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FAMILIES OF L_p-SPACES WITH INDUCTIVE AND PROJECTIVE TOPOLOGIES

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Let $(X, \mathscr{A}, \mu)$ be a measure space, and $S \subset [1, \infty)$. This paper investigates basic properties of $L^P(S) = \bigcap_{t \in S} L_t(\mu)$ and $L^I(S)=$ span of $\bigcup_{t \in S} L_t(\mu)$, when they are endowed with appropriate projective and inductive topologies.

If X is μ-finite or μ is a counting measure, then $L^P(S)$, $L^I(S)$ are projective and inductive limits in the usual sense. In this case the extensive abstract theory of inductive and projective limits applies. In the general case, however, this theory does not appear applicable. Using special properties of L_p-spaces a basic duality is established between $L^P(S)$ and $L^I(S')$, for the general case, where S' is the set of conjugates to elements of S.

Next such properties as metrizability, normability and completeness for $L^P(S)$, $L^I(S)$ are considered. The question of when $L^P(S) = L^P(T)$ is also considered, and it is shown that there is a certain maximal set T for which this is true. Similarly for $L^I(S)$.

In § 4 we compare the weak topology for $L^I(S)$ with its inductive topology obtained by giving each inductee the weak topology. We are unable to make a complete comparison but do show that the two topologies are quite close. The corresponding problem for $L^P(S)$, mentioned in Proposition 2.2. is simple.

Let us give some basic definitions. By a measure space is meant a triple $(X, \mathscr{A}, \mu)$ in which $\mathscr{A}$ is a σ-algebra of subsets of the set X and μ is a measure on $\mathscr{A}$. $\mathscr{M}$ is the set of all $\mathscr{A}$-measurable complex-valued functions on X and $L_p(\mu)$, $1 \leqq p \leqq \infty$, is defined as usual.

If (E, E') is a dual pair of vector spaces we use the symbols $\sigma(E, E')$, $\tau(E, E')$ and $\beta(E, E')$ in the usual fashion to denote the corresponding weak, Mackey and strong topologies for E (cf., [3] or [5]). If $f, g \in \mathscr{M}$ and $fg \in L_1(\mu)$ we write

$$< f, g > = \int_X fg \, d\mu \, .$$

For $S \subset [1, \infty]$ we let $S' = \{s' : 1 \leqq s' \leqq \infty, 1/s + 1/s' = 1 \text{ for some } s \in S\}$. If $f \in \mathscr{M}$, Rf, If denote, respectively, the real and imaginary parts of f and, if f is real valued, f^+, f^- denote its positive and negative parts. For $A \in \mathscr{A}$ χ_A denotes the characteristic function of A.

DEFINITION 1.1. Let $S \subset [1, \infty]$, $S \neq \varnothing$. Define

$$L^I(S) = \text{span in } \mathscr{M} \text{ of } \bigcup_{t \in s} L_t(\mu) \,,$$

and

$$L^P(S) = \bigcap_{t \in s} L_t(\mu) \,.$$

For each $t \in S$ define

$$u_t : L_t(\mu) \longrightarrow L^I(S) \,,$$
$$v_t : L^P(S) \longrightarrow L_t(\mu)$$

to be the natural injections. Let $\mathscr{S}_t$, $\mathscr{W}_t$, respectively, be the strong (norm) and weak topologies for $L_t(\mu)$, $1 \leqq t \leqq \infty$.

Let $\mathscr{S}^I(S)$, $\mathscr{W}^I(S)$ be the inductive topologies for $L^I(S)$ with respect to the families $\{(L_t(\mu), \mathscr{S}_t, u_t) : t \in S\}$ and $\{L_t(\mu), \mathscr{W}_t, u_t) : t \in S\}$, respectively. (See [5, p. 54]. In [3, p. 79], $L^I(S)$ is called an "inductive limit" with either of these two topologies; however, this phrase is used differently in [5].)

Let $\mathscr{S}^P(S)$, $\mathscr{W}^P(S)$ be the projective topologies for $L^P(S)$ with respect to the families $\{(L_t(\mu), \mathscr{S}_t, v_t) : t \in S\}$ and $\{(L_t(\mu), \mathscr{W}_t, v_t): t \in S\}$, respectively. (See [5], p. 51], or [3, p. 84]. Again a difference of terminology exists.)

2. The basic duality.

We give here proofs for the basic duality between $L^P(S)$ and $L^I(S')$.

PROPOSITION 2.1. (i) *Suppose X is μ-σ-finite. If $S \subset [1, \infty]$, then $\mathscr{W}^I(S)$, $\mathscr{S}^I(S)$, are separated. If $S \subset [1, \infty)$, then the dual of $L^I(S)$ under either of $\mathscr{W}^I(S)$, $\mathscr{S}^I(S)$ consists of the maps $g \rightarrow \langle g, f \rangle$, where $f \in L^P(S')$ is unique.*

(ii) *Suppose X is not μ-σ-finite. If $S \subset [1, \infty)$, $\mathscr{W}^I(S)$, $\mathscr{S}^I(S)$ are separated. If $S \subset (1, \infty)$, then the $\mathscr{W}^I(S)$, $\mathscr{S}^I(S)$-dual of $L^I(S)$ consists of the maps $g \rightarrow \langle g, f \rangle$, where $f \in L^P(S')$ is unique.*

Proof. For the second statement in either (i) or (ii) recall that, as $\mathscr{S}^I(S)$, $\mathscr{W}^I(S)$ are inductive topologies, a linear form F on $L^I(S)$ is continuous if and only if $F \circ u_t$ is continuous for each $t \in S$ ([3, p. 74]). Using Riesz representation and the fact that the simple functions of μ-finite support are dense in all of the spaces $(L_t, \mathscr{W}_t)$, $(L_t, \mathscr{S}_t)$, $t \in S$, it is not hard to show that such F's are exactly those of the form $F(g) = \langle g, f \rangle$ for some unique $f \in L^P(S')$.

For the first statement take $0 \neq g \in L^I(S)$. Suppose, for example, that $(Rg)^+ \neq 0$. As the support of g is μ-σ-finite, there is a set $B \subset \{x \in X: Rg(x) > 0\}$ such that $0 < \mu(B) < \infty$. Let F be the linear form on $L^I(S)$ given by

$$F(h) = \int_B h d\mu \ .$$

Then $F(g) \neq 0$. As $F \circ u_t$ is both $\mathscr{W}_t$, $\mathscr{S}_t$-continuous, $t \in S$, F is continuous. Thus $\mathscr{W}^I(S)$, $\mathscr{S}^I(S)$ are separated.

Suppose $L^P(S')$ is the dual of $L^I(S)$ as in 2.1. A separated inductive topology formed from barrelled spaces is barrelled ([3, p.81]); hence, $\mathscr{S}^I(S) = \tau(L^I(S), L^P(S')) = \beta(L^I(S), L^P(S'))$. Clearly $\sigma(L^I(S), L^P(S')) \subset \mathscr{W}^I(S) \subset \mathscr{S}^I(S)$; but it is not so clear as to whether or not the first containment here can be replaced by equality.

Let $S \subset [1, \infty]$. Both $\mathscr{W}^P(S)$ and $\mathscr{S}^P(S)$ are separated ([3, p. 85]) and $\mathscr{W}^P(S) = \sigma(L^P(S), D_0)$, where D_0 is the dual of $(L^P(S), \mathscr{W}^P(S))$ ([3, p. 99]).

PROPOSITION 2.2. *Take $S \subset [1, \infty]$. Then each $f \in L^I(S')$ defines a continuous linear form on $(L^P(S), \mathscr{W}^P(S))$ and on $(L^P(S), \mathscr{S}^P(S))$ by $g \rightarrow < g, f >$. Now suppose $S \subset [1, \infty)$ and that either X is μ-σ-finite or that $1 \notin S$. Then $L^I(S')$ is the dual of $(L^P(S), \mathscr{W}^P(S))$. In this case $\mathscr{W}^P(S) = \sigma(L^P(S), L^I(S'))$.*

Proof. If $f = \sum_{i=1}^n f_i \in L^I(S')$, where $f_i \in L_{t_i}(\mu)$, $t_i \in S'$, $1 \leqq i \leqq n$, then the linear forms $g \rightarrow < g, f_i >$ are continuous on $L_{t_i'}(\mu)$, $1/t_i' + 1/t_i = 1$. As $v_{t_i'}: L^P(S) \rightarrow L_{t_i'}(\mu)$ is continuous, $1 \leqq i \leqq n$, $g \rightarrow < g, f >$ is continuous.

