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TORSION-FREE AND DIVISIBLE MODULES OVER MATRIX RINGS

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A short exact sequence $0 \to K \to F \to E \to 0$ of left modules over a ring A is 1-pure if $aK = K \cap aF$ for all $a \in A$, and *pure* if for any right A-module M, the map $M \otimes K \to M \otimes F$ is injective. A module E is torsion free (Hattori) if its presence on the right forces 1-purity, and flat if it forces purity. Similarly, we have on the left the notions of divisibility (Hattori) and absolute purity. Considering the functor $E \to E^n$ taking A-modules to modules over the matrix ring $M_n(A)$, a sequence is called *n-pure* if its image under this functor is 1-pure; *n-torsion-free* and *n-divisible* modules are similarly defined. It is shown that purity, flatness, and absolute purity, respectively, are equivalent to the requirement that n-purity, n-torsion-freeness, and n-divisibility should hold for all n. n-divisibility and absolute purity are preserved under direct sums, products and certain inductive limits; n-torsion-freeness and flatness under direct sums and inductive limits, but not products. A condition is given guaranteeing that products of at most a given cardinality preserve n-torsion-freeness. It is shown that if every left ideal of A is generated by at most n elements, then n-torsion-freeness is equivalent to flatness. The behavior of these properties under localization is studied, and it is shown that if A is locally a domain then the two notions of purity agree if and only if w. gl. dim. $(A) \leqq 1$.

A will always denote a ring with identity; all modules will be unitary and left modules unless otherwise stated. If no confusion can arise $F \otimes E$ will mean $F \otimes_A E$; similarly for $\mathrm{Hom}(F, E)$, $\mathrm{Tor}_m(F, E)$, and $\mathrm{Ext}^m(F, E)$.

1. Matrices. For a positive integer n, let $M_n(A)$ denote the ring of $n \times n$ matrices over A (we shall sometimes use $B = M_n(A)$ for convenience of notation) and $M_n(E)$ the left $M_n(A)$-module of $n \times n$ matrices over E, where scalar multiplication looks like usual matrix multiplication. Let $e_{ij} \in M_n(A)$ be the matrix having 1 in the (i, j) position and zeros elsewhere.

When considering E^n as a left $M_n(A)$-module, it is convenient to think of the elements as "column vectors", so we will denote an n-tuple of E^n as $(x_1, x_2, \cdots, x_n)'$, the prime denoting transpose. Note that $M_n(E)$ is a direct sum (as $M_n(A)$-modules) of n copies of E^n.

The well-known theorem of Morita states that the category of left A-modules is category isomorphic to the category of left $M_n(A)$-modules by the (exact) functor $E \to E^n$. (The inverse functor is $M \to e_{11}M$ for M a left $M_n(A)$-module. The A-modules $e_{ii}M$, $i = 1, \cdots n$, are all isomorphic and $M \cong (e_{11}M)^n$ by the map $x \to (e_{11}(e_{11}x), e_{11}(e_{12}x), \cdots, e_{11}(e_{1n}x))'$. Also for E a left A-module, $E \cong e_{11}E^n$.) The following result is thus obvious.

PROPOSITION 1.1. *E is an injective left A-module if and only if E^n is an injective left $M_n(A)$-module (if and only if $M_n(E)$ is an injective left $M_n(A)$-module).*

PROPOSITION 1.2. *For E a right A-module and F a left A-module, $E^n \otimes_B F^n \cong E \otimes_A F$ (as groups). In the situation $({}_DE_A, {}_AF_C)$ this is an isomorphism of $(M_n(D), M_n(C))$-bimodules.*

Proof. Note that $E^n \otimes_B F^n$ is generated by the elements of the form $(0, \cdots, 0, x_i, 0, \cdots) \otimes (0, \cdots, y_j, 0, \cdots 0)'$ for $x_i \in E$, $y_j \in F$, and x_i appearing as the i^{th} coordinate, y_j as the j^{th}. But

$$\begin{aligned}(0, \cdots, x_i, \cdots, 0) \otimes (0, \cdots, y_j, \cdots, 0)' &= (0, \cdots, x_i, \cdots, 0) \otimes e_{ji}(y_j, 0, \cdots, 0)' \\ &= (0, \cdots, x_i, \cdots, 0)\, e_{ji} \otimes (y_j, 0, \cdots, 0)' \\ &= (\delta_{ij}x_i, 0, 0, \cdots, 0) \otimes (y_j, 0, \cdots, 0)',\end{aligned}$$

where δ_{ij} is the Kronecker delta. Thus the elements $(x, 0, \cdots, 0) \otimes (y, 0, \cdots, 0)'$, for $x \in E$, $y \in F$, generate $E^n \otimes_B F^n$.

There is a group homomorphism $E^n \otimes_B F^n \to E \otimes_A F$ such that $(x_1, x_2, \cdots, x_n) \otimes (y_1, y_2, \cdots, y_n)' \to \sum_{i=1}^n x_i \otimes y_i$. It is clear from the above that the inverse of this is the map $E \otimes_A F \to E^n \otimes_B F^n$ such that $x \otimes y \to (x, 0, \cdots, 0) \otimes (y, 0, \cdots, 0)'$.

The final assertion follows from the form of the isomorphism.

Since Tor is categorically defined, we have

COROLLARY 1.3. $\operatorname{Tor}_m^B(E^n, F^n) \cong \operatorname{Tor}_m^A(E, F)$.

COROLLARY 1.4. w.gl. dim A = w.gl. dim $M_n(A)$.

2. Purity. We shall call a short exact sequence

$$(*) \qquad 0 \longrightarrow K \longrightarrow F \longrightarrow E \longrightarrow 0$$

of left A-modules 1-*pure* if $aF \cap K = aK$ for every $a \in A$; that is, if and only if $A/aA \otimes K \to A/aA \otimes F$ is an injection for every $a \in A$. Also $(*)$ is *pure* (or K is a *pure submodule* of F) if for every right

A-module G, the map $G \otimes K \to G \otimes F$ is an injection. (This latter definition is due to P. M. Cohn [5]. In [19], Warfield uses "RD-pure" for our "1-pure".) Every pure exact sequence is clearly 1-pure. Note that it is sufficient to consider only finitely generated (presented) modules G in the definition of pure, since any module is a direct limit of finitely generated (presented) modules and tensor products commute with direct limits.

