PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 1, 1970

A CHARACTERIZATION OF THE NIL RADICAL OF A RING

WILLIAM J. WICKLESS

Let R be a ring and S a subring of R. Let φ be a ring homomorphism mapping S onto a division ring Γ. Choose an ideal $P \subsetneqq R$ maximal with respect to the property $(P \cap S)^{\phi} = (0)$. P is a prime ideal of R. If P is any prime ideal of R which can be obtained in the above manner write $P = P(\Gamma, S, \varphi)$.

It is shown that all primitive ideals are of the form $P = P(\Gamma, S, \varphi)$ and that a ring R is nil if and only if it has no prime ideals of the form $P = P(\Gamma, S, \varphi)$. It is shown that the nil radical of any ring is the intersection of all prime ideals $P = P(\Gamma, S, \varphi)$.

It is shown that if $P = P(\Gamma, S, \varphi)$ for all prime ideals $P \subsetneqq R$ then the nil and Baer radicals coincide for all homomorphic images of R. If the nil and Baer radicals coincide for all homomorphic images of R, it is shown that any prime ideal P of R is contained in a prime ideal $P' = P'(\Gamma, S, \varphi)$.

Finally, by consideration of prime ideals $P = P(\Gamma, S, \varphi)$, two theorems are proved giving information about rings satisfying very special conditions.

2. Certain prime ideals in rings. Let R be any ring and S a subring of R. Suppose φ is a ring homomorphism mapping S onto a division ring Γ. We may choose an ideal $P \subsetneqq R$ maximal with respect to the property $(P \cap S)^{\varphi} = (0)$. It is an easy exercise to check that P will be a prime ideal of R. If P is any prime ideal of R which is a maximal ideal such that $(P \cap S)^{\varphi} = (0)$ for some subring $S \subsetneqq R$ and some ring homomorphism $\varphi: S \to \Gamma$, Γ a division ring, we write $P = P(\Gamma, S, \varphi)$. Throughout, for any ring R, we let $J(R)$, $N(R)$, $\beta(R)$ denote respectively the Jacobson, nil, and Baer radicals of R. We start with the following simple fact.

THEOREM 1. *Let R be a ring and P a primitive ideal of R. Then $P = P(\Gamma, S, \varphi)$.*

Proof. Let $P = (0: M)$ for some simple right R module M. Let Γ be the centralizer of M. Γ is a division ring. As R/P is primitive it is well known ([3], Th. 3, p. 33) that there exists a subring $S' \subsetneqq R/P$ and a homomorphism $\varphi': S' \to \Gamma$. It is easy to check $P = P(\Gamma, S, \varphi)$ with $S = (S')\pi^{-1}$, $\varphi = \pi\varphi'$, π the natural map from R onto R/P.

We next consider the structure of rings which have no prime ideals of the form $P = P(\Gamma, S, \varphi)$.

THEOREM 2. *A ring R is nil if and only if it has no prime ideals P of the form $P = P(\Gamma, S, \varphi)$.*

Proof. If R is nil then every subring $S \subseteqq R$ is nil and cannot be mapped onto a division ring. Thus, R has no prime ideals of the form $P(\Gamma, S, \varphi)$.

Now assume R has no prime ideal of the form $P(\Gamma, S, \varphi)$. This requires that every subring $S \subseteqq R$ is a Jacobson radical ring, for if S is any ring with $J(S) \neq S$, we can find a subring $S' \subseteqq S$ which can be mapped homomorphically onto a division ring Γ-let Γ be the centralizer of a simple S module for example.

We now show if R is a ring such that $J(S) = S$ for all subrings S then R is nil. We wish to thank Professor S. A. Amitsur for the following simple proof of this fact. Let $u \in R$ and $\langle u \rangle$ denote the subring of R generated by u. We know $J(\langle u \rangle) = \langle u \rangle$. Let $\langle u \rangle^*$ denote the ring $\langle u \rangle$ with an identity adjoined in the usual way. Now $\langle u \rangle^*$ is a homomorphic image of $Z[x]$, the ring of polynomials in an indeterminate x with integral coefficients. By a result of Goldman ([2], Th. 3), we know that the Jacobson radical of any homomorphic image of $Z[x]$ is nil. Thus $J(\langle u \rangle^*)$ is nil, and $\langle u \rangle = J(\langle u \rangle) = J(\langle u \rangle^*) \cap \langle u \rangle$ is nil. Thus u is nilpotent. As u was an arbitrary element of R we have R is nil. This proves the theorem.

We now obtain a result about the nil radical of an arbitrary ring.

THEOREM 3. *For any ring R, $N(R) = \bigcap_{\alpha \in T} P_\alpha$, where $\{P_\alpha \mid \alpha \in T\}$ is the set of all prime ideals of R of the form $P = P(\Gamma, S, \varphi)$.*

Proof. Let $P = P(\Gamma, S, \varphi)$ be any prime ideal of the above type. As $N(R)$ is nil, it is easy to check that we have $[(N(R) + P) \cap S]^\varphi = (0)$. As P was a maximal ideal in R such that $(P \cap S)^\varphi = (0)$, we must have $N(R) \subseteqq P$. Thus $N(R) \subseteqq \bigcap_{\alpha \in T} P_\alpha$.