For the second statement, note that in this case a local base at 0 in $(L^P(S), \mathscr{W}^P(S))$ is formed by the sets of the form $\{x : x \in L^P(S), |< x, y >| \leqq 1 \text{ for all } y \in F\}$ where F runs through the finite sets in $\bigcup_{t \in S'} L_t(\mu)$. Equivalently we may let F run through the finite sets in $L^I(S')$. Thus $\mathscr{W}^P(S)$ is precisely the weak topology induced by $L^I(S')$ and $L^I(S')$ must be the appropriate dual.

Take $S \subset [1, \infty)$ and let D_1 be the dual of $(L^P(S), \mathscr{S}^P(S))$. By the above, $D_1 \supset L^I(S')$ and we would like to know when equality holds. If $1 \in S$ and X is not μ-σ-finite, then we cannot expect equality for the general case (cf., [4], Chapter 11, problem 46). On the other hand, it turns out that equality holds if either $1 \notin S$ or X is μ-σ-finite. If the L_p-spaces are linearly ordered under containment, for example if X is μ-finite or if μ is a counting measure, this is easy to see: In this case $(L^P(S), \mathscr{S}^P(S))$ is a projective limit in the sense of [5, p. 52] (let the $g_{\alpha\beta}$ be identity maps here). It follows from [5, Chapter IV, Th. 4.4], that D_1 is algebraically isomorphic to $L^I(S')$. The proof that $D_1 = L^I(S')$ in the general case appears to require use of special properties of L_P-spaces.

THEOREM 2.3. *Let* $(X, \mathscr{A}, \mu)$ *be a* μ*-*σ*-finite measure space,* $S \subset [1, \infty)$. *Then the dual of* $(L^P(S), \mathscr{S}^P(S))$ *consists of the maps* $g \to \langle g, f \rangle$, *where* $f \in L^I(S')$ *is unique.*

Proof. The argument parallels the classical argument for Riesz representation, differing at certain crucial points (cf., [4, chapter 11, § 7]). The uniqueness of f follows from the usual considerations. We divide the rest of the proof into three parts. Let F be an $\mathscr{S}^P(S)$-continuous linear form on $L^P(S)$.

I. We first show that there is a measurable function f such that $F(g) = \langle g, f \rangle$, for all $g \in L^P(S)$. Let $\{X_n\}_1^\infty \subset \mathscr{A}$ be an increasing sequence such that $\bigcup_n X_n = X$ and $\mu(X_n) < \infty$, $n = 1, 2, \cdots$. As $\mu(X_n) < \infty$ we may apply our earlier remarks to conclude the existence of a unique $f_n \in L^I(S')$ whose support is contained in X_n and such that $F(g) = \langle g, f_n \rangle$ for all $g \in L^P(S)$ which vanish outside X_n. Then f_{n+1} agrees a.e. with f_n on X_n, $n = 1, 2, \cdots$, and we may suppose $f_{n+1} = f_n$ on X_n. Define f on X by $f = f_n$ on X_n, $n = 1, 2, \cdots$. Take any non-negative $g \in L^P(S)$. Let $g_n = g\chi_{X_n}$ so that $g_n \uparrow g$ and $\|g - g_n\|_t \xrightarrow[n]{} 0$ for all $t \in S$. As F is continuous,

$$\begin{aligned} F(g) &= \lim_n F(g_n) = \lim_n \int g_n f_n d\mu \\ &= \lim_n \int g_n f d\mu \\ &= \lim_n \int g_n[(Rf)^+ - (Rf)^- + i(If)^+ - i(If)^-]d\mu \\ &= \int g f d\mu, \text{ by monotone convergence .} \end{aligned}$$

If $g \in L^P(S)$ is arbitrary, we break its real and imaginary parts into positive and negative parts and use the linearity of F to complete the proof of I.

Now for $t \in S$, $\delta > 0$ define

$$V(t, \delta) = \{g \in L^P(S) : \|g\|_t < \delta\} .$$

As F is continuous it is bounded on some $\mathscr{S}^P(S)$-neighborhood of 0. Hence there exists $\delta > 0$ and $t_1, \cdots, t_n \in S$ such that $1 \leqq t_1 < t_2 < \cdots < t_n$ and F is bounded on $\bigcap_{i=1}^n V(t_i, \delta)$.

II. We now show that $f \in L^I(S')$ under the assumption that $1 < t_1$. For $g \in L^P(S)$ define $\|g\| = \max_{1 \leqq i \leqq n} \|g\|_{t_i}$ so that $\| \;\; \|$ is a norm on $L^P(S)$ and F is then $\|\ \|$-continuous. If $n = 1$, then $f \in L_{t'_1}(\mu) \subset L^I(S')$ because F is then $\|\ \|_{t_1}$-continuous on $L^P(S)$ and $L^P(S)$ is dense in $(L_{t_1}, \mathscr{S}_{t_1})$ (e.g., $L^P(S)$ contains the simple functions of μ-finite support). We suppose $n > 1$. By breaking f into real, imaginary, positive and negative parts it suffices to show $f \in L^I(S')$ under the assumption $f \geqq 0$.

Let $A_1 = \{x \in X: f(x) \geqq 1\}$, $A_2 = \{x \in X: f(x) < 1\}$. Let $f_i = f\chi_{A_i}$ and define the linear form F_i on $L^P(S)$ by $F_i(g) = \langle g, f_i \rangle$, $i = 1, 2$. Each F_i is $\|\ \|$-continuous so there exists $M_i > 0$ such that

$$\left| \int \phi f_i d\mu \right| \leqq M_i \|\phi\| \text{ for all } \phi \in L^P(S)\ . \tag{1}$$

To show $f \in L^I(S')$ we show each $f_i \in L^I(S')$, $i = 1, 2$.

If $f_1 = 0$ a.e., $f_1 \in L^I(S')$. Otherwise let $\{\psi_j^{(1)}\}_{j=1}^\infty$ be a sequence of simple functions of μ-finite support such that $\psi_j^{(1)}(x) \geqq 1$ or $\psi_j^{(1)}(x) = 0$ for all $x \in X$ and $\psi_j^{(1)} \uparrow f_1^{t'_n}$, where $1/t_i + 1/t'_i = 1, 1 \leqq i \leqq n$. Let $\phi_j^{(1)} = (\psi_j^{(1)})^{1/t_n}$, $j = 1, 2, \cdots$, so that $\phi_j^{(1)}, \psi_j^{(1)} \in L^P(S)$. Then

$$\|\phi_j^{(1)}\|_{t_i}^{t_i} = \int (\psi_j^{(1)})^{t_i/t_n} \leqq \int \psi_j^{(1)}$$

so

$$\|\phi_j^{(1)}\|_{t_i} \leqq \left[\int \psi_j^{(1)} \right]^{1/t_i}, 1 \leqq i \leqq n, j = 1, 2, \cdots\ . \tag{2}$$

Also $\phi_j^{(1)} f_1 \geqq \phi_j^{(1)} (\psi_j^{(1)})^{1/t'_n} = \psi_j^{(1)}$, $j = 1, 2, \cdots$ and, hence,

$$\int \psi_j^{(1)} \leqq \int \phi_j^{(1)} f_1 \leqq M_1 \|\phi_j^{(1)}\|\ , \tag{3}$$

by (1). For some $i \in \{1, \cdots, n\}$ there exists a subsequence $\{j_k\}$ of the integers such that $\|\phi_{j_k}^{(1)}\| = \|\phi_{j_k}^{(1)}\|_{t_i}$, $k = 1, 2, \cdots$. Hence, by (3) and (2)

$$\int \psi_{j_k}^{(1)} \leqq M_1 \|\phi_{j_k}^{(1)}\|_{t_i} \leqq M_1 \left[\int \psi_{j_k}^{(1)} \right]^{1/t_i}$$

or

$$\int \psi_{j_k}^{(1)} \leqq M_1^{t'_i}, k = 1, 2, \cdots\ .$$

By monotone convergence this implies

$$\int f_1^{t'_n} \leqq M_1^{t'_i}$$

so $f_1 \in L_{t'_n}(\mu) \subset L^I(S')$.