For a positive integer m, we say that $(*)$ is *m-pure* if $0 \to K^m \to F^m \to E^m \to 0$ is a 1-pure extension of $M_m(A)$-modules. Note that this is equivalent to requiring 1-purity of $0 \to M_m(K) \to M_m(F) \to M_m(E) \to 0$.

PROPOSITION 2.1. *If $n \geqq m$ and $(*)$ is n-pure, then $(*)$ is m-pure.*

Proof. For $a \in M_m(A)$, show that $aF^m \cap K^m = aK^m$ by imbedding m-tuples in n-tuples, using the n-purity of $(*)$, and then extracting the needed m-tuples.

THEOREM 2.2. *$(*)$ is pure if and only if $(*)$ is n-pure for all $n \geqq 1$.*

Proof. First suppose that $(*)$ is pure and let n be a positive integer, G a right $M_n(A)$-module. Then $G \cong M^n$ for a right A-module M. Since $(*)$ is pure, the sequence $0 \to M \otimes K \to M \otimes F$ is exact, so by Proposition 1.2, $0 \to M^n \otimes_B K^n \to M^n \otimes_B F^n$ is exact, or equivalently, $0 \to G \otimes_B K^n \to G \otimes_B F^n$ is exact. Thus $0 \to K^n \to F^n \to E^n \to 0$ is pure, so 1-pure, so $(*)$ is n-pure.

Conversely, we must show that $G \otimes K \to G \otimes F$ is an injection for any finitely presented right A-module G. However, for such a G there exist free right A-modules F_2 (on n generators) and F_1 (on m generators), with $n > m$, such that $F_2 \xrightarrow{\tau} F_1 \to G \to 0$ is exact. Let b be the $m \times n$ matrix associated with the homomorphism τ, and let $M_{m,n}(A)$ be the right $M_n(A)$-module of $m \times n$ matrices. Now $F_2^n \xrightarrow{\tau^n} F_1^n \to G^n \to 0$ is exact, $F_2^n \cong M_n(A)$, $F_1^n \cong M_{m,n}(A)$ (as right $M_n(A)$-modules) and the image of τ^n is carried under the isomorphism to $bM_n(A)$, so $G^n \simeq M_{m,n}(A)/bM_n(A)$. Let $\iota : M_{m,n}(A) \to M_n(A)$ be the $M_n(A)$-linear injection taking an $m \times n$ matrix a to the $n \times n$ matrix $\iota(a)$ obtained by tacking on $n - m$ rows of zeros. Noting that the $M_n(A)$-linear map $\alpha : M_{m,n}(A)/bM_n(A) \to M_n(A)/\iota(b)M_n(A)$ by $\alpha(a + bM_n(A)) = \iota(a) + \iota(b)M_n(A)$ has an abvious left inverse, we see that the vertical maps in the following commutative square are injections:

$$\begin{array}{ccc} M_{m,n}(A)/bM_n(A) \otimes K^n & \longrightarrow & M_{m,n}(A)/bM_n(A) \otimes F^n \\ \downarrow{\scriptstyle \alpha \otimes 1} & & \downarrow{\scriptstyle \alpha \otimes 1} \\ M_n(A)/\iota(b)M_n(A) \otimes K^n & \longrightarrow & M_n(A)/\iota(b)M_n(A) \otimes F^n . \end{array}$$

The n-purity of $(*)$ yields the injectivity of the lower map and thus of the upper map. Hence $G^n \otimes K^n \to G^n \otimes F^n$ is an injection, so $G \otimes K \to G \otimes F$ is an injection.

COROLLARY 2.3. *(Cohn's criterion for purity.) K is pure in F if and only if whenever $a_{ij} \in A$, $y_j \in F$, $x_i \in K$ $(i, j = 1, \cdots, n)$ and $x_i = \sum_{j=1}^{n} a_{ij} y_j$ for each i, then there are $z_j \in K$ $(j = 1, \cdots, n)$ such that $x_i = \sum_{j=1}^{n} a_{ij} z_j$ for each i.*

Note that in order to prove $(*)$ pure it is sufficient, by Proposition 2.1 and Theorem 2.2, to prove that $(*)$ is n-pure for $n \in J$, where J is a cofinal subset of the positive integers.

COROLLARY 2.4. *For a positive integer m, K is pure in F if and only if K^m is pure in F^m; i.e., $(*)$ is pure if and only if $0 \to K^m \to F^m \to E^m \to 0$ is pure.*

Proof. The necessity was shown in the above proof. For the sufficiency, note that $M_n(M_m(A)) \cong M_{mn}(A)$ (as rings) and $(E^m)^n \cong E^{mn}$ as modules over this ring. Now apply the theorem and the above remark.

That is, purity of an A-submodule is equivalent to purity of the equivalent $M_n(A)$-submodule. We shall see that the same cannot be said of 1-purity.

This corollary also says that if $0 \to K^m \to F^m \to E^m \to 0$ is pure for some $m \geqq 1$, it is pure for every $m \geqq 1$.

For a group G, let $M(G)$ denote the set of infinite matrices over G which are eventually zero except for a constant down the diagonal. Then $M(A)$ is a ring and $M(E)$ an $M(A)$-module.

For N the set of positive integers, let $E^{(N)}$ be the left $M(A)$-module of families $(x_n)_{n \in N}$ from E with finite support. Note that if $a \in M(A)$, there is an $n \in N$, a matrix $(a_{ij}) \in M_n(A)$, and an element $c \in A$ such that

$$a = \left(\begin{array}{c|c} a_{ij} & 0 \\ \hline 0 & C \end{array}\right)$$

where C is the infinite diagonal matrix $C = \operatorname{diag}(c, c, c, \cdots)$.

COROLLARY 2.5. *$(*)$ is pure if and only if $0 \to M(K) \to M(F) \to M(E) \to 0$ (respectively $0 \to K^{(N)} \to F^{(N)} \to E^{(N)} \to 0$) is a 1-pure extension of $M(A)$-modules.*

A left A-module P is said to be *RD-projective* (*pure-projective*) if for any 1-pure (pure) exact sequence $(*)$ of left A-modules, the induced map $\operatorname{Hom}(P, F) \to \operatorname{Hom}(P, E)$ is a surjection. Every *RD-*

projective module is pure-projective. Warfield has shown in [19] (where he defines the notion) that a module is RD-projective if and only if it is a direct summand of a direct sum of cyclically presented modules (after Kaplansky [14], we shall call an A-module *cyclically presented* if it is of the form A/Aa for some $a \in A$), and that a module is pure-projective if and only if it is a summand of a direct sum of finitely presented modules. In particular, any finitely presented module is pure-projective. The following result was suggested by Warfield.

