\mathscr{B}$, modulo sets of measure 0. We omit the details since the situation will be clearer after the definition of $L^1(\mathscr{A})$. However we will make one comment on the lattice operation in $\mathscr{A}$. For any sets $\{A_i\}$ from $\mathscr{A}$ going with sets $\{B_i\}$ from $\mathscr{B}$, let $\vee A_i$ be a set going with $\cup B_i$. Of course $\vee A_i$ is unique in $\mathscr{A}$ modulo sets of density 0, but in addition the following is true.

THEOREM 2. *Let $A_1, A_2, \cdots,$ be sets from $\mathscr{A}$. Let D be any set having density equal to* dens $(\vee A_i)$ *and suppose $D \supseteqq A_i$* (dens), $i = 1, 2, \cdots$. *Then $D = \vee A_i$* (dens).

Proof. Since $D - \vee A_i \subseteqq D - \bigcup_{i=1}^{N} A_i$ (dens), and since

$$\operatorname{dens}\left(D - \bigcup_{i=1}^{N} A_1\right) = \operatorname{dens}(D) - \operatorname{dens}\left(\bigcup_{i=1}^{N} A_i\right) ,$$

we see that $\text{dens}(D - \vee A_i) = 0$, i.e., $D \subseteqq \vee A_i$ (dens). But since $\text{dens}(D) = \text{dens}(\vee A_i)$ we have also $\vee A_i \subseteqq D$ (dens). Hence $D = \vee A_i$ (dens).

This characterizes $\vee A_i$, modulo sets of density 0, as the smallest set having a density and containing all the sets $A_1, A_2, \cdots$.

3. Applications to sequences (mod 1).

Let T be the unit circle in the complex plane, and let $\{z_n\}$ be a sequence of points in T. We will say that $\{z_n\}$ has the distribution μ if μ is a probability measure on T such that for any "interval," I, of T (i.e., any connected subset of T) satisfying $\mu(\partial I) = 0$, we have

$$\mu(I) = \lim_N \frac{1}{N} \text{card}\,\{j \mid z_j \in I; j = 1, \cdots, N\}\,.$$

If we let $\mathscr{F}$ be the algebra of sets generated by those I for which $\mu(\partial I) = 0$, the preceding work is applicable and we get as special cases the following two theorems.

THEOREM 3. *If $\{z_n\}$ has the distribution μ, and if $\mu = \mu_{at} + \mu_c$ where μ_{at} is the atomic part of μ, then there exists a set A of integers, unique to within a set of density 0 such that, for any interval I satisfying $\mu(\partial I) = 0$,*

$$\mu_{at}(I) = \lim_N \frac{1}{N} \text{card}\,\{j \mid j \in A; z_j \in I; j = 1, \cdots, N\}\,.$$

THEOREM 4. *If $\{z_n\}$ has the distribution μ, and if $\mu = \mu_1 + \mu_2$ where μ_1 is absolutely continuous with respect to Lebesgue measure and μ_2 is singular with respect to Lebesgue measure, there exists a set A of integers, unique to within a set of density 0 such that*

$$\mu_1(I) = \lim_N \frac{1}{N} \text{card}\,\{j \mid j \in A; z_j \in I; j = 1, \cdots, N\}$$

and

$$\mu_2(I) = \lim_N \frac{1}{N} \text{card}\,\{j \mid j \in A^c; z_j \in I, j = 1, \cdots, N\}$$

for any interval I for which ∂I is μ-null.

Proof. The proof uses only the fact that $\mu_1 \perp \mu_2$: there are disjoint Borel sets B_1 and B_2 such that $\mu_1(B) = \mu(B_1 \cap B)$, and $\mu_2(B) = \mu(B_2 \cap B)$. Since the intervals having μ-null boundaries generate $\mathscr{B}$, B_1 and B_2 are admissible. The conclusion of the theorem follows.

4. Representation of $L^1(X)$ as a space of sequences. In this section we define a space $L^1(\mathscr{A})$ of sequences (actually equivalence classes of sequences) which is isometrically isomorphic to $L^1(X)$.