To show $f_2 \in L^I(S')$ we take $\{\psi_j^{(2)}\}_1^\infty$ to be a sequence of simple nonnegative functions of μ-finite support such that $\psi_j^{(2)} \uparrow f_2^{t'_1}$. Let

$\phi_j^{(2)} = (\psi_j^{(2)})^{1/t_1}$, $j = 1, 2, \cdots$. Then

$$\| \phi_j^{(2)} \|_{t_i}^{t_i} = \int (\psi_j^{(2)})^{t_i/t_1} \leqq \int \psi_j^{(2)}$$

so

$$\| \phi_j^{(2)} \|_{t_i} \leqq \left[\int \psi_j^{(2)} \right]^{1/t_i}, 1 \leqq i \leqq n, j = 1, 2, \cdots . \tag{4}$$

As before, $\phi_j^{(2)} f_2 \geqq \phi_j^{(2)} (\psi_j^{(2)})^{1/t_1'} = \psi_j^{(2)}$ so

$$\int \psi_j^{(2)} \leqq \int \phi_j^{(2)} f_2 \leqq M_2 \| \phi_j^{(2)} \| , \tag{5}$$

by (1). Again, for some $i \in \{1, \cdots, n\}$ there is a subsequence $\{j_l\}$ of the integers such that $\| \phi_{j_l}^{(2)} \| = \| \phi_{j_l}^{(2)} \|_{t_i}$, $l = 1, 2, \cdots$. By (5) and (4)

$$\int \psi_{j_l}^{(2)} \leqq M_2 \| \phi_{j_l}^{(2)} \|_{t_i} \leqq M_2 \left[\int \psi_{j_l}^{(2)} \right]^{1/t_i}$$

or

$$\int \psi_{j_l}^{(2)} \leqq M_2^{t_i'} .$$

By monotone convergence of the $\psi_{j_l}^{(2)}$'s we get $f_2 \in L_{t_1'}(\mu) \subset L^I(S')$, completing the proof of II.

III. We now show $f \in L^I(S')$ when $t_1 = 1$. Let $B_m = \{x \in X : f(x) \geqq m\}$, $m = 1, 2, \cdots$. For some m $\mu(B_m) < \infty$ for suppose the contrary. Then for every n there exists $C_n \in \mathscr{A}$ such that $1 \leqq \mu(C_n) < \infty$ and $f(x) \geqq n$ for all $x \in C_n$. Let $g_n = [\delta/2\mu(C_n)] \chi_{C_n}$ so that $\| g_n \|_t = \delta/2$ for all $t \in [1, \infty)$. Then F is bounded on $\{g_n\}_1^\infty$. But

$$\int g_n f d\mu \geqq \frac{\delta}{2\mu(C_n)} \int_{C_n} n d\mu = \frac{n\delta}{2}$$

which is unbounded. Therefore we may choose m such that $\mu(B_m) < \infty$. Let $f_1 = f\chi_{B_m}$ and $f_2 = f - f_1$. Then $f_2 \in L_\infty(\mu) \subset L^I(S')$ and we must show that $f_1 \in L^I(S')$ also.

The argument now proceeds as in the third paragraph of part II. A difficulty occurs in the event that $\| \phi_{j_k}^{(1)} \| = \| \phi_{j_k}^{(1)} \|_{t_i} = \| \phi_{j_k}^{(2)} \|_1$, $k = 1, 2, \cdots$, because then the inequalities which follow are not valid. We modify them as follows:

$$\int \psi_{j_k}^{(1)} d\mu \leqq M_1 \| \phi_{j_k}^{(1)} \|_1 \leqq M_1 [\mu(B_m)]^{1-1/t_n} \| \phi_{j_k}^{(1)} \|_{t_n}$$

$$\leqq K \left[\int \psi_{j_k}^{(1)} d\mu \right]^{1/t_n}, K = M_1 [\mu(B_m)]^{1-1/t_n}$$

so

$$\int \psi_{j_k}^{(1)} d\mu \leqq K^{t_n'},\ k = 1, 2, \cdots ,$$

yielding, as before, $f_1 \in L_{t_n'}(\mu) \subset L^I(S')$. This completes the proof of 2.3.

THEOREM 2.4. *The conclusion of* 2.3 *is valid if* $(X, \mathscr{A}, \mu)$ *is an arbitrary measure space and* $S \subset (1, \infty)$.

Proof. We apply the previous theorem and the constructs which occur in I, II of its proof. Given the linear form F we fix a neighborhood $\bigcap_{i=1}^n V(t_i, \delta)$ on which F is bounded, $1 < t_1 < \cdots < t_n$, $t_i \in (S)$. Take any μ-σ-finite $E \in \mathscr{A}$. There exists a unique $f_E \in L^I(S')$ whose support is contained in E such that $F(g) = \int g f_E\, d\mu$ for all $g \in L^P S$ which vanish a.e. on $X \sim E$. We write $f_E = f_E^{(1)} + i f_E^{(2)}$, where f_E^{ℓ} are real. Let

$$f_E^{\ell,1} = f_E^{(\ell)} \chi_{[f_E^{(\ell)} \geqq 1]} ,$$
$$f_E^{\ell,2} = f_E^{(\ell)} \chi_{[0 \leqq f^{(\ell)} < 1]} ,$$
$$f_E^{\ell,3} = f_E^{(\ell)} \chi_{[-1 < f^{(\ell)} \leqq 0]} ,$$
$$f_E^{\ell,4} = f_E^{(\ell)} \chi_{[f_E^{(\ell)} \leqq -1]} ,$$

where, for example, $[0 \leqq f^{(\ell)} < 1]$ denotes $\{x \in X\colon 0 \leqq f^{(\ell)}(x) < 1\}, \ell = 1, 2$. If $A \subset E$, $A \in \mathscr{A}$, then $f_A^{\ell,m} = f_E^{\ell,m}$ a.e. on A, $\ell = 1, 2$, $m = 1, \cdots 4$. Also we have seen that $f_E^{\ell,m} \in L_{t_n'}(\mu)$ if $m = 1, 4$ and $f_E^{\ell,m} \in L_{t_n'}(\mu)$ if $m = 2, 3$; $\ell = 1, 2$. Now for any μ-σ-finite $E \in \mathscr{A}$ define

$$\lambda(E) = \sum_{\ell=1}^{2} \Big[\sum_{m=1,4} \int \Big| f_E^{\ell,m} \Big|^{t_n'} d\mu + \sum_{m=2,3} \int \Big| f_E^{\ell,m} \Big|^{t_1'} d\mu \Big] .$$

Since the M_i's of equation (1) of 2.3 may be replaced by the norm $\| F \|$ of F with respect to the norm $\| \ \| = \max_{1 \leqq i \leqq n} \{\| \ \|_{t_i}\}$ on $L^P(S)$, we get from the proof of II that

$$\lambda(E) \leqq 8 \max\{\| F \|^{t_1'}, \| F \|^{t_n'}\}$$

for all μ-σ-finite $E \in \mathscr{A}$. The argument now proceeds along the classical lines which appear, for example, in [4, Ch. 11, Th. 7.30]. We obtain a μ-σ-finite set H on which λ achieves its maximum value. If H_0 is any μ-σ-finite set containing H we get that $f_{H_0}^{\ell,m} = f_H^{\ell,m}$ a.e. on H_0, so $f_H = f_{H_0}$ a.e. on H_0. Setting $f = f_H$ one obtains $F(g) = \langle g, f \rangle$ via the usual argument ([4, Ch. 11, 7.30]).

3. Some properties of $L^P(S), L^I(S)$. In this section we examine some basic topological properties of $L^P(S), L^I(S)$. We also consider the question of when $L^P(S) = L^P(T)$ and $L^I(S) = L^I(T)$. If S is a subset of the real line S^C shall denote its convex hull.

THEOREM 3.1. *$(L^P(S), \mathscr{S}^P(S))$ is a Fréchet space when $S \subset [1, \infty]$.*

Proof. Let $S_0 = (S \sim \{\infty\})^C$ and let S_1 be a countable subset of the interval S_0 such that $S_1^C = S_0$. Let S_2 be S_1 if $\infty \notin S$ and $S_1 \cup \{\infty\}$ if $\infty \in S$. S_2 is countable and we claim that $\mathscr{S}^P(S)$ is generated by the norms $\{\| \ \|_t\}_{t \in S_2}$. Let $\{f_n\}$ be a net in $L^P(S)$ such that $f_n \to 0$ in $\mathscr{S}^P(S)$. Take $t \in S_2$. If $t = \infty$, then $\infty \in S$ and $\|f_n\|_t \to 0$. If $t \neq \infty$, there exist finite $t_1, t_2 \in S$ such that $t_1 \leqq t \leqq t_2$. Letting

$$\phi_n(\gamma) = \log(\|f_n\|_\gamma^\gamma)$$

we get $\phi_n(t) \leqq \alpha_n \phi_n(t_1) + (1\text{-}\alpha_n)\phi_n(t_2)$, where $0 \leqq \alpha_n \leqq 1$ ([1, 13.19]) so $\phi_n(t) \leqq \phi_n(t_1) + \phi_n(t_2) \to -\infty$ as n gets large. Consequently $\|f_n\|_t \xrightarrow[n]{} 0$ for all $t \in S_2$. In a similar fashion one can show that if $\|f_n\|_t \xrightarrow[n]{} 0$ for all $t \in S_2$ then $\|f_n\|_t \xrightarrow[n]{} 0$ for all $t \in S$, i.e., $f_n \xrightarrow[n]{} 0$ in $\mathscr{S}^P(S)$. Thus $\mathscr{S}^P(S)$ is generated by a countable family of norms and is consequently metrizable.