PROPOSITION 2.6. *For a ring A, the following conditions are equivalent.*

(a) *Every 1-pure extension $(*)$ of left A-modules is pure.*

(b) *Every finitely presented left A-module is a summand of a direct sum of cyclically presented modules.*

(c) *Every finitely presented left A-module is RD-projective.*

(d) *Every pure-projective left A-module is RD-projective.*

(a′) *Every 1-pure extension $(*)$ of right A-modules is pure.*

(b′) *Every finitely presented right A-module is a summand of a direct sum of cyclically presented modules.*

(c′) *Every finitely presented right A-module is RD-projective.*

(d′) *Every pure-projective right A-module is RD-projective.*

Proof. By the remarks above, (b) $\Leftrightarrow$ (c), (d) $\Rightarrow$ (c), (b′) $\Leftrightarrow$ (c′) and (d′) $\Rightarrow$ (c′). Clearly (a) $\Rightarrow$ (d) and (a′) $\Rightarrow$ (d′).

To prove that (b′) $\Rightarrow$ (a), we must show that if $(*)$ is a 1-pure extension of left A-modules and H a finitely presented right A-module, then $0 \to H \otimes K \to H \otimes F$ is exact. But by (b′), $H \oplus L = G = \oplus_{i \in I} A/a_iA$, for $a_i \in A$, I some index set. Then $0 \to A/a_iA \otimes K \to A/a_iA \otimes F$ is exact for each $i \in I$ by hypothesis, so $0 \to \oplus (A/a_iA \otimes K) \to \oplus (A/a_iA \otimes F)$, or $0 \to G \otimes K \to G \otimes F$, is exact. That is, $0 \to (H \oplus L) \otimes K \to (H \oplus L) \otimes F$ is exact, so the desired conclusion follows. Similarly, (b) $\Rightarrow$ (a′).

For example, Warfield has shown in [19] that if A is a valuation ring then condition (b) is satisfied. This leads us to localization.

If S is a multiplicatively closed subset of the center of A (with $0 \notin S, 1 \in S$), let $S^{-1}A$ denote the ring of fractions of A having elements of S as denominators and $S^{-1}E$ the $S^{-1}A$-module of fractions of E. (In particular, if S is the complement of a maximal ideal $\mathfrak{M}$ of A we employ the usual notation: $S^{-1}A = A_{\mathfrak{M}}$ and $S^{-1}E = E_{\mathfrak{M}}$.) Since the center of A is also the center of $M_n(A)$, we can form the ring of fractions $S^{-1}M_n(A)$. It is easy to check that the map $(a_{ij})/s \to (a_{ij}/s)$ from $S^{-1}M_n(A)$ to $M_n(S^{-1}A)$ is a ring isomorphism. Then $S^{-1}E^n \simeq (S^{-1}E)^n$ as modules over this ring.

LEMMA 2.7. *If S is a multiplicative subset of the center of A and $(*)$ is an n-pure extension of left A-modules, then $0 \to S^{-1}K \to S^{-1}F \to S^{-1}E \to 0$ is an n-pure extension of left $S^{-1}A$-modules.*

Proof. This follows immediately from the definition for $n = 1$; then use the above remarks.

PROPOSITION 2.8. *Let A be a commutative ring. Then $(*)$ is 1-pure (pure) if and only if $0 \to K_{\mathscr{M}} \to F_{\mathscr{M}} \to E_{\mathscr{M}} \to 0$ is a 1-pure (pure) extension of $A_{\mathscr{M}}$-modules for every maximal ideal $\mathscr{M}$ of A.*

Proof. Let $\mathscr{M}$ be a maximal ideal. If $(*)$ is 1-pure, then $0 \to K_{\mathscr{M}} \to E_{\mathscr{M}} \to F_{\mathscr{M}} \to 0$ is 1-pure by the lemma, with $S = A - \mathscr{M}$. And if $(*)$ is pure, then it is n-pure for every $n \geqq 1$, so again by the lemma, $0 \to K_{\mathscr{M}} \to F_{\mathscr{M}} \to E_{\mathscr{M}} \to 0$ is n-pure for every $n \geqq 1$ and thus pure by Theorem 2.2.

The proofs of sufficiency follow easily, since $G \otimes K \to G \otimes F$ is an injection if and only if $(G \otimes K)_{\mathscr{M}} \to (G \otimes F)_{\mathscr{M}}$ is an injection for every $\mathscr{M}$ and since S^{-1} commutes with tensor products.

Hence if A is commutative, every 1-pure extension of A-modules is pure if and only if for every maximal ideal $\mathscr{M}$, every 1-pure extension of $A_{\mathscr{M}}$-modules is pure. (For the demonstration of this, recall that if C is an $A_{\mathscr{M}}$-module then C can also be considered an A-module and $C_{\mathscr{M}} \cong C$ as $A_{\mathscr{M}}$-modules.) Thus when investigating the equivalence of 1-pure and pure, one may assume that A is local.

COROLLARY 2.9. *If A is commutative and* w.gl. dim. $(A) \leqq 1$, *then any 1-pure extension of A-modules is pure.*

Proof. Endo has shown in [7] that w. gl. dim. $(A) \leqq 1$ if and only if each $A_{\mathscr{M}}$ is a valuation ring.

Warfield has shown in [19] that the converse is true if A is a commutative integral domain.

3. Torsion-free and flat modules. For $a \in A$, let $_r(a)$ denote the right annihilator ideal of a in A. After Hattori [11] we say that a left A-module E is *torsion-free* if whenever $a \in A$, $x \in E$ and $ax = 0$, then $x \in {}_r(a)\,E$. That is, E is torsion-free if whenever $ax = 0$ we can write $x = \sum_{i=i}^{n} b_i\, y_i$ with each $y_i \in E$, $b_i \in A$, and $ab_i = 0$. This definition agrees with the usual one if A is an integral domain, for in this case $_r(a) = 0$. Hattori obtained the following characterization [11].

