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EIGENVALUES IN THE BOUNDARY OF THE NUMERICAL RANGE

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We study eigenvalues λ of a continuous linear operator T on a complex Banach space X that lie in the boundary of the numerical range of T. We show that the kernel of $T-\lambda I$ is orthogonal, in the sense of G. Birkhoff, to the range of $T-\lambda I$.

M. R. Fortet [5, Th. III, p. 32] proves that if T is a continuous linear operator of norm one on a strictly convex Banach space, then the kernel of $T-I$ is orthogonal to the range of $T-I$. Proposition 1 is a generalisation of this result, since the numerical radius is less than or equal to the norm [2, Th. 4.1]. Proposition 1 is also related to the theorem of N. Nirschl and H. Schneider that an eigenvalue in the boundary of the numerical range has ascent one [8, Th. 4, p. 362] and [2, Th. 10.10].

If T is a continuous linear operator on a Banach space X (over the complex field), the *numerical range* $V(T, \mathscr{B})$ of T is the set

$$\{F(T): F \in \mathscr{B}^*, \|F\| = F(I) = 1\}$$

where $\mathscr{B}$ is the Banach algebra of all continuous linear operators on X, $\mathscr{B}^*$ is the dual Banach space of $\mathscr{B}$, and I is the identity operator on X [2, Chapter 3] and [1, § 3]. The *spatial numerial range* [2, Definition 9.1] $V(T)$ of T is the set

$$\{f(Tx): f \in X^*, x \in X, \|f\| = \|x\| = f(x) = 1\} .$$

The numerical range of T is equal to the closed convex hull of the spatial numerical range, that is, $V(T, \mathscr{B}) = \overline{\text{co}}\ V(T)$ [2, Th. 3.9] and [1, Th. 6]. The spectrum, and hence the set of eigenvalues of T, is contained in the numerical range of T [2, Th. 2.6]. A linear subspace Y of X is said to be *orthogonal* to a linear subspace Z of X if $\|y\| \leqq \|y+z\|$ for all y in Y and all z in Z [6] and [4, p. 93].

There is no loss of generality in assuming that 0 is the eigenvalue in the boundary of the numerical range, as we assume henceforth, because we may achieve this by adding a scalar multiple of the identity to T.

PROPOSITION 1. *Let T be a continuous linear operator on a complex Banach space X. If 0 is in the boundary of the numerical range of T, that is, $0 \in \partial\, \overline{\text{co}}\ V(T)$, then the kernel of T is orthogonal* *to the range of T. In particular $T^{-1}\{0\} \oplus (TX)^-$ is closed in X.*

Proof. Since 0 is in the boundary of $V(T, \mathscr{B})$, a closed convex subset of the complex plane, we may assume that $\max\{\mathrm{Re}\,\lambda: \lambda \in V(T, B)\} = 0$, by multiplying T by a suitable complex number of modulus 1. Assuming this, we have $\|\exp \alpha T\| \leqq 1$ for all nonnegative real numbers α by [2, Th. 3.4]. If T is one-to-one, the kernel of T is null and the result follows because 0 is orthogonal to all vectors. We now assume that T is not one-to-one. Let y be an element of unit norm in X annihilated by T, and let

$$D(y) = \{f \in X^*: \|f\| = f(y) = 1\}\ .$$

Then $D(y)$ is a nonempty $\sigma(X^*, X)$-compact convex subset of X^*, by the Hahn-Banach Theorem and Alaoglu's Theorem, and $\exp \alpha T^*$ is a $\sigma(X^*, X)$—continuous affine mapping on $D(y)$ for each nonnegative real α, since $\|\exp \alpha T\| \leqq 1$ and $Ty = 0$. Further $\{\exp \alpha T^*: \alpha \text{ is real}, \alpha \geqq 0\}$ is a commutative semigroup on $D(y)$. The Markov-Kakutani fixed point theorem [4, Th. V. 10.6, p. 456] implies that there is an f in $D(y)$ such that $\exp \alpha T^* f = f$ for all nonnegative real α. The use of a fixed point theorem was suggested to me by the application of a generalization of Brouwer's fixed point theorem due to Kakutani in the proof of Theorem 1 of [3]. Taking the right hand derivative of $\exp \alpha T^*$ at $\alpha = 0$, and applying the equation $\exp \alpha T^* f = f$, we obtain $T^* f = 0$. Therefore $\|y + z\| \geqq |f(y + z)| = f(y) = \|y\|$ for all z in TX, and so the kernel of T is orthognonal to the range of T. That $T^{-1}\{0\} \oplus (TX)^-$ is closed in X, follows in a routine way from the result that $T^{-1}\{0\}$ is orthogonal to TX, and hence to $(TX)^-$. This completes the proof.