To prove completeness suppose that $\{f_n\}_1^\infty$ is an $\mathscr{S}^P(S)$-Cauchy sequence in $L^P(S)$. Then for each $t \in S$ $\{f_n\}$ is $\| \ \|_t$-Cauchy so there exists $f^{(t)} \in L_t(\mu)$ such that $\|f_n - f^{(t)}\|_t \xrightarrow[n]{} 0$. It suffices to show that if $t_1, t_2 \in S$ then $f^{(t_1)} = f^{(t_2)}$ a.e. For then we define $f = f^{t_1}$ a.e. on X and evidently $f_n \longrightarrow f$ in $\mathscr{S}^P(S)$. Take $t_1, t_2 \in S$. As $\|f_n - f^{(t_1)}\|_{t_1} \xrightarrow[n]{} 0$, there is a subsequence $\{f_{n'}\}$ of $\{f_n\}$ which converges to $f^{(t_1)}$ a.e. But $\|f_{n'} - f^{(t_2)}\|_{t_2} \xrightarrow[n']{} 0$ so there is a subsequence $\{f_{n''}\}$ of $\{f_{n'}\}$ converging to $f^{(t_2)}$ a.e. Hence $f^{(t_1)} = f^{(t_2)}$ a.e.

COROLLARY 3.2. *Take $S \subset [1, \infty)$ and assume that either X is μ-σ-finite or that $1 \notin S$. Then $\mathscr{S}^P(S) = \tau(L^P(S), L^I(S')) = \beta(L^P(S), L^I(S'))$.*

Proof. By 3.1, $(L^P(S), \mathscr{S}^I(S))$ is a Fréchet space and hence barrelled. Theorem 2.3 and 2.4 apply.

COROLLARY 3.3. *$(L^I(S), \mathscr{S}^I(S))$ is complete if $S \subset (1, \infty)$.*

Proof. The strong dual of a metric space is complete.

Given $S \subset [1, \infty)$, there is a maximal subset $\hat{S}$ of $[1, \infty)$ such that $L^P(S) = L^P(\hat{S})$, $\mathscr{S}^P(S) = \mathscr{S}^P(\hat{S})$ and $\mathscr{W}^P(S) = \mathscr{W}^P(\hat{S})$. $\hat{S} \supset S$ and is even an interval. Similar statements hold for $L^I(S)$. We now establish these facts.

DEFINITION 3.4. Let $(X, \mathscr{A}, \mu)$ be a measure space, $S \subset [1, \infty)$. Let $s_1 = \text{glb}\, S$, $s_2 = \text{lub}\, S$ (we allow $s_2 = \infty$). We consider the following conditions on $(X, \mathscr{A}, \mu)$:

(i) $1 \leqq t_1 < t_2 < \infty$ implies $L_{t_1}(\mu) \supset L_{t_2}(\mu)$;

(ii) $1 \leqq t_1 < t_2 < \infty$ implies $L_{t_1}(\mu) \subset L_{t_2}(\mu)$.

If (i) but not (ii) holds define $\hat{S} = [1, s_2) \cup S$ and $\tilde{S} = (s_1, \infty) \cup S$. If (ii) but not (i) holds define $\hat{S} = (s_1, \infty) \cup S$, $\tilde{S} = [1, s_2) \cup S$. If both (i) and (ii) hold, define $\hat{S} = \tilde{S} = [1, \infty)$ and, if neither (i) nor (ii) hold, define $\hat{S} = \tilde{S} = S^C$.

It is easy to see that 3.4(i) may be replaced by the condition.

(i)′ For some t_1, t_2 such that $1 \leqq t_1 < t_2 < \infty$ it is the case that $L_{t_1}(\mu) \supset L_{t_2}(\mu)$.

For suppose (i)′ holds and that $1 \leqq s_1 < s_2 < \infty$. If there exists $f \in L_{s_2}(\mu)$ such that $f \notin L_{s_1}(\mu)$, take $s = \text{glb}\ \{t: \int ||f||^t d\mu < \infty\}$. Then $s_1 \leqq s \leqq s_2$ (apply [1, 13.19]). Now take $u \in [s, s_2]$ such that $\int |f|^u d\mu < \infty$ and $ut_1/t_2 < s$. Letting

$$g = |f|^{u/t_2}$$

gives $g \in L_{t_2}$ but $g \notin L_{t_1}$, a contradiction. Thus no such f exists and 3.4(i) holds. Similarly 3.4(ii) is equivalent to

(ii)′ For some t_1, t_2 such that $1 \leqq t_1 < t_2 < \infty$ it is the case that $L_{t_1}(\mu) \subset L_{t_2}(\mu)$.

THEOREM 3.5. 3.4(i) *is equivalent to*

(i)″ *Every μ-σ-finite set in $\mathscr{A}$ is μ-finite.*

3.4(ii) *is equivalent to*

(ii)″ *Every member of $L_1(\mu)$ is essentially bounded.*

If 3.4(i) *is false, then, given $1 \leqq u < \infty$ and $1 < v < \infty$, there exist measurable functions f_1, f_2 on X such that*

$$f_1 \in \bigcap_{u<t<\infty} L_t(\mu),\ f_1 \notin \bigcup_{1 \leqq t \leqq u} L_t(\mu)$$

and

$$f_2 \in \bigcap_{v \leqq t<\infty} L_t(\mu),\ f_2 \notin \bigcup_{1 \leqq t < v} L_t(\mu)\ .$$

If 3.4(*ii*) *is false, then, given $1 \leqq u < \infty$ and $1 < v < \infty$, there exist measurable functions g_1, g_2 on X such that*

$$g_1 \in \bigcap_{1 \leqq t \leqq u} L_t(\mu),\ g_1 \notin \bigcup_{u<t<\infty} L_t(\mu)$$

and

$$g_2 \in \bigcap_{1 \leqq t<v} L_t(\mu),\ g_2 \notin \bigcup_{v \leqq t<\infty} L_t(\mu)\ .$$

Proof. If (i)″ holds then every function in $\bigcup_{1 \leqq t < \infty} L_t(\mu)$ has

μ-finite support so 3.4(i) holds. Suppose (i)″ is false. We shall show that the functions f_1, f_2 then exist. This, in turn, shows that 3.4(i) implies (i)″. If (i)″ is false, then there exist pairwise disjoint sets $A_1, A_2, \cdots \in \mathscr{A}$ such that $\mu(A_n) = \lambda_n$, where $1 \leqq \lambda_n < \infty$, $n = 1, 2, \cdots$. One now defines

$$f_1(x) = \begin{cases} \left[\dfrac{1}{\lambda_n n}\right]^{1/u} & \text{if } x \in A_n \\ 0 & \text{if } x \notin \bigcup_1^\infty A_n \end{cases}$$

and

$$f_2(x) = \begin{cases} \left[\dfrac{1}{\lambda_n (n+1)\,[\log(n+1)]^2}\right]^{1/v}, & \text{if } x \in A_n\,. \\ 0 & \text{if } x \notin \bigcup_1^\infty A_n \end{cases}$$

For example, to see that $f_2 \notin \bigcup_{1 \leqq t < v} L_t(\mu)$, take $t \in [1, v)$ and note that $\lambda_n^{1-t/v} \geqq 1$ so

$$\int |f_2|^t \, d\mu \geqq \sum_{n=1}^{\infty} \left[\frac{1}{(n+1)\,[\log(n+1)]^2}\right]^{t/v} = \infty\,.$$

(ii)″ implies 3.4(ii) because

$$\int |f|^{t_2} \, d\mu \leqq \Big[\, \| f \|_\infty \Big]^{t_2 - t_1} \int |f|^{t_1} \, d\mu\,.$$

We shall show that if (ii)″ is false, then the functions g_1, g_2 exist. This shows that 3.4(ii) implies (ii)″ and completes the proof of the theorem. If (ii)″ is false then $L_1(\mu)$ and, hence $L_v(\mu)$ contains an essentially unbounded function f. As

$$0 < \int_{[f \geqq n]} |f|^v \, d\mu \xrightarrow[n]{} 0\,,$$

there exist pairwise disjoint sets $A_n \in \mathscr{A}$ such that

$$0 < \int_{A_n} |f|^v \, d\mu < e^{-n}, \quad n = 1, 2, \cdots$$

and $|f(x)| \geqq 1$ for all $x \in \bigcup_{n=1}^{\infty} A_n$. Let

$$b_n = \left[\int_{A_n} |f|^v \, d\mu\right]^{-1/v}$$

so $b_n > 1$, $n = 1, 2, \cdots$. Define

$$g_2(x) = f(x)\Big[\sum_{n=1}^{\infty} b_n \chi_{A_n}(x)\Big] \text{for all } x \in X\,.$$

Then for $\alpha \geqq 0$

$$\int |g_2|^{v+\alpha}\, d\mu = \sum b_n^{v+\alpha} \int_{A_n} |f|^{v+\alpha}\, d\mu$$
$$\geqq \sum b_n^v \int_{An} |f|^v\, d\mu = \infty\ .$$

For $\delta > 0$

$$\int |g_2|^{v-\delta}\, d\mu = \sum b_n^{v-\delta} \int_{A_n} |f|^{v-\delta}\, d\mu$$
$$= \sum \Big[\int_{A_n} |f|^v\, d\mu \Big]^{\delta/v} \frac{\int_{An} |f|^{v-\delta}\, d\mu}{\int_{An} |f|^v\, d\mu}$$
$$\leqq \sum \Big[\int_{An} |f|^v\, d\mu \Big]^{\delta/v}$$
$$< \sum e^{-n\delta/v} < \infty\ .$$

Thus g_2 satisfies the required conditions.