PROPOSITION 3.1. *E is torsion-free if and only if* $\mathrm{Tor}_1(A/aA, E) = 0$ *for every $a \in A$.*

Thus the direct sum of a family of A-modules is torsion-free if and only if each is torsion-free. Any free module, and even any flat module, is torsion-free. Note that E is a torsion-free A-module if and only if E^n is a torsion-free A-module. We use the preceding proposition to extend another Hattori result.

PROPOSITION 3.2. *For a left A-module E, the following are equivalent.*

(a) *E is torsion-free.*

(b) *There exists a 1-pure extension $(*)$ of left A-modules with F torsion-free.*

(c) *Every exact sequence $(*)$ is 1-pure.*

Proof. (c) $\Rightarrow$ (b) since there always exists an exact sequence $(*)$ with F free. Consider the above proposition and the exact sequence $\mathrm{Tor}_1(A/aA, F) \to \mathrm{Tor}_1(A/aA, E) \to A/aA \otimes K \to A/aA \otimes F$ for the implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a).

EXAMPLE. If n and m are positive integers, then $E = Z/mZ$ can be made into a module over Z/mnZ. Then E is torsion-free if and only if m and n are relatively prime.

We shall call E *n-torsion-free* if E^n is a torsion-free $M_n(A)$-module. Note that a free module is n-torsion-free for all $n \geqq 1$ and that the direct sum of a family of A-modules is n-torsion-free if and only if each is n-torsion-free. We obtain an easily-proven result analogous to the one above.

PROPOSITION 3.3. *For a left A-module E, the following are equivalent.*

(a) *E is n-torsion-free.*

(b) *There exists an n-pure extension $(*)$ with F n-torsion-free.*

(c) *Every exact sequence $0 \to K' \to F' \to E^n \to 0$ of left $M_n(A)$-modules is 1-pure.*

(d) *Every exact sequence $(*)$ is n-pure.*

Recalling that if $n \geqq m$ then an n-pure extension is m-pure, we have

COROLLARY 3.4. *If $n \geqq m$, an n-torsion-free module is m-torsion-free.*

Another similar well-known result follows from the definitions of flat and pure.

PROPOSITION 3.5. *For a left A-module E, the following are equivalent.*

(a) *E is flat.*

(b) *There exists a pure extension* (∗) *with F flat.*

(c) *Every exact sequence* (∗) *is pure.*

COROLLARY 3.6. *E is a flat A-module if and only if E^n is a flat $M_n(A)$-module.*

Proof. Choose (∗) exact with F free and recall that (∗) is pure if and only if $0 \to K^n \to F^n \to E^n \to 0$ is pure.

COROLLARY 3.7. *If F is flat and K a submodule, then F/K is flat if and only if K is pure in F.*

Combining the above proposition, Theorem 2.2, and Proposition 3.3, we have

THEOREM 3.8. *A module is flat if and only if it is n-torsion-free for all* $n \geqq 1$.

Restating the definition in terms of elements, we see that E is n-torsion-free if and only if the following condition is satisfied:

If $x_j \in E$ and $a_{ij} \in A$ $(i = 1, \cdots, n;\ j = 1, \cdots, n)$ are such that $\sum_{j=1}^{n} a_{ij}x_j = 0$ for each i, then there are $y_u \in E$, $b_{ju} \in A$ $(j = 1 \cdots, n;\ u = 1, \cdots, m)$ such that $x_j = \sum_{u=1}^{m} b_{ju}y_u$ for each j and $\sum_{j=1}^{n} a_{ij}b_{ju} = 0$ for each i and u.

Actually we need not require a square scalar matrix; that is, we could let $i = 1, \cdots, r$ and $j = 1, \cdots, s$ for any $r, s \leqq n$.

Using this and Theorem 3.8, Chase's criterion for flatness [4, Proposition 1.2 (c)] is easily derived.

The following proposition is a direct generalization of the fact that over a principal ideal ring every torsion-free module is flat.

PROPOSITION 3.9. *If every finitely generated right ideal of A is generated by at most n elements, then any n-torsion-free left A-module is flat.*

Proof. Let E be an n-torsion-free left A-module. To show E flat, it suffices to show $\mathscr{I} \otimes E \to A \otimes E = E$ is an injection for $\mathscr{I}$ any finitely generated right ideal. By hypothesis, $\mathscr{I} = a_1A + a_2A + \cdots + a_nA$, so if $z \in \mathscr{I} \otimes E$ we can write $z = \sum_{i=1}^{n} a_i \otimes x_i$, for $x_i \in E$. But if $\sum_{i=1}^{n} a_i x_i = 0$ there are elements $y_j \in E$, $a_{ij} \in A$ $(i = 1,$

$\cdots n$; $j = 1, \cdots m$) such that $x_i = \sum_{j=1}^{m} b_{ij} y_j$ for each i and $\sum_{i=1}^{n} a_i b_{ij} = 0$ for each j. Thus

$$\sum_{i=1}^{n} a_i \otimes x_i = \sum_{i=1}^{n} a_i \otimes (\sum_{j=1}^{m} b_{ij} y_j) = \sum_{j=1}^{m} (\sum_{i=1}^{n} a_i b_{ij}) \otimes y_j = 0 ,$$

so the map is an injection.

COROLLARY 3.10. *If every finitely generated right ideal of A is generated by at most n elements, then any torsion-free left $M_n(A)$-module is flat.*

Proof. If M is a torsion-free left $M_n(A)$-module, then $M \cong E^n$ for a left A-module E which is by definition n-torsion-free.

By the proposition, E is a flat A-module, so by Corollary 3.6, $E^n \cong M$ is a flat $M_n(A)$-module.

Considering the "other side" of the tensor product, we have the following similarly proven result.

PROPOSITION 3.11. *If a left A-module is generated by no more than n elements and is n-torsion-free, then it is flat.*

In particular, any cyclic torsion- free module or any torsion-free principal ideal is flat.

C. U. Jensen [13] has proved the following proposition for the case where A has no zero divisors, but his proof can be easily adapted to our more general situation.

PROPOSITION 3.12. *A torsion-free left A-module E is flat if and only if for all (finitely generated) right ideals U and V we have $(U \cap V)E = UE \cap VE$.*

We would like to have a similar criterion telling when a torsion-free module is n-torsion-free, but we have only been able to show that if E is $2n$-torsion-free then $(U \cap V)E = UE \cap VE$ for U and V right ideals which are generated by at most n elements.