If f is a sequence of real numbers, f will be called *measurable* if for all real numbers x in a dense set, we have

$$D_x = \{j \mid f(j) < x\} \in \mathscr{A} .$$

The function $\alpha(x) = \operatorname{dens}(D_x)$ is nondecreasing and of course can be extended to a nondecreasing function defined for all x. We will loosely refer to any such extension as the *distribution* of f. A measurable sequence f will be called *integrable* if its distribution α satisfies

$$\text{(i)} \quad \int_{-\infty}^{\infty} d\alpha(x) = 1 \quad \text{and} \quad \text{(ii)} \quad \int_{-\infty}^{\infty} |x| d\alpha(x) < \infty .$$

If f is integrable, let $M(f) = \int_{-\infty}^{\infty} x d\alpha(x)$. Let L be the set of all integrable sequences.

LEMMA 11. *If $f \in L$ has the distribution α, then $D_x \in \mathscr{A}$ for every x for which α is continuous.*

Proof. We give only a sketch. Suppose α is continuous at x_0. Then for all x, y with $x < x_0 < y$ and $D_x, D_y \in \mathscr{A}$, we have $D_x \subseteqq D_{x_0} \subseteqq D_y$, so it is clear that $\operatorname{dens}(D_{x_0})$ exists. We must show that D_{x_0} goes with a Borel set B_{x_0}. Since D_x and D_y are in $\mathscr{A}$, they go with Borel sets B_x and B_y with $B_x \subseteqq B_y(\mu)$. Set $B_{x_0} = \cap B_{y_i}$, the intersection being taken over a decreasing sequence $\{y_i\}$ converging to x_0. Then it is easily verified that D_{x_0} goes with B_{x_0}, proving the lemma.

We define a map ϕ from L to $L^1(X)$ as follows: let $f \in L$ and construct a sequence of partitions $P_n = (\cdots, a_{-1,n}, a_{0,n}, a_{1,n}, \cdots)$ of the real line having the following properties:

(i) $\operatorname{mesh}(P_n) < 2^{-n} (n = 1, 2, \cdots)$

(ii) $a_{2i,n+1} = a_{i,n} (n = 1, 2, \cdots; i = 0, \pm 1, \pm 2, \cdots)$

(iii) $a_{i,n}$ is a point of continuity of α.

Define

$$D_{i,n} = \{j \mid a_{i,n} \leqq f(j) < a_{i+1,n}\}$$

and let $B_{i,n}$ be a Borel set such that $D_{i,n}$ goes with $B_{i,n}$. Define, in $L^1(X)$,

$$g_n = \sum_{i=-\infty}^{\infty} a_{i,n} \chi B_{i,n} .$$

$\{g_n\}$ is clearly a Cauchy sequence in $L^1(X)$. Let $\phi(f) = \lim g_n$. The limit may be taken either a.e. or in $L^1(X)$.

LEMMA 12. *$\phi(f)$ does not depend on the choice of the partitions P_n.*

The proof is omitted. Suffice it to say that $\phi(f)$ could have been defined by a "spectral" integral of the form

$$\phi(f)(\cdot) = \int_{-\infty}^{\infty} xd(\chi_{B_x}(\cdot)) ,$$

where $D_x = \{j \mid f(j) < x\}$ goes with B_x.

THEOREM 5. *If f and g are in L and if f has the distribution α, then $\phi(f) = \phi(g)$ if and only if*

$$\{j \mid f(j) < x\} = \{j \mid g(j) < x\} \text{ (dens)}$$

for every x for which α is continuous.

Proof. Assume the condition is satisfied. If we choose partition points at which both α and β (the distribution of g) are continuous, we get immediately $\phi(f) = \phi(g)$.

Conversely, suppose that $\phi(f) = \phi(g)$ and that x is a point of continuity of both α and of β. Then as in the proof of Lemma 11 we see that both the sets $\{j \mid f(j) < x\}$ and $\{j \mid g(j) < x\}$ go with $\{z \mid \phi(f)(z) < x\}$ and hence are equal (dens). It then follows that $\alpha = \beta$ on a dense set so the points of continuity of α are precisely the points of continuity of β. This proves the theorem.

We now set $f \equiv g$ if $\phi(f) = \phi(g)$, and let $[f]$ be the equivalence class containing f. The collection of all equivalence classes $[f]$ $(f \in L)$ will be denoted by $L^1(\mathscr{A})$.