REMARKS 2. In general the space $T^{-1}\{0\} \oplus (TX)^-$ of Proposition 1 is not equal to X. For example let X be $\mathscr{C}[0, 1]$, the space of continuous complex valued functions on $[0, 1]$ with the supremum norm, let g be a continuous real valued function on $[0, 1]$ that is zero at 0 and positive on $(0, 1]$, and let T be the operation of multiplication by g in X. Then T is a hermitian operator on X [2, Chapter 2], since $\|\exp itg\| = 1$ for all real t, so that the numerical range of T is contained in the real line [2, Lemma 5.2]. Further $T^{-1}\{0\} \oplus (TX)^- = (TX)^-$ is the set of functions in X that vanish at 0.

Proposition 1 gives another proof of the result that an eigenvalue in the boundary of $\overline{\mathrm{co}}\, V(T)$ has ascent one [8] and [2, Th. 10.10].

PROPOSITION 3. *Let T be a nonzero continuous linear operator on a complex Banach space X, and let 0 be in the spectrum of T and in the boundary of the numerical range of T, that is,*

$$0 \in \sigma(T) \cap \partial \overline{\mathrm{co}}\, V(T) .$$

If TX is closed in X, then 0 is an eigenvalue of T, $X = T^{-1}\{0\} \oplus TX$, and 0 is an isolated point of the spectrum of T.

Proof. By Proposition 1, $T^{-1}\{0\} \oplus TX$ is closed in X so that if it is not equal to X there is a nonzero continuous linear functional f on X that is zero on $T^{-1}\{0\} \oplus TX$. Let Y^0 denote the annihilator in X^* of a subset Y of X. Then $(TX)^0 = T^{*-1}\{0\}$ where T^* is the adjoint of T [9, Th. 4.6-*C*, *p.* 226]. Since TX is closed in X which is complete, $T^*X^* = (T^*X^*)^- = T^{-1}\{0\}^0$ [9, Problem 7, p. 227]. By construction f is thus in $(T^*X^*)^-$ and in $T^{*-1}\{0\}$. Now T^* is a continuous linear operator on X^* with 0 in the boundary of the numerical range of T^*. That 0 is in the boundary of the numerical range of T^* follows from the equality $V(T^*, \mathscr{B}(X^*)) = V(T, \mathscr{B})$, which is an immediate consequence of Theorem 9.4(i) and Corollary 9.6(ii) of [2]. On the space X^* the operator T^* satisfies the assumptions of Proposition 1 so that the intersection of $(T^*X^*)^-$ and $T^{*-1}\{0\}$ is $\{0\}$ by Proposition 1. This gives a contradiction as we have previously shown that f, which is not zero, is in this intersection. Hence $X = T^{-1}\{0\} \oplus TX$. Since the spectrum of T is contained in the numerical range of T [2, Th. 2.6], 0 is in the boundary of the spectrum of T. Therefore TX is not equal to X by [7, Lemma 2.2], and so the kernel of T is nonnull and 0 is an eigenvalue of T.

Regarded as an operator on the Banach space TX, T is invertible and so $(\lambda I - T)$ restricted to TX is invertibile for all λ in a neighborhood of 0 in the complex plane. On the space $T^{-1}\{0\}$, the operator T has spectrum $\{0\}$. Since $X = T^{-1}\{0\} \oplus TX$, $\lambda I - T$ is invertibile on X for all λ in a neighbourhood of 0 but not at 0. This shows that 0 is an isolated point in the spectrum of T and completes the proof.

REMARKS 4. If T satisfies the hypotheses of Proposition 1, and if $(T^*X^*)^- = T^{-1}\{0\}^0$, then part of the proof of Proposition 3 shows that $X = (TX)^- \oplus T^{-1}\{0\}$.

From the assumptions of Proposition 3 it does not follow that the range of T is orthogonal to the kernel of T. Let Y and Z be closed linear subspaces of a complex Banach space X such that $X = Y \oplus Z$, Y is orthogonal to Z, and Z is not orthogonal to Y (spaces with these properties exists; see [6]). Let E be the projection from X onto Y annihilating Z. Then the norm of E is one, so that the eigenvalue 1 of E is in the boundary of the numerical range of E. Further $(1 - E)X = Z$ is not orthogonal to $(I - E)^{-1}\{0\} = Y$.

REMARK 5. If we add the hypothesis that the Banach space X is reflexive, then $(T^*X^*)^- = T^{-1}\{0\}^0$ for all continuous linear operators T on X [9, § 4.6, p. 226] so that if 0 is in the boundary of the numerical range of T, we have $X = (TX)^- \oplus T^{-1}\{0\}$ by Remark 4. As a corollary to this we have the following result.

Let X be a reflexive complex Banach space, and let T be a continuous linear operator on X such that 0 is in the boundary of the numerical range of T. Then 0 is an eigenvalue of T if, and only if, TX is not dense in X, that is, if and only if 0 is an eigenvalue of T^*.

This follows immediately from the equation $X = (TX)^- \oplus T^{-1}\{0\}$ which holds for T since X is reflexive.

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