EXAMPLE. Let A be a commutative integral domain with p and q two inequivalent primes such that $\mathscr{A} = A_p + A_q \neq A$. Then $\mathscr{A}$ is torsion-free but not 2-torsion-free. For letting

$$a = \begin{pmatrix} -q & p \\ 0 & 0 \end{pmatrix} \in M_2(A) \text{ and } x = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathscr{A}^2, \text{ then } ax = 0 .$$

The right annihilator of a in $M_2(A)$ consists of all matrices of the form $\begin{pmatrix} pb & pc \\ qb & qc \end{pmatrix} = \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix}$ for $b, c \in A$. It is easy to see that

$x \notin r(a)\mathscr{A}^2$.

For example, if K is a commutative field and $A = K[X, Y]$—the polynomials in two indeterminates with coefficients from K—then X and Y are inequivalent primes, so the ideal of polynomials with no constant term is torsion-free but not 2-torsion-free.

EXAMPLE. Let K be a commutative field, $K[[X]]$ the ring of formal power series with coefficients in K, and let A be the subring of $K[[X]]$ consisting of all power series without terms of degree 1 (*cf.* [3], Chapter VIII, Exercise 10). Then A is a local domain having maximal ideal the set of power series of order at least two. This maximal ideal is generated by X^2 and X^3, so if it were 2-torsion-free it would be flat, hence free (since A is local). Since it is not free, this gives another example of a torsion-free module which is not 2-torsion-free.

Note that these examples show that we cannot replace "pure" by "1-pure" in Corollary 2.4.

Unfortunately, we do not yet have an example of a module which is 2-torsion-free but not 3-torsion-free, so it is conceivable that 2-torsion-free may be equivalent to flat.

If $f: A \to C$ is a homomorphism of rings and E a left C-module, then C and E can be made into left A-modules. If C is an n-torsion free A-module and E an n-torsion-free C-module, then E is an n-torsion-free A-module. As is shown by the following example, however, the converse is not true, even for the specialized case where $n = 1$, $C = M_2(A)$, and $f: A \to M_2(A)$ is the canonical imbedding.

EXAMPLE. Let Z be the ring of integers, let p and q be two inequivalent primes of Z and let $\mathscr{A}$ be the principal left ideal of $M_2(Z)$ generated by $a = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$. Let $E = M_2(Z)/\mathscr{A}$. Then E is torsion-free (hence flat) as a Z-module but not as an $M_2(Z)$-module.

Localization. If S is a multiplicative subset of the center of A and E a torsion-free left A-module, then $S^{-1}E$ is a torsion-free left $S^{-1}A$-module. Thus, since $M_n(-)$ and E^n commute with S^{-1}, if E is n-torsion-free (flat) so is $S^{-1}E$. We want to show the "converse". First we need some notation and preliminaries.

Let A be commutative, let $B = M_n(A)$, $C = S^{-1}A$, and $D = S^{-1}M_n(A)$. Recall [2] that if E and F are A-modules then $S^{-1}(E \otimes_A F) \cong S^{-1}E \otimes_C S^{-1}F$.

LEMMA 3.13. *If $\mathscr{A}$ is an ideal of A, then*

$$S^{-1}(\mathrm{Tor}_1^A(A/\mathscr{A}, E)) \cong \mathrm{Tor}_1^C(S^{-1}A/S^{-1}\mathscr{A}, S^{-1}E).$$

Proof. We have the exact sequence $0 \to \mathscr{A} \to A \to A/\mathscr{A} \to 0$ so we obtain exact sequences

$$0 \longrightarrow \mathrm{Tor}_1^C(S^{-1}A/S^{-1}\mathscr{A},\ S^{-1}E) \longrightarrow S^{-1}\mathscr{A} \otimes_C S^{-1}E \longrightarrow S^{-1}E$$

and

$$0 \longrightarrow S^{-1}(\mathrm{Tor}_1^A(A/\mathscr{A},\ E)) \longrightarrow S^{-1}(\mathscr{A} \otimes_A E) \longrightarrow S^{-1}E.$$

LEMMA 3.14. *If F is a right $M_n(A)$-module, E a left $M_n(A)$-module, then $S^{-1}(F \otimes_B E) \cong S^{-1}F \otimes_D S^{-1}E$ as $S^{-1}A$-modules. (Note that on the left we are considering $S \subset A$, and on the right, $S \subset M_n(A)$.)*

Proof. The $S^{-1}A$-linear map $S^{-1}F \otimes_D S^{-1}E \to S^{-1}(F \otimes_B E)$ taking $x/s \otimes y/t \to x \otimes y/st$ has as inverse the map $x \otimes y/s \to 1/s\,(x/1 \otimes y/1)$.

LEMMA 3.15. *If $\mathscr{A}$ is a right ideal of $M_n(A)$ and E a left $M_n(A)$-module, then*

$$S^{-1}(\mathrm{Tor}_1^B(M_n(A)/\mathscr{A},\ E)) \cong \mathrm{Tor}_1^D(S^{-1}M_n(A)/S^{-1}\mathscr{A},\ S^{-1}E).$$

Proof. Similar to the proof of Lemma 3.13, but use Lemma 3.14.

THEOREM 3.16. *Let A be a commutative ring and E an A-module. Then E is n-torsion-free if and only if $E_{\mathscr{M}}$ is an n-torsion-free $A_{\mathscr{M}}$-module for every maximal ideal $\mathscr{M}$ of A.*

Proof. Only the sufficiency requires proof. First let $n = 1$. Let $a \in A$, let $\mathscr{M}$ be a maximal ideal of A and $S = A\text{-}\mathscr{M}$. Then $S^{-1}(\mathrm{Tor}_1^A(A/aA,\ E)) \cong \mathrm{Tor}_1^{A_{\mathscr{M}}}(A_{\mathscr{M}}/aA_{\mathscr{M}},\ E_{\mathscr{M}})$ by Lemma 3.13; if $E_{\mathscr{M}}$ is assumed torsion-free this last group equals zero. Letting $\mathscr{M}$ range through the set of maximal ideals A, $\mathrm{Tor}_1^A(A/aA,\ E) = 0$, so E is torsion-free. For $n > 1$ the proof is the same but uses Lemma 3.15.