LEMMA 13. *If $f \in L$ then there is a sequence $f^* \in L$ with $f = f^*$* (dens) *such that*

$$M(f) = \lim_N \frac{1}{N} \sum_{j=1}^{N} f^*(j) .$$

Here we have used the notation $f = g$ (dens) to mean that the set $\{j \mid f(j) \neq g(j)\}$ has density 0. If $f = g$ (dens) then surely $f \equiv g$, but the converse need not be true as the example $f(j) = 1/j$ and $g(j) = 0$ shows.

Proof. First assume that α is continuous at 0. We can clearly treat the positive and negative parts of f separately so we assume also that $f \geqq 0$. Let $0 = \delta_1, \delta_2, \delta_3, \cdots$ be an increasing sequence of numbers at which α is continuous and such that $\lim \delta_n = \infty$. Let

$$f_n(j) = \begin{cases} f(j), & \delta_n \leqq f(j) < \delta_{n+1} \\ 0 & \text{otherwise} . \end{cases}$$

Since

$$\lim_N \frac{1}{N}\sum_{j=1}^N f_n(j) = \int_{\delta_n}^{\delta_{n+1}} x d\alpha(x) ,$$

we can modify f_n on a set of density 0 (actually a finite set) so that for *all* N the modified function f_n^* satisfies

$$\frac{1}{N}\sum_{j=1}^N f_n^*(j) < \int_{\delta_n}^{\delta_{n+1}} x d\alpha(x) + 2^{-n} .$$

Let $f^* = \sum f_n^*$. Then

$$\{j \mid f(j) \neq f^*(j)\} \subseteqq \{j \mid f(j) > N\} \text{ (dens)}$$

for every N so that $f = f^*$ (dens). Moreover

$$\frac{1}{N}\sum_{j=1}^N f^*(j) = \sum_{n=1}^\infty \left(\frac{1}{N}\sum_{j=1}^N f_n^*(j)\right) .$$

Again the interchange of limits is justified by the dominated convergence, and we have

$$\lim_N \frac{1}{N}\sum_{j=1}^N f^*(j) = \int_0^\infty x d\alpha(x)$$

proving the lemma for those f for which α is continuous at 0. For any f choose x_0 at which α is continuous. Consider $g = f - x_0$, apply the preceding to get g^* and set $f^* = g^* + x_0$. Clearly f^* has all the desired properties.

THEOREM 6. *The space $L^1(\mathscr{A})$ is a vector space.*

Proof. We must show that L is closed with respect to addition and to multiplication by real numbers and that furthermore $f + g \equiv f' + g'$ and $cf \equiv cf'$ whenever $f \equiv f'$ and $g \equiv g'$. Suppose f and g are in L. We must show that for all numbers x in a dense set, $A_x = \{j \mid f(j) + g(j) < x\} \in \mathscr{A}$. For each rational number r_i, let $A_i = \{j \mid f(j) < x - r_i; g(j) < r_i\}$. For any $M > 0$ and $\varepsilon > 0$, choose N so large that a subcollection of the numbers $r_1, r_2, \cdots, r_N$ partitions the interval $[-M, M]$ with a mesh less than ε. Then for any $j \in A_{x-\varepsilon} - \bigcup_{i=1}^N A_i$ we have $f(j) + g(j) < x - \varepsilon$ and either $f(j) < x - r_i$ for all $r_i \in [-M, M]$, $f(j) \geqq x - r_i$ for all $r_i \in [-M, M]$ or for some $r_i, r_j \in [-M, M]$, $f(j) < x - r_i$, $f(j) \geqq x - r_j$. In the first case $f(j) < x - M + \varepsilon$, in the second $f(j) > x + M - \varepsilon$, and in the third, since $j \notin A_i$, $g(j) \geqq r_i$. Since we may suppose $|r_i - r_j| < \varepsilon$ we get in the third case $f(j) + g(j) \geqq x - \varepsilon$. Hence the third case is impossible and we have

$$A_{x-\varepsilon} - \bigcup_{i=1}^{N} A_i \subseteqq \{j \,|\, f(j) > x + M - \varepsilon\} \cup \{j \,|\, f(j) < x - M + \varepsilon\} .$$