To obtain g_1 take $t_n \downarrow u$. By the above, there exists $h_n \in \bigcap_{1 \leqq t < t_n} L_t(\mu)$ such that $h_n \notin L_{t_n}(\mu)$, $n = 1, 2, \cdots$, and for all $x \in X$ either $h_n(x) = 0$ or $h_n(x) \geqq 1$. Let

$$a_n = 2^{-n} \Big[\sup \Big\{ \Big[\int h_n^u\, d\mu \Big]^s : \frac{1}{u} \leqq s \leqq 1 \Big\} \Big]^{-1}.$$

Define $S_k = \sum_{n=1}^k a_n h_n$, $k = 1, 2, \cdots$. For $k > m$ we have

$$||S_k - S_m|| \leqq \sum_{n=m+1}^{k} a_n ||h_n||_u$$
$$\leqq \sum_{n=m+1}^{k} 2^{-n} \longrightarrow 0, \quad \text{as } m, k \longrightarrow \infty\ .$$

Hence there exists $g_1 \in L_u(\mu)$ such that $||S_k - g_1||_u \underset{k}{\longrightarrow} 0$. There exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ such that $S_{n_k} \longrightarrow g_1$ a.e. As $\{S_n\}$ is a pointwise increasing sequence, $g_1(x) \geqq S_n(x)$ for all $x \in X$, $n = 1, 2, \cdots$. To see that $g_1 \notin \bigcup_{u < t < \infty} L_t(\mu)$, take $t \in (u, \infty)$. Take $t_n \in (u, t)$. If $g_1 \in L_t$ then, since $g_1 \in L_u(\mu)$, we have $g_1 \in L_{t_n}(\mu)$. But

$$\int |g_1|^{t_n} d\mu \geqq \int S_n^{t_n}\, d\mu \geqq a_n^{t_n} \int h_n^{t_n}\, d\mu$$
$$= \infty\ ,$$

a contradiction. Therefore $g_1 \notin \bigcup_{u < t < \infty} L_t(\mu)$. On the other hand, if $t \in [1, u]$ then $1/u \leqq 1/t \leqq 1$. As $h_n(x)$ is either 0 or $\geqq 1$, we have

$$\begin{aligned}||g_1||_t &\leqq \sum_{n=1}^{\infty} a_n ||h_n||_t \\ &\leqq \sum_{n=1}^{\infty} a_n \Big[\int h_n^u \, d\mu \Big]^{1/t} \\ &\leqq \sum_{n=1}^{\infty} 2^{-n} < \infty .\end{aligned}$$

This proves the theorem.

Incidently, it is not hard to show that 3.4(ii) is equivalent to (ii)$'''$. For every pairwise disjoint sequence $\{A_n\} \subset \mathscr{A}$ such that $\mu(A_n) > 0$ for all n, it is the case that $\liminf \mu(A_n) > 0$.

But we shall not use this in the sequel.

THEOREM 3.6. *Take* $S, T \subset [1, \infty)$. *Then*

(a) $L^P(S) = L^P(\hat{S})$, $L^I(S) = L^I(\tilde{S})$.

(b) $L^P(S) = L^P(T)$ *if and only if* $\hat{S} = \hat{T}$; $L^I(S) = L^I(T)$ *if and only if* $\tilde{S} = \tilde{T}$.

(c) $\hat{S}$ *is the largest subset of* $[1, \infty)$ *determining* $L^P(S)$, i.e., $L^P(T) = L^P(S)$ *implies* $T \subset \hat{S}$. *Similarly for* $\tilde{S}$ *and* $L^I(S)$.

(d) *If* $L^P(S) = L^P(T)$, *then* $\mathscr{W}^P(S) = \mathscr{W}^P(T)$ *and* $\mathscr{S}^P(S) = \mathscr{S}^P(T)$.

(e) *If* $L^I(S) = L^I(T)$, *then* $\mathscr{W}^I(S) = \mathscr{W}^I(T)$ *and* $\mathscr{S}^I(S) = \mathscr{S}^I(T)$.

Proof. (a) It suffices to show that $L^P(S) = L^P(S^C)$ and $L^I(S) = L^I(S^C)$. Clearly, $L^P(S) \supset L^P(S^C)$ and the reverse containment follows from the fact that $L_{t_1}(\mu) \cap L^t{}_2(\mu) \subset L_t(\mu)$ whenever $1 \leqq t_1 \leqq t \leqq t_2 < \infty$ ([1, 13, 19]). It is also clear that $L^I(S^C) \supset L^I(S)$. The reverse containment follows from the fact that if $f \in L_t(\mu)$, where $1 \leqq t_1 \leqq t \leqq t_2 < \infty$ and $t_1, t_2 \in S$, then $f\chi_{A_1} \in L_{t_1}(\mu)$ and $f\chi_{A_2} \in L_{t_2}(\mu)$, where $A_1 = \{x \in X: |f(x)| \geqq 1\}$, $A_2 = \{x \in X: |f(x)| < 1\}$.

(b) The sufficiency follows from (a) and the necessity from Theorem 3.5: For example, suppose neither 3.4(i) nor (ii) hold and $\hat{S} = [a, b)$, $1 \leqq a < b < \infty$. Take $v = a$ and f_2 according to 3.5. Now take $v = b$ and g_2 according to 3.5. Then $f_2 + g_2 \in L_t(\mu)$ if and only if $t \in [a, b)$. Consequently, $L^P(\hat{T}) = L^P(\hat{S})$ implies $f_2 + g_2 \in L^P(\hat{T})$ which, in turn, implies $\hat{T} = \hat{S}$.

(c) For example, if $L^P(T) = L^P(S)$, then $T \subset \hat{T} = \hat{S}$ by (b).

(d) It suffices to show that $\mathscr{W}^P(S) = \mathscr{W}^P(\hat{S})$ and $\mathscr{S}^P(S) = \mathscr{S}^P(\hat{S})$. Clearly $\mathscr{S}^P(\hat{S}) \supset \mathscr{S}^P(S)$. To show equality it suffices to show that for each $t \in \hat{S}$ the injection

$$v_t\colon (L^P(S), \mathscr{S}^P(S)) \longrightarrow (L_t(\mu), \mathscr{S}_t)$$

is continuous. (It is well-defined since $L^P(S) = L^P(\hat{S})$ by (a).) The argument of 3.1 shows that v_t is continuous for all $t \in S^C$. If 3.4(i) holds, there may exist $t_0 \in \hat{S}$ such that $1 \leqq t_0 < t$ for all $t \in S$. We must show that in this case v_{t_0} is continuous. Take $\{f_n\} \subset L^P(S)$ such that $f_n \longrightarrow 0$ in $\mathscr{S}^P(S)$. Take $t_1 \in S$. Then $1 \leqq t_0 < t_1$ and $\|f_n\|_{t_1} \xrightarrow[n]{} 0$. The support of each f_n is μ-σ-finite so the union U of these supports is μ-σ-finite. By 3.5, $\mu(U) < \infty$ and we may suppose $\mu(U) > 0$. Then

$$\|f_n\|_{t_0} \leqq \frac{\mu(U)^{1/t_0}}{\mu(U)^{1/t_0}} \|f_n\|_{t_1} \xrightarrow[n]{} 0$$

so v_{t_0} is continuous. If 3.4(ii) holds, there may exist $s_0 \in \hat{S}$ such that $1 \leqq s < s_0$ for all $s \in S$. We must show that in this case v_{s_0} is continuous. Again let $\{f_n\} \subset L^P(S)$ be such that $f_n \longrightarrow 0$ in $\mathscr{S}^P(S)$ and take $s_1 \in S$. Then $1 \leqq s_1 < s_0$ and $\|f_n\|_{s_1} \xrightarrow[n]{} 0$. We claim that $\{f_n\}_1^\infty$ is uniformly essentially bounded. Otherwise, for each n there exists k_n such that $|f_{k_n}(x)| \geqq 2^{2n}$ for x in some set A_n with $\mu(A_n) > 0$. Letting $f = \sum (1/2^n) |f_{k_n}|$ we get

$$\|f\|_{s_1} \leqq \sum \frac{1}{2^n} \|f_{k_n}\|_{s_1} < \infty$$

while $f(x) \geqq 2^n$ for $x \in A_n$. Thus $f^{s_1} \in L_1$ but is not essentially bounded. By 3.5, this cannot happen. Thus there exists $M > 0$ such that $\|f_n\|_\infty \leqq M$ for all n. It follows that

$$\int |f_n|^{s_0} d\mu \leqq M^{s_0 - s_1} \int |f_n|^{s_1} d\mu \xrightarrow[n]{} 0 ,$$

so v_{s_0} is continuous.