COROLLARY 3.17. *E is flat if and only if $E_{\mathscr{M}}$ is flat for every $\mathscr{M}$.*

Hence every torsion-free A-module is flat if and only if for every maximal ideal $\mathscr{M}$, every torsion-free $A_{\mathscr{M}}$-module is flat.

If A is commutative and w.gl.dim. $A \leqq 1$, we have shown that the notions of 1-pure and pure coincide, so by Proposition 3.2 and Proposition 3.5, the notions of torsion-free and flat also coincide. The converse is not true, for if $A = Z/4Z$ then every torsion-free A-module is flat, but w. gl. dim $A = \infty$. (Perhaps torsion-free agrees with flat

if and only if finitistic w. gl. dim. $A \leqq 1$.) For an integral domain A, 1-pure coincides with pure if and only if torsion-free coincides with flat if and only if w. gl. dim. $A \leqq 1$ (i. e., A is Prufer). We have the following slightly stronger result. Recall first that a ring is said to be *left PP* (*PF*) if every principal left ideal is projective (flat).

PROPOSITION 3.18. *If A is a commutative PF ring, then w. gl. dim $A \leqq 1$ if and only if every torsion-free A-module is flat.*

Proof. We need only show the sufficiency. Fieldhouse has shown [10] that A is *PF* if and only if $A_{\mathscr{M}}$ is an integral domain for each maximal ideal $\mathscr{M}$. So if torsion-free agrees with flat over A, the same is true over $A_{\mathscr{M}}$, which must thus be a Prufer ring. Since a local Prufer domain is a valuation ring, we have w. gl. dim. A $\leqq 1$.

Rephrasing this result: w. gl. dim. $A \leqq 1$ if and only if A is *PF* and torsion-free agrees with flat.

Inductive limits. Let I be a directed set and let (A_i, Φ_{ji}) be an inductive system of rings. For each $i \in I$, let E_i be a left A_i-module such that (E_i, f_{ji}) is an inductive system of modules and such that if $i \leqq j$, $x_i \in E_i$, $a_i \in A_i$, then $f_{ji}(a_i x_i) = \Phi_{ji}(a_i) f_{ji}(x_i)$. Then $E = \varinjlim E_i$ can be made into a left module over $A = \varinjlim A_i$ (see [1]). Let $f_i : E_i \to E$ and $\Phi_i : A_i \to A$ denote the canonical maps. Note that if $n \geqq 1$ then $\varinjlim M_n(A_i) \cong M_n(\varinjlim A_i)$ as rings and $\varinjlim E_i^n \cong (\varinjlim E_i)^n$ as modules over this ring.

THEOREM 3.19. *If each E_i is an n-torsion-free A_i-module, then E is an n-torsion-free A-module.*

Proof. By the above remark, it suffices to prove the theorem for $n = 1$. Let $a \in A$ and let $k \in I$ be such that $\Phi_k(a_k) = a$ for some $a_k \in A_k$. Let $J = \{i \in I \mid i \geqq k\}$. Then J is cofinal in I so inductive limits over J are the same as over I. There is a family $(a_i)_{i \in J}$ such that $\Phi_{ji}(a_i) = a_j$ if $i \leqq j$ and $\Phi_i(a_i) = a$ for every $i \in J$. Then it is easy to show that $a \varinjlim A_i \cong \varinjlim a_i A_i$ as A-modules and hence (since tensor products commute with inductive limits)

$\operatorname{Tor}_1^A(A/aA, E) \cong \varinjlim \operatorname{Tor}_1^{A_i}(A_i/a_iA_i, E_i)$. Since each A_i is assumed to be torsion-free over A_i, $\operatorname{Tor}_1^A(A/aA, E) = \varinjlim 0 = 0$, so E is torsion-free.

An interesting example of this is obtained by fixing A and E and taking the inductive limit over the positive integers: $M_\omega(A) = \varinjlim M_n(A)$ and $M_\omega(E) = \varinjlim E^n$, where the maps $M_m(A) \to M_n(A)$ and $E^n \to E^m$, defined if and only if m is a multiple of n, say $m = np$,

are the inbeddings $B \to M_p(B)$ by $b \to \text{diag}\,(b, b, \cdots, b)$ and $G \to G^p$ by $x \to (x, x, \cdots, x)'$. Note that since M_n commutes with M_m and inductive limits, it also commutes with M. That is,

$M_n(M_\omega(A)) \cong M_\omega(A)$ and $(E^\omega)^n \cong E^\omega$.

Proposition 3.20. *For a left A-module E, the following are equivalent.*

(a) *E is flat.*

(b) *E^ω is a torsion-free $M_\omega(A)$-module.*

(c) *E^ω is a flat $M_\omega(A)$-module.*

Proof. (a) $\Rightarrow$ (b). If E is flat then E is n-torsion-free for each $n \geqq 1$ or E^n is torsion-free over $M_n(A)$. Thus (b) follows from the preceding theorem.

(b) $\Leftrightarrow$ (c). Since $(E^\omega)^n \cong E^\omega$, if E^ω is torsion-free it is n-torsion-free for any $n \geqq 1$, hence flat.

(b) $\Rightarrow$ (a). We show E flat by showing that linear relations in E arise from linear relations in A [4, Proposition 2.3 (b)]. Suppose $a_i \in A$, $x_i \in E\,(i = 1, \cdots, n)$ and $\sum_{i=1}^n a_i x_i = 0$. Let $x = (x_1, x_2, \cdots, x_n)' \in E^n$ and let $a \in M_n(A)$ be the matrix having $a_1, a_2, \cdots, a_n$ across the first row and zeros elsewhere. Then $ax = 0$ in E^n. Map a to $M_\omega(A)$ and x to E^ω, use the fact that E^ω is torsion-free and obtain elements from E and A which yield the desired linear relations.

Recall the definitions of $M(A)$, $M(E)$ and $E^{(N)}$. It can be similarly shown that E is flat (over A) if and only if $M(E)$ (respectively $E^{(N)}$) is torsion-free (over $M(A)$).

Direct products. Although arbitrary direct sums of torsion-free modules are torsion-free, this is not true of direct products, as is shown by the following example.