Now since $\bigcup_{i=1}^{N} A_i \subseteqq \vee A_i$ (dens), we see that dens $(A_{x-\varepsilon} - \vee A_i) = 0$, i.e., $A_{x-\varepsilon} \subseteqq \vee A_i$ (dens). Let $T(x) = \text{dens}\,(\vee A_i)$. Clearly $A_x \supseteqq A_i$ for all i, so we get for every x and every $\varepsilon > 0$,

$$\underline{\text{dens}}\,(A_x) \geqq \text{dens}\,(\vee A_i) = T(x)$$

and

$$\overline{\text{dens}}\,(A_{x-\varepsilon}) \leqq \text{dens}\,(\vee A_i) = T(x) .$$

Here we have used the notation $\underline{\text{dens}}$ and $\overline{\text{dens}}$ for lower and upper density respectively. Replacing x by $x + \varepsilon$ in the second inequality we get

$$T(x) \leqq \underline{\text{dens}}\,(A_x) \leqq \overline{\text{dens}}\,(A_x) \leqq T(x + \varepsilon) .$$

This shows that A_x has a density if x is a point of continuity of T and Theorem 2 then gives $A_x \in \mathscr{A}$, since $A_x = \vee A_i$ (dens).

Note that for such x, the set A_x goes with the Borel set $\{z \,|\, \phi(f) + \phi(g) < x\} (= \{z \,|\, \phi(f) + \phi(g) \leqq x\}(\mu))$. Hence $\phi(f+g) = \phi(f) + \phi(g)$. The only nontrivial point left is to prove that $-f \in L$ whenever $f \in L$. But $\{j \,|\, -f(j) < x\} = \{j \,|\, f(j) < -x\} = \{j \,|\, f(j) \leqq -x\}^c = \{j \,|\, f(j) < -x\}^c \in \mathscr{A}$ whenever $-x$ is a point of continuity of α. This finishes the proof.

At the outset one is tempted to call a sequence f, "measurable," if for *all* x, $\{j \,|\, f(j) < x\} \in \mathscr{A}$. However, consider the following: let D be a set not in $\mathscr{A}$ and define $f(j) = -1/j$ and

$$g(j) = \begin{cases} \dfrac{1}{j}, & j \notin D \\ \dfrac{1}{j} - \dfrac{1}{j^2}, & j \in D . \end{cases}$$

Then $\{j \,|\, f(j) + g(j) < 0\} = D \notin \mathscr{A}$, so that $f + g$ is not "measurable" even though f and g are.

THEOREM 7. *If* $L^1(\mathscr{A})$ *is given the norm* $|[f]| = M(|f|)$ *then* ϕ *is an isometric isomorphism of* $L^1(\mathscr{A})$ *and* $L^1(X)$.

Proof. It is clear from what has already been done that ϕ is an isometric isomorphism of $L^1(\mathscr{A})$ into $L^1(X)$. We must show that ϕ is onto, so let $\psi \in L^1(X)$ and suppose ψ has distribution α, $\alpha(x) = \mu\{z \,|\, \psi(z) < x\}$. Choose partitions P_n as before in the definition of ϕ.

For each n let $A'_{i,n}$ go with the Borel set $\{z \mid \psi(z) < a_{i,n}\}$. We can suppose that for all i, $A'_{i,n} \subseteqq A'_{i+1,n}$ and that $A'_{i,n+1} \subseteqq A'_{j,n}$ whenever $a_{i,n+1} \leqq a_{j,n}$. Set $A_{i,n} = A'_{i+1,n}$ and define

$$f_n(j) = \sum_i a_{i,n} \chi_{A_{i,n}}(j)$$

and let $f(j) = \lim_n f_n(j)$. Now if $a_{k,n} < x < a_{k+1,n}$,

$$A'_{k,n} = \bigvee_{i=1}^{k-1} A_{i,n} \subseteqq \{q \mid f(q) < x\} \subseteqq \bigvee_{i=1}^{k+1} A_{i,n} = A'_{i+2,n} \text{ (dens)} .$$

This shows that if x is a point of continuity of α, the set $\{j \mid f(j) < x\}$ is in $\mathscr{A}$. This is what we wanted to prove.