We now show $x \notin N(R) \rightarrow x \notin \bigcap_{\alpha \in T} P_\alpha$. Let $x \notin N(R)$. Then (x), the ideal generated by x in R, is not nil. By Theorem 2 we have $S \subseteqq (x)$ and $\varphi: S \rightarrow \Gamma$, S a subring of (x), Γ a division ring, φ a ring homomorphism onto. Let $P = P(\Gamma, S, \varphi)$. Clearly $P \in \{P_\alpha \mid \alpha \in T\}$ and $x \notin P$. This proves the theorem.

We now wish to consider rings in which all prime ideals are of the form $P = P(\Gamma, S, \varphi)$. We obtain the following partial result.

THEOREM 4. *Let R be a ring such that P prime in $R \rightarrow P = P(\Gamma, S, \varphi)$. Then for all ideals $I \subseteqq R$ we have $N[R/I] = \beta[R/I]$. If $N[R/I] = \beta[R/I]$ for all ideals $I \subseteqq R$ we have P prime in $R \rightarrow P \subseteqq P'(\Gamma, S, \varphi)$.*

Proof. Let R be such that P prime in $R \rightarrow P = P(\Gamma, S, \varphi)$. Let I be any ideal of R. We first note there is a one-to-one correspondence between all prime ideals $P/I = P/I(\Gamma, S, \varphi)$ of the ring R/I and all prime ideals of the form $P(\Gamma, S, \varphi)/I$ in R/I where $P(\Gamma, S, \varphi)$ is a prime ideal in R containing I. Let $P/I = P/I(\Gamma, S, \varphi)$ where S is a subring of R/I. Write S as S'/I for S' a subring of R. Then $P/I(\Gamma, S, \varphi) = P(\Gamma, S', \pi\varphi)/I$ where π is the natural homomorphism mapping S' onto S. Conversely, if $P = P(\Gamma, S, \varphi)$ is a prime ideal of R containing I then $P(\Gamma, S, \varphi)/I = P/I(\Gamma, S + I/I, \lambda\varphi')$ where λ is the natural homomorphism from S onto $S + I/I$ and $\varphi': S + I/I \rightarrow \Gamma$ is given by $(s + I)^{\varphi'} = s^{\varphi}$.

Thus we have: $N[R/I] = \bigcap_{\alpha}[P/I(\Gamma, S, \varphi)]_{\alpha} = \bigcap_{\alpha}P(\Gamma, S, \varphi)_{\alpha}/I = \beta[R/I]$. (Recall, by our assumption on R, $\{P(\Gamma, S, \varphi)_{\alpha} \supseteqq I\}$ is the set of all prime ideals of R containing I.)

To prove the second statement of our theorem let $N[R/I] = \beta[R/I]$ for all I and let P be any prime ideal of R. We have $N[R/P] = \beta[R/P] = (0)$, thus, by Theorem 2, R/P has a prime ideal $P' = P'(\Gamma, S, \varphi)$. We have $P \subseteqq P'$, which finishes the proof of the theorem.

We conclude by proving two theorems about rings satisfying very special conditions. If $P = P(\Gamma, S, \varphi) \subseteqq R$, we may extend P to a maximal right ideal T such that $(T \cap S)^{\varphi} = (0)$. T will be a prime right ideal in the sense that if U is a right ideal of R, $U \nsubseteqq T$ and $x \in R$ with $Ux \subseteqq T$, then $x \in T$. (This is weaker than the usual definition of prime right ideal which requires $x = 0$.) We have the following theorem.

THEOREM 5. *If R is a ring such that every prime right ideal is two sided, then every nil right ideal of R is contained in $N(R)$.*

Proof. Let A be a nil right ideal of R with $A \nsubseteqq N(R)$. Then $A + RA$ is not nil and thus, by Theorem 2, contains a subring S which may be mapped homomorphically onto a division ring Γ by a map φ. As A is nil, we have $(A \cap S)^{\varphi} = (0)$. We may extend A to a maximal right ideal T such that $(T \cap S)^{\varphi} = (0)$. By the assumption of our theorem we know T is two sided. But then, $R + RA \subseteqq T$, a contradiction.

THEOREM 6. *Let R be a ring such that if S is a subring of*

R, I an ideal of S, then there exist T an ideal of R such that $T \cap S = I$. Then $J(R)$ is nil.

Proof. It is enough to show that the ring $J = J(R)$ contains no subrings S which can be mapped by a ring homomorphism φ onto a division ring Γ. Assume that S is such a subring. Consider in the ring J a prime ideal $P = P(\Gamma, S, \varphi)$.

Now J/P contains the subring $S + P/P$ which can be mapped onto Γ by $\pi\varphi$ where π is the natural map from S to $S + P/P$. It is easy to check that the ring J inherits the condition of our theorem. Therefore, as P was maximal in J such that $(P \cap S)^{\varphi} = (0)$, we must have Kernel $\pi\varphi = (0)$. Thus $S + P/P \cong \Gamma$, a contradiction since J/P is a radical ring. Thus J is nil.

REFERENCES

1. N. J. Divinsky, *Rings and Radicals*, Univ. of Toronto Press, Toronto, 1965.
2. O. Goldman, *Hilbert rings and the Hilbert nullstellensatz*, Math. Z. **54** (1951).
3. N. Jacobson, *Structure of rings*, A.M.S. Coll. Publ. 37, Amer. Math. Soc., Providence, 1964.

Received November 3, 1969. The preparation of this paper was supported in part by NSF Grant #GP-13164.

UNIVERSITY OF CALIFORNIA, LOS ANGELES