EXAMPLE. Let K be a commutative field and V a vector space over K with a countably infinite base. Let A be the ring $K \times V$ with addition componentwise and multiplication defined by $(a, x) \cdot (b, y) = (ab, ay + bx)$ for $a, b \in K$ and $x, y \in V$. Note that as an ideal of A, V cannot be finitely generated, and that if $x \in V$ the annihilator of x in A is V. Then it is easy to show that the direct product of a countably infinite number of copies of A is not a torsion-free A-module.

Hattori has shown [11] that for a ring A, arbitrary products of torsion-free left A-modules are torsion-free if and only if ${}_r(a)$ is a finitely generated right ideal for every $a \in A$. Our next proposition is a refinement of this result. First however, we make a definition.

Let ζ be an infinite cardinal number. A right ideal $\mathscr{A}$ of A is said to have *property* (F, ζ) if for any right ideal $V \subseteq \mathscr{A}$ which is generated by a set of cardinality at most ζ there is a finitely generated right ideal C such that $V \subseteq C \subseteq \mathscr{A}$. Note that $\mathscr{A}$ has property (F, ζ) for every infinite cardinal ζ if and only if $\mathscr{A}$ is finitely generated.

PROPOSITION 3.21. *Let ζ be an infinite cardinal and I a set of cardinality ζ; for each $i \in I$ let A_i equal A considered as a left module over itself. Then the following are equivalent.*

(a) $\prod_{i \in I} A_i$ *is a torsion-free left A-module.*

(b) $\prod_{i \in I} E_i$ *is a torsion-free left A-module for any family $(E_i)_{i \in I}$ of torsion-free left A-modules.*

(c) ${}_r(a)$ *has property (F, ζ) for every $a \in A$.*

Proof. It is obvious that (b) implies (a), and the proof that (a) implies (c) is straightforward. To show that (c) implies (b), let $(E_i)_{i \in I}$ be a family of torsion-free left A-modules and suppose $a \in A$, $(x_i)_{i \in I} \in \prod E_i$ are such that $a\,(x_i)_{i \in I} = 0$. Then for each i, $ax_i = 0$, so there is an integer m_i and elements

$$b_{ui} \in r(a),\ y_{ui} \in E_i\ (u = 1, \cdots, m_i) \text{ such that } x_i = \sum_{u=1}^{m_i} b_{ui}\, y_{ui}\ .$$

Let V be the right ideal of A generated by the set

$$\{b_{ui} \mid i \in I,\ u = 1, \cdots, m_i\}\ .$$

which has cardinality at most ζ. Since $V \subseteq {}_r(a)$, which is assumed to have property (F, ζ), V must be contained in a finitely generated ideal $C = b_1 A + \cdots + b_m A$, with $C \subseteq {}_r(a)$. Hence there exist $c_{uij} \in A$ such that $b_{ui} = \sum_{j=1}^{m} b_j c_{uij}$, so $(x_i)_{i \in I} = \sum_{j=1}^{m} b_j\, (\sum_{u=1}^{m_i} c_{uij} y_{ui})_{i \in I}$. Since each $b_j \in {}_r(a)$, $\prod E_i$ is torsion-free.

A similar result holds for n-torsion-free modules: ζ-size products of n-torsion-free A-modules are n-torsion-free if and only if ${}_r(a)$ satisfies property (F, ζ) for every a in $M_n(A)$.

It can also be shown, using a diagonalization argument similar to Hattori's [11, Proposition 8], that arbitrary products of n-torsion-free left A-modules are n-torsion-free if and only if ${}_r(a)$ is a finitely generated right ideal of $M_n(A)$ for every a in $M_n(A)$.

EXAMPLE. We conclude this section with an example of a ring in which every ${}_r(a)$ has property (F, ω) but not all ${}_r(a)$ are finitely generated, where ω is the cardinality of Z. Hence over this ring countable products of torsion-free modules are torsion-free.

Let I be an uncountably infinite set and for each $i \in I$ let K_i be

a commutative field. Let V be the ideal of $\prod_{i\in I} K_i$ consisting of families with at most denumerable support. Let A be the ring $Z \times V$, with multiplication

$$(n, b) \cdot (m, c) = (nm, nc + mb + bc) \text{ for } n, m \in Z, b, c \in V,$$

and

$$\text{let } E = A/V.$$

Now consider the ring $C = A \times E$ with multiplication

$$(a, c + V) \cdot (b, d + V) = (ab, ad + cb + V) \text{ for } a, b, c, d \in A.$$

C is a commutative ring with identity which satisfies the above claim. Since we have two rings involved, we use the notation ${}_{rA}(a)$ (respectively ${}_{rC}(a)$) to denote the annihilator ideal of a in A (respectively in C). Note that if $a \in A$ then ${}_{rA}(a)$ is a principal ideal. It then follows that if $a \in A\text{-}V$ or $b \in V$ then ${}_{rC}(a, b + V)$ is principal, and that if $a \in V$ and $b \in A\text{-}V$ then ${}_{rC}(a, b + V) = {}_{rC}(a, 1 + V) = V_a \times E$ where $V_a = {}_{rA}(a) \cap V$. The ideal $V_a \times E$ is not finitely generated but it does satisfy property (F, ω).

4. **Divisible and absolutely pure modules.** For $a \in A$, let $l(a)$ denote the left annihilator ideal of a in A. In the same paper already referred to [11], Hattori has defined a left A-module K to be *divisible* if whenever $a \in A, x \in E$ and $l(a)x = 0$, then $x \in aK$. That is, K is divisible if and only if $aK = \{x \in K \mid l(a) \subseteq \mathrm{Ann}_A(x)\}$. Hattori has also noted that K is divisible if and only if $\mathrm{Ext}^1(A/Aa, K) = 0$ for every $a \in A$. Thus every injective module is divisible and direct sums and products of divisible modules are divisible. Dual to the results of the previous section, we have the following definitions and propositions.

PROPOSITION 4.1. *For a left A-module K, the following are equivalent.*

(a) *K is divisible.*

(b) *Every exact extension $(*)$ is 1-pure.*

(c) *There exists a 1-pure extension $(*)$ with F divisible.*

Let $I(K)$ denote the injective envelope of K.