As before it is clear that $\mathscr{W}^P(S) \subset \mathscr{W}^P(\hat{S})$ and to show equality it suffices to show that each v_t is $\mathscr{W}^P(S)$, $\mathscr{W}_t$-continuous, $t \in \hat{S}$. If X is μ-σ-finite or $1 \notin S$ then the dual of $L^P(S)$ is $L^I(S')$ under both $\mathscr{W}^P(S)$ and $\mathscr{S}^P(S)$. Also $\mathscr{W}^P(S) = \sigma(L^P(S), L^I(S'))$. The $\mathscr{W}^P(S)$, $\mathscr{W}_t$-continuity of each v_t now follows from their $\mathscr{S}^P(S)$, $\mathscr{S}_t$-continuity (cf. [3, p. 39]). If X is not μ-σ-finite and $1 \in S$, then it is not clear that $L^P(S)$ has the same dual under both $\mathscr{W}^P(S)$ and $\mathscr{S}^P(S)$, so this argument does not apply. In this case we take a typical subbasic $\mathscr{W}^P(\hat{S})$-neighborhood of 0, say

$$V(g) = \{f \in L^P(S) : |< f, g >| < 1\}$$

where $g \in L_{t'}(\mu)$, $t' \in (\hat{S})'$. If $t' = \infty$, then $t' \in S'$ and $V(g) \in \mathscr{W}^P(S)$. For $t' \neq \infty$ take $t \in \hat{S}$ such that $1/t + 1/t' = 1$. Suppose there exists $t_1 \in S$ such that $1 \leqq t \leqq t_1$. Let $A_1 = \{x : |g(x)| \geqq 1\}$, $A_2 = \{x : |g(x)| < 1\}$ and $g_i = g\chi_{A_i}$, $i = 1, 2$. Then $g = g_1 + g_2$, $g_1 \in L_{t_1'}(\mu)$ and $g_2 \in L_\infty(\mu)$ giving that

$$V' = \{f \in L^P(S) : | < f, g_i > | < \frac{1}{2}, i = 1, 2\}$$

is in $\mathscr{W}^P(S)$. But $V' \subset V(g)$ so $V(g) \in \mathscr{W}^P(S)$. If such a t_1 does not exist then 3.4(ii) must hold. In this case g is essentially bounded so $g \in L_\infty(\mu)$ and again $V(g) \in \mathscr{W}^P(S)$. It follows that $\mathscr{W}^P(S) = \mathscr{W}^P(\hat{S})$.

(e) As before it suffices to show that $\mathscr{S}^I(S) = \mathscr{S}^I(\tilde{S})$ and $\mathscr{W}^I(S) = \mathscr{W}^I(\tilde{S})$. Let us show the former, for example. $\mathscr{S}^I(\tilde{S})$ is a locally convex topology for $L^I(S)$ such that

$$u_t : (L_t(\mu), \mathscr{S}_t) \longrightarrow (L^I(S), \mathscr{S}^I(\tilde{S}))$$

is continuous for all $t \in \tilde{S}$. As $S \subset \tilde{S}$ each $u_t, t \in S$, is continuous when $L^I(S)$ has the topology $\mathscr{S}^I(\tilde{S})$. But $\mathscr{S}^I(S)$ is the finest locally convex topology for $L^I(S)$ such that u_t, $t \in S$, is continuous. Hence $\mathscr{S}^I(S) \supset \mathscr{S}^I(\tilde{S})$. But since $\tilde{S} \supset S$ a direct comparison of the basic neighborhoords of 0 gives $\mathscr{S}^I(\tilde{S}) \supset \mathscr{S}^I(S)$ (cf., [3, p. 79]).

THEOREM 3.7. *Let $(X, \mathscr{A}, \mu)$ be a measure space, $S \subset [1, \infty]$, and let* (i), (ii) *refer to the conditions of Definition* 3.4.

(a) *If* (i), (ii) *hold, $\mathscr{S}^P(S)$ is normable.*

(b) *If* (i) *holds and $\infty \in S$, $\mathscr{S}^P(S)$ is normable.*

(c) *If* (ii) *is false and $\infty \notin S$, $\mathscr{S}^P(S)$ is normable if and only if $\hat{S}$ is closed and bounded.*

(d) *In all other cases $\mathscr{S}^P(S)$ is normable if and only if* lub $S \in S$.

Proof. If $t \in S$, $\in > 0$ we let

$$V(t, \in) = \{f \in L^P(S) : ||f||_t < \in\} .$$

We apply frequently, below, the considerations which occur in the proof of 3.6 (d).

(a) If (i), (ii) hold and $\infty \notin S$, then the considerations of the proof of 3.6(d) show that $|| \ ||_1$ generates $\mathscr{S}^P(S)$, which is therefore normable. If $\infty \in S, \mathscr{S}^P(S)$ is generated by $|| \ ||_1$ and $|| \ ||_\infty$. In this case $V(1, 1) \cap V(\infty, 1)$ is a bounded $\mathscr{S}^P(S)$-neighborhood of 0 and $\mathscr{S}^P(S)$ is normable.

(b) In this case $\mathscr{S}^I(S)$ is generated by $|| \ ||_1$ and $|| \ ||_\infty$ and hence normable.

(c) We suppose (ii) is false and $\infty \notin S$. If $\hat{S}$ is closed and bounded, say, $\hat{S} = [u, v]$, where $1 \leqq u \leqq v < \infty$, then $|| \ ||_u$, $|| \ ||_v$ generate $\mathscr{S}^P(S)$ (since $\{u, v\}^\wedge = \hat{S}$), which is therefore normable.

Now suppose (ii) is false, $\infty \notin S$ and $\mathscr{S}^P(S)$ is normable. We must show that $\hat{S}$ is closed and bounded. $\hat{S}$ is an interval with left and right end-points, u, v, say, where $1 \leqq u \leqq v \leqq \infty$. If (i) holds, then $u = 1 \in \hat{S}$. Suppose (i) is false and that $u \notin \hat{S}$. We show that in this case every $\mathscr{S}^P(S)$-neighborhood of 0 is unbounded, yielding a contradiction. Let $W = \bigcap_{i=1}^n V(t_i, \varepsilon_i)$ be an arbitrary basic $\mathscr{S}^P(S)$-neighborhood of 0. Then there exists $t \in S$ such that $u < t < t_i$, $i \leqq i \leqq n$. By 3.5 there exists a nonnegative $f \in \bigcap_{i=1}^n L_{t_i}(\mu)$ such that $f \notin L_t(\mu)$. There exists $\delta > 0$ such that $\|\delta f\|_{t_i} < \varepsilon_i$, $1 \leqq i \leqq n$. Set $g = \delta f$. As f has μ-σ-finite support there exists a sequence $\{g_n\}$ of simple functions with μ-finite support such that $0 \leqq g_m \uparrow g$. Then $\{g_m\} \subset L^P(S)$ and $\|g_m - g\|_{t_i} \xrightarrow[m]{} 0$, $1 \leqq i \leqq n$. Thus $\{g_m\}_{m=N}^{\infty} \subset W$ for some N. But $g \notin L_t(\mu)$ and $0 \leqq g_m \uparrow g$ imply that $\|g_m\|_t \xrightarrow[m]{} \infty$. Consequently,

$$mV(t, 1) \not\supset W, \quad m = 1, 2, \cdots$$

and W is not bounded. This contradiction assures that $u \in \hat{S}$.