5. **Examples.** In the first example we exhibit two algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ that are independent; i.e., for any $A_1 \in \mathscr{A}_1$, $A_2 \in \mathscr{A}_2$, the set $A_1 \cap A_2$ has density and $\text{dens}(A_1 \cap A_2) = \text{dens}(A_1) \text{dens}(A_2)$. Let T be the unit circle, let $X = T \times T$, let $\mu \times \mu$ be normalized Lebesgue (i.e., Haar) measure on X, let $\tilde{\mathscr{B}}$ be the Borel sets of X, and let $\tilde{\mathscr{F}}$ be the algebra generated by the rectangles of X. Let $\{z, w) \in X$ be such that the sequence $\{(z, w)^n\} = \{(z^n, w^n)\}$ is uniformly distributed in X. Let $\mathscr{A}$ be the algebra determined by this sequence and let $\mathscr{A}_1$ and $\mathscr{A}_2$ be the algebras determined by $\{z^n\}$ and $\{w^n\}$, respectively. Let $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$ go with Borel sets B_1 and $B_2 \in \mathscr{B}$. We will show that, in $\mathscr{A}$, A_1 goes with $B_1 \times T$ and A_2 goes with $T \times B_2$. Then it will follow that $A_1 \cap A_2$ goes with $(B_1 \times T) \cap (T \times B_2) = B_1$ so that $\text{dens}(A_1 \cap A_2) = (\mu \times \mu)(B_1 \times B_2) = \text{dens}(A_1) \text{dens}(A_2)$. But the fact that A_1 goes with $B_1 \times T$ is clear: let A'_1 go with $B_1 \times T$ and consider $I \times T \in \mathscr{F}$. Then

$$\begin{aligned} \mu(B_1 \cap I) &= (\mu \times \mu)((B_1 \times T) \cap (I \times T)) \\ &= \lim_N \frac{1}{N} \text{card}\,\{j \mid (z, w)^j \in I \times T; j \in A'_1; j = 1, \cdots, N\} \\ &= \lim_N \frac{1}{N} \text{card}\,\{j \mid z^j \in I; j \in A'_1; j = 1, \cdots, N\} . \end{aligned}$$

Hence $A'_1 = A_1$ (dens).

In this second example, we let $X = [0, 1]$ and let $\mathscr{B}$ be the Borel sets of X. We are concerned with the sequence $\{z_n\}$ defined by

$$z_n = \sum_{j=2}^{\infty} \frac{n_j}{j!}$$

where n_j is the remainder obtained by dividing n by j, $0 \leqq n_j < j$.

Let r be a rational number in $[0, 1)$. We show that $\{z_n\}$ is uniformly distributed in X by showing that

$$\lim_{N} \frac{1}{N} \operatorname{card} \{n \mid z_n \leqq r; n = 1, \cdots, N\} = r .$$

Let k and K be such that $r = k/K!$ Note that $z_n \leqq k/K!$ if and only if $n^2/2! + \cdots + n_K/K! < k/K!$, for suppose the latter. Then $n_2/2! + \cdots + n_K/K! \leqq k - 1/K!$, so that $z_n \leqq k - 1/K! + \sum_{j=K+1}^{\infty} j - 1/j! = k/K!$. The converse follows from the fact that for any n, $n_j = n$ for $j > n$. It follows that the set $\{n \mid z_n \leqq k/K!\}$ is periodic with period $K!$ so its density is

$$\frac{1}{K!} \operatorname{card} \left\{n \,\middle|\, \frac{n_2}{2!} + \cdots + \frac{n_K}{K!} < k/K!; n = 0, \cdots, K! - 1\right\} = k/K! = r .$$

THEOREM 8. *The algebra $\mathscr{A}$, constructed by means of the sequence $\{z_n\}$, contains all periodic sequences.*

Proof. Let $K \geqq 2$ and let p be between 0 and $K - 1$. Let

$$I = \bigcup_{i=0}^{(K-1)!-1} \left(\frac{i}{(K-1)!} + \frac{p}{K!}, \frac{i}{(K-1)!} + \frac{p+1}{K!}\right) .$$

It is not difficult to see that $n_K \equiv [K!\, z_n] \pmod{K}$ so if $z_n \in I$ then $n_K = p$ and conversely. Hence $\{n \mid z_n \in I\} = \{n \mid n \equiv p \pmod{K}\}$. This proves the theorem.

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REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.

2. J. Kubilius, *Probabilistic methods in the theory of numbers*, Translations of Mathematical Monographs, vol. 11, American Mathematical Society, Providence, R. I., 1964.

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