COROLLARY 4.2. *K is divisible if and only if $0 \to K \to I(K) \to E \to 0$ is 1-pure.*

We shall say that K is *n-divisible* if K^n is a divisible $M_n(A)$-module. Since K injective implies K^n injective, hence divisible, an injective module is n-divisible for every $n \geqq 1$. Since

$$\oplus_{i\in I} K_i^n \cong (\oplus_{i\in I} K_i)^n \text{ and } \prod_{i\in I} K_i^n \cong (\prod_{i\in I} K_i)^n$$

as $M_n(A)$-modules, $\oplus K_i$ (respectively $\prod K_i$) is n-divisible if and only if each K_i is n-divisible. Hence K is n-divisible if and only if $M_n(K)$ is a divisible $M_n(A)$-module.

PROPOSITION 4.3. *For a left A-module K, the following are equivalent.*

(a) *K is n-divisible.*

(b) *There exists an n-pure extension $(*)$ with F n-divisible.*

(c) *Every exact sequence $0 \to K^n \to F' \to E' \to 0$ of left $M_n(A)$-modules is 1-pure.*

(d) *Every exact sequence $(*)$ is n-pure.*

COROLLARY 4.4. *If $n \geqq m$, an n-divisible module is m-divisible.*

Proof. Use Proposition 2.1.

COROLLARY 4.5. *K is n-divisible if and only if $0 \to K \to I(K) \to E \to 0$ is n-pure.*

COROLLARY 4.6. *A submodule K of an n-divisible module F is n-divisible if and only if $(*)$ is n-pure.*

After B. H. Maddox [17 and 18] we call K *absolutely pure* if it is a pure submodule of every module in which it can be imbedded. That is, K is absolutely pure if and only if every extension $(*)$ is pure. Maddox has shown that any injective module is absolutely pure and that any pure submodule of an injective module is absolutely pure. Thus, since K is pure in $I(K)$ if and only if K^n is pure in $I(K)^n$ (which is injective), we see that K is an absolutely pure A-module if and only if K^n is an absolutely pure $M_n(A)$-module. Note that an absolutely pure module is divisible.

THEOREM 4.7. *A left A-module K is absolutely pure if and only if it is n-divisible for every $n \geqq 1$.*

Proof. If K is absolutely pure, then for every $n \geqq 1$, K^n is absolutely pure, hence divisible.

Conversely, if K is n-divisible for every $n \geqq 1$, then $0 \to K \to I(K) \to E \to 0$ is n-pure for every $n \geqq 1$, hence pure (by Theorem 2. 2) so K is pure in $I(K)$, or K is absolutely pure.

COROLLARY 4. 8. *For an A-module K, the following are equivalent.*

(a) *K is absolutely pure.*
(b) *There exists a pure extension (*) with F absolutely pure.*
(c) *Every extension (*) is pure.*

COROLLARY 4.9. *If $(K_i)_{i\in I}$ is a family of absolutely pure A-modules, then $\prod_{i\in I} K_i$ is absolutely pure.*

Proof. Recall that $\prod K_i$ is n-divisible if and only if each K_i is n-divisible.

COROLLARY 4.10. *K is an absolutely pure A-module if and only if $M(K)$ is a divisible $M(A)$-module.*

We have a partial answer to the question of inductive limits.

PROPOSITION 4.11. *If (K_i, f_{ji}) is an inductive system of modules over the inductive system of rings (A_i, Φ_{ji}), if each f_{ji} is an injection and if each K_i is a divisible A_i-module, then $\varinjlim K_i$ is divisible over $\varinjlim A_i$.*

Proof. This follows directly from the definitions.

COROLLARY 4.12. *In the same situation, if each K_i is n-divisible, then $\varinjlim K_i$ is n-divisible.*

Proof. K^n and $M_n(-)$ commute with inductive limits.

COROLLARY 4.13. *If each K_i is absolutely pure, then $\varinjlim K_i$ is absolutely pure.*

Maddox obtained this last result in the case where all $A_i = A$. The next proof follows one by Maddox, which followed one by E. Matlis.

PROPOSITION 4.14. *For a ring A, the following are equivalent.*
(a) *Every quotient module of a divisible left A-module is divisible.*
(b) *The sum of two divisible submodules of a left A-module is divisible.*

Proof. If K and H are divisible left A-modules, then so is $K \oplus H$, so $K + H$ is divisible if (a) is assumed. Conversely, assume (b) and let H be a submodule of a divisible module M. Let L and N be two copies of M and let $P = L \oplus N$ let Δ be the diagonal of P and let

$D = \Delta \cap (H \oplus H)$. With $\Phi\colon P \to P/D$ the cannonical surjection it is easy to see that $P/D = \Phi(L) + \Phi(N)$; since $\Phi | L$ and $\Phi | N$ are injections, $M \cong \Phi(L) \cong \Phi(N)$ so by hypothesis P/D is divisible. But

$$P/D = ((L \oplus 0) + D)/D \oplus \Delta/D ,$$

so $\Delta/D \cong M/H$ is divisible.

(Note that the only property of divisibility used in this proof is that $K \oplus H$ is divisible if and only if K and H are divisible. Hence we can prove the proposition for any property satisfying this condition.)

Hattori has shown in [11] that all quotients of divisible left A-modules are divisible if and only if A is left PP. So if $M_n(A)$ is left PP then quotients of n-divisible left A-modules are n-divisible. Hence if $M_n(A)$ is left PP for every $n \geqq 1$ then every quotient module of an absolutely pure left A-module is absolutely pure. For example, if A is left semi-hereditary then each $M_n(A)$ is left semi-hereditary and thus left PP. (Surprisingly, the converse is true; that is, if $M_n(A)$ is left PP for every $n \geqq 1$ then A is left semi-hereditary. This can be seen by using [16, Corollary 2.3].) Hence if A is left semi-hereditary, quotients and finite sums of absolutely pure left A-modules are absolutely pure, and by one of Maddox' theorems, every left A-module has a unique maximal absolutely pure submodule. We do not know whether any of these imply that A is left semi-hereditary.

A classical theorem says that if A is an integral domain then a torsion-free module is injective if and only if it is divisible. We conclude with an example due to Maddox which shows that this result does not hold in our more general setting.

EXAMPLE. For $i \in N$, the set of positive integers, let $A_i = Z/4Z$; then let $A = \prod_{i \in N} A_i$ and $E = \bigoplus_{i \in N} A_i$. E is an A-module which is flat and absolutely pure, hence torsion-free and divisible, but E is not injective.

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