We now argue that $\infty > v \in \hat{S}$ in an analogous fashion, showing that the contrary would imply that no $\mathscr{S}^P(S)$-neighborhood of 0 is bounded. For suppose $v \notin \hat{S}$. Let $W = \bigcap_{i=1}^n V(t_i, \varepsilon_i)$ be an arbitrary basic $\mathscr{S}^P(S)$-neighborhood of 0. Then there exists $t \in \hat{S}$ such that $t_i < t < v, i = 1, 2 \cdots$. By 3.5, there exists a non negative $f \in \bigcap_{i=1}^n L_{t_i}(\mu)$ such that $f \notin L_t(\mu)$. As before we take simple functions g_m with μ-finite support such that $0 \leqq g_m \uparrow \delta f$, for appropriate $\delta > 0$. One obtains $\{g_m\}_{m=N}^{\infty} \subset W$ for some N while $\|g_m\|_t \to \infty$, showing that W is not bounded, a contradiction. This completes the proof of (c). (d) Suppose (ii) is false and $\infty \in S$. If (i) is true, case (b) applies so we may suppose (i) is false. Let $s_1 = \operatorname{lub} S$. If $s_1 \in S$, it is not hard to show from the considerations in the proof of 3.6 (d) that $\| \ \|_{s_1}, \| \ \|_\infty$ generate $\mathscr{S}^P(S)$ so $\mathscr{S}^P(S)$ is normable. On the other hand suppose $\mathscr{S}^P(S)$ is normable. One can show $s_1 \in S$ by the argument which occurs in the second paragraph of the proof of (c). This works even if some $t_i = \infty$ because the function f obtained from the proof of 3.5 is essentially bounded and $\|g_n\|_\infty \leqq \|g\|_\infty$. (If all t_i's are ∞, f may still be chosen to have μ-σ-finite support by the negation of 3.5(i)''.)

Suppose (ii) is true and again let $s_1 = \operatorname{lub} S$. By (a) we may as well suppose (i) is false. Suppose $s_1 \in S$. If $\infty \notin S$, $\| \ \|_{s_1}$ generates $\mathscr{S}^P(S)$ and if $\infty \in S$, $\| \ \|_{s_1}$ and $\| \ \|_\infty$ generate $\mathscr{S}^P(S)$ so, in either case, $\mathscr{S}^P(S)$ is normable. If $\mathscr{S}^P(S)$ is assumed normable, one argues that $s_1 \in S$ as above. This completes the proof of 3.7.

The conditions of 3.7 also determine when $(L^I(S), \mathscr{S}^I(S))$ is metrizable for the case when $S \subset (1, \infty)$. For then $(L^I(S), \mathscr{S}^I(S))$ is the strong dual of the metrizable space $(L^P(S'), \mathscr{S}^P(S'))$ and so is metrizable if and only if the latter space is normable. In this case, of course, $(L^I(S), \mathscr{S}^I(S))$ is normable.

THEOREM 3.8. *Take* $S \subset (1, \infty)$. $B \subset L^I(S)$ *is* $\mathscr{S}^I(S)$*-bounded if and only if there exists a finite set* $F \subset S$ *such that* $B \subset L^I(F)$ *and* B *is bounded in the norm of* $(L^I(F), \mathscr{S}^I(F))$.

Proof. Let "0" denote polar and let $V(t, \varepsilon) \subset L^P(S')$ be as in the proof of 3.7. As B is bounded, there exists $\varepsilon > 0$ and $t_1', \cdots, t_n' \in S'$ such that $B^0 \supset \bigcap_{i=1}^n V(t_i', \varepsilon)$. $(L^P(\{t_i'\}), \mathscr{S}^P(t_i'\}))$ is normable (by 3.7) with norm $\| \ \| = \sup \{\| \ \|_{t_i'} : 1 \leqq i \leqq n\}$. Letting $1/t_i + 1/t_i' = 1$, $1 \leqq i \leqq n$, we set $F = \{t_1, \cdots, t_n\}$ so that $F \subset S$. $\mathscr{S}^I(F)$ is normable by

$$\| y \| = \sup \{ | < x, y > | : x \in L^P(F'), \| x \| \leqq 1 \} .$$

Also

$$B \subset B^{00} \subset [\bigcap_{i=1}^n V(t_i', \varepsilon)]^0$$

$$= \{y \in L^I(S) : \sup_{\substack{x \in L^P(S') \\ \| x \|_{t_i'} < 1 \\ 1 \leqq i \leqq n}} | < y, x > | \leqq \varepsilon\}$$

$$(*) \qquad \subset \{y \in L^I(S) : \sup_{\substack{x \in L^P(F') \\ \| x \| \leqq 1}} | < y, x > | \leqq \varepsilon\}$$

because $L^P(S')$ is dense in $(L^P(F'), \mathscr{S}^P(F'))$ (e.g., $L^P(S')$ contains the simple functions of μ-finite support). But the right-hand set in $(*)$ clearly consists of elements from the dual of $(L^P(F'), \mathscr{S}^P(F'))$. Hence

$$B \subset \{y \in L^I(F) : \| y \| \leqq \in \} ,$$

establishing the necessity of the condition for B to be bounded. The sufficiency is clear since the natural imbedding $i : (L^I(F), \mathscr{S}^I(F)) \longrightarrow (L^I(S), \mathscr{S}^I(S))$ is continuous.

Suppose that X is μ-finite so that 3.4 (i) holds and $L^I(S)$ is an inductive limit in the usual sense. Then it is known that a sequence $\{x_n\}_1^\infty$ $\mathscr{S}^I(S)$-converges to 0 if and only if for some $s \in S$ $\| x_n \|_s \xrightarrow[n]{} 0$ ([2, p. 454]). We do now know if this is true in the general case.

4. The weak topology in $L^I(S)$. If $L^I(S)$ and $L^P(S')$ are dual we have seen that $\sigma(L^I(S), L^P(S')) \subset \mathscr{W}^I(S)$. In this section we attempt to compare these two topologies. Our results indicate that they are quite close. However even in the case when S consists of two elements and $(X, \mathscr{A}, \mu)$ is the real line with Lebesgue measure we do not know if they are equal.

THEOREM 4.1. *Take* $S \subset (1, \infty)$ *and suppose that either* 3.4 (i) *or* (ii) *holds. Then a bounded net in* $L^I(S)$ *converges to zero in* $\mathscr{W}^I(S)$ *if and only if it converges to zero in* $\sigma(L^I(S), L^P(S'))$. *By bounded is meant with respect to any topology of the dual pair* $(L^I(S), L^P(S'))$.

Proof. Let $(\phi_d, d \in D, \geqq)$ be a bounded net in $L^I(S)$ converging to zero in $\sigma(L^I(S), L^I(S'))$. To show that $\phi_d \longrightarrow 0$ in $\mathscr{W}^I(S)$ we shall show that for some $t' \in S'$ $< \phi_d, h > \underset{d}{\longrightarrow} 0$ for all $h \in L_{t'}(\mu)$. (One may then apply the corollary in [3, p. 79]). By 3.8, there is a finite set $F \subset S$ such that $\{\phi_d\}_{d \in D} \subset L^I(F)$ and is bounded there. By 3.4 (i) and (ii) we may suppose $F = \{t\}$. There exists $M > 0$ such that $\|\phi_d\|_t \leqq M$ for all $d \in D$.

Now take any $h \in L_{t'}(\mu)$ and $\varepsilon > 0$. The support A of h is μ-σ-finite so we write $A = \bigcup_{n=1}^{\infty} A_n$, where $\mu(A_n) < \infty$, $n = 1, 2, \cdots$. We may suppose $A_n \subset A_{n+1}$, $n = 1, 2, \cdots$. Let $B_n = \{x \in X : |h(x)| \leqq n\}$. Finally let $C_n = A_n \cap B_n$ so that $\bigcup_{n=1}^{\infty} C_n = A$ and $h\chi_{C_n} \in L^P(S')$ for all n. Take N so large that

$$\int_{X \sim C_n} |h|^{t'} d\mu < \left(\frac{\varepsilon}{2M}\right)^{t'} .$$

As $h\chi_{C_n} \in L^P(S')$, there exists $d_0 \in D$ such that

$$|< \phi_d, h\chi_{C_n} >| < \varepsilon/2 \text{ for all } d \geqq d_0 .$$

Thus for $d \geqq d_0$ we have

$$\begin{aligned} |< \phi_d, h >| &\leqq |< \phi_d, h\chi_{C_n} >| + \left|\int_{X \sim C_n} \phi_d h d\mu\right| \\ &< \varepsilon/2 + \|\phi_d\|_t \, \|h\chi_{X \sim C_n}\|_{t'} \\ &< \varepsilon/2 + M\frac{\varepsilon}{2M} = \varepsilon . \end{aligned}$$

Thus, when 3.4 (i) or (ii) hold, $\sigma(L^I(S), L^P(S'))$ agrees with $\mathscr{W}^I(S)$ on bounded sets. In particular, the same sequences converge in $\sigma(L^I(S), L^P(S'))$ as in $\mathscr{W}^P(S)$.

THEOREM 4.2. *Take* $S \subset [1, \infty)$ *and suppose that either* X *is* μ-σ-*finite or that* $1 \notin S$. *Let* $(\phi_d, d \in D_1 \geqq)$ *be a net of nonnegative* *functions in* $L^I(S)$. *The* $\phi_d \to 0$ *in* $\sigma(L^I(S), L^P(S'))$ *if and only if* $\phi_d \to 0$ *in* $\mathscr{W}^I(S)$.

Proof. Assuming that $\phi_d \to 0$ in $\sigma(L^I(S), L^P(S'))$, we shall show that $\phi_d \to 0$ in $\mathscr{W}^I(S)$. By applying 3.6 we may assume S is countable, say $S = \{t_1, t_2, \cdots\}$. If $t \in [1, \infty)$, $t' \in (1, \infty]$ denotes the conjugate of t. For each $t \in S$ let $F_{t'}$ be a finite set in $L_{t'}(\mu)$. Let

$$V_t = \{g \in L_t(\mu) : \ | < g, f > | < 1 \text{ for all } f \in F_{t'}\} .$$

Let V be the convex hull in $L^I(S)$ of $\bigcup_{t \in S} V_t$ so that V is a typical basic $\mathscr{W}^I(S)$-neighborhood of 0. To prove the theorem we shall show that ϕ_d is eventually in V. For each $i = 1, 2, \cdots$ let

$$f_i = \max\{(Rf)^+, (Rf)^-, (If)^+, (If)^- : f \in F_{t'_i}\} .$$

Then $f_i(x) \geqq 0$ for all x. It suffices to show that there exists $d_0 \in D$ such that the following holds:

(1) For all $d \geqq d_0$ there exists positive $c_1, \cdots, c_m$ $(m = m(d))$ such that

$$\sum c_i = 1 \text{ and } \phi_d = \sum_{i=1}^{m} c_i \psi_i, \text{ where } \psi_i \in L^I(S) \text{ and}$$

$$0 \leqq < \psi_i, f_{k_i} > < \frac{1}{4} ,$$

$1 \leqq i \leqq m$ and suitable integers k_i .

To prove (1) set $g_i = \min\{f_k : 1 \leqq k \leqq i\}$, $i = 1, 2, \cdots$. Then $0 \leqq g_i(x)$ for all $x \in X$ and $g_i \in \bigcap_{1 \leqq k \leqq i} L_{t'_k}(\mu)$. Also $g_i \downarrow g$ where $g \in L^P(S')$. There exists $d_0 \in D$ such that

$$0 \leqq < \phi_d, g > < \frac{1}{8} \quad \text{for all } d \geqq d_0 .$$

We shall show that this is the d_0 of (1). Take any $d \geqq d_0$. $\phi_d \in L_{t_j}(\mu)$ for some j so

$$0 \leqq < \phi_d, g_j > < \infty .$$

As $\phi_d g_i \downarrow \phi_d g$ as $i \to \infty$, we have

(2) $$0 \leqq < \phi_d, g_n > < \frac{1}{8} \text{ for some } n .$$

(Actually (2) holds for all n sufficiently large but we shall not use this.)

Now set

$$A_1 = \{x \in X : f_1(x) \leqq f_i(x), 1 \leqq i \leqq n\}$$

$$A_2 = \{x \in X:\ x \notin A_1;\ f_2(x) \leqq f_i(x),\ 1 \leqq i \leqq n\}$$
$$\vdots$$
$$A_n = \{x \in X:\ x \notin \bigcup_{i=1}^{n-1} A_i;\ f_n(x) \leqq f_i(x),\ 1 \leqq i \leqq n\}\ .$$

Then $g_n = \sum_{i=1}^n f_i\chi_{A_i}$ and $\bigcup_{i=1}^n A_i = X$.

Suppose first that in (2) we have $< \phi_d, g_n > = 0$. Then

$$\int_{A_i} \phi_d f_i d\mu = 0, \quad 1 \leqq i \leqq n\ ,$$

so

$$\int_X [n\, \phi_d\, \chi_{A_i}] f_i d\mu = 0, \quad 1 \leqq i \leqq n\ . \tag{3}$$

Writing $\phi_d = \sum_{i=1}^n 1/n\ [n\, \phi_d \chi_{A_i}]$, combined with (3), proves (1) in this case.

Now suppose $< \phi_d, g_n > > 0$ in (2). By renumbering the f_i 's, if necessary, we may suppose

$$\int_{A_i} \phi_d f_i d\mu > 0, \quad 1 \leqq i \leqq m\ , \tag{4}$$

where $1 \leqq m \leqq n$, and

$$\int_{A_i} \phi_d f_i d\mu = 0, \quad m < i \leqq n\ . \tag{5}$$

We suppose $m < n$; the case $m = n$ is handled by an obvious adjustment. Set

$$\beta_i = \frac{\int_{A_i} \phi_d f_i d\mu}{2 < \phi_d, g_n >}, \quad 1 \leqq i \leqq m\ . \tag{6}$$

By (4), (5)

$$\sum_{i=1}^m \beta_i = \frac{1}{2}\ . \tag{7}$$

Now

$$\int_X \left[\frac{1}{\beta_i}\, \phi_d \chi_{A_i}\right] f_i d\mu = 2 < \phi_d, g_n >, \quad \text{by (6)} \tag{8}$$
$$< \frac{1}{4}, \quad \text{by (2)},\ 1 \leqq i \leqq m$$

Also, by (5),

$$\int_X [2(n-m)\phi_d \chi_{A_i}] f_i \, d\mu = 0, \; m < i \leqq n . \tag{9}$$

We write

$$\phi_d = \sum_{i=1}^{m} \beta_i \left[\frac{1}{\beta_i} \phi_d \chi_{A_i} \right] + \sum_{i=m+1}^{n} \frac{1}{2(n-m)} [2(n-m)\, \phi_d \chi_{A_i}] .$$

This, combined with (7), (8) and (9), proves (1).

Let $\{\phi_d\}$ be a net in $L^I(S)$ which converges to 0 in $\sigma(L^I(S), L^P(S'))$. If we know that $\{(R\phi_d)^{\pm}\}$ and $\{(I\phi_d)^{\pm}\}$ also converged to 0 in $\sigma(L^I(S), L^P(S'))$, then 4.2 would imply convergence of $\{\phi_d\}$ to 0 in $\mathscr{W}^I(S)$. We close with an example of a net in $L^I(S)$ to which neither 4.1 nor 4.2 are applicable even though 3.4(i) holds.

EXAMPLE 4.3. Let $X = [0, 2\pi)$ and let μ be Lebesgue measure so that 3.4(i) holds. Take $S = (p, \infty)$ for some $1 \leqq p < \infty$. Let D be the set of all finite subsets of $L^P(S')$ directed by inclusion. Take arbitrary $d \in D$ and let $|d|$ denote the cardinal of d. Take $n = n(d)$ so large that

$$\left| \int_0^{2\pi} (\sin nx)\, g(x)\, d\mu(x) \right| < \frac{1}{|d|^2}, \quad \text{for all } g \in d .$$

Define $\phi_d \in L^I(S)$ by $\phi_d(x) = |d| \sin nx$. To see that $\phi_d \to 0$ in $\sigma(L^I(S), L^P(S'))$ take any $g \in L^P(S')$ and $\varepsilon > 0$. Let $d_0 \in D$ be such that $g \in d_0$ and d_0 contains at least $1/\varepsilon$ elements. Then

$$|< \phi_d, g >| < \frac{1}{|d|} < \varepsilon \text{ for all } d \supset d_0 .$$

On the other hand, $< \phi_d^+, 1 > \; = \; < \phi_d^-, 1 > \; = 2|d|$ for all $d \in D$ so $\phi_d^{\pm} \not\to 0$ in $\sigma(L^I(S), L^P(S'))$. Finally for any $t \in S$ $\|\phi_d\|_t \geqq \|\phi_d\|_1 = 4|d|$ for all $d \in D$ so $\{\phi_d\}$ is not eventually bounded. We do not know whether or not $\{\phi_d\}$ converges to zero in $\mathscr{W}^I(S)$.

REFERENCES

1. E. Hewitt and K. Stromberg, *Real and abstract analysis*, New York, 1965.
2. L. V. Kantorovich and G. P. Akilov, *Functional analysis in normed spaces*, New York, 1964.
3. A. P. Robertson and W. Robertson, *Topological vector spaces*. Cambridge, 1966.
4. H. L. Royden, *Real analysis*, New York, 1968.
5. H. H. Schaefer, *Topological vector spaces*, New York, 1966.

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