PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 1, 1970

CARATHEODORY THEOREMS IN CONVEX PRODUCT STRUCTURES

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Various attempts have been made to place convexity in an axiomatic setting. Recently J. Eckhoff has considered the classic theorem of Radon in several different settings. Most of his work is done in what we call an Eckhoff space, i.e., in a finite product of euclidean spaces where convex sets are defined as the cartesian products of usual convex sets in each component space. The purpose of this paper is to investigate the closely related theorem of Caratheodory and its generalizations in this setting.

The papers of F. W. Levi [5] and Dauzer, Grunbaum, Klee [3] have various approaches to axiomatic settings of convexity, and a good bibliography for before 1961. See the papers of Eckhoff [4] and Bonnice-Reay [2] for more recent results and references.

1. Eckhoff spaces. The pair $(E, \mathscr{C})$ denotes an *Eckhoff space* provided (1) E is a direct cartesian product $E = \prod_{i=1}^{n} E_i$ where each E_i is a d_i-dimensional euclidean space with $\mathscr{C}_i$ the family of all convex sets of E_i, and (2) $\mathscr{C} = \{\prod_{i=1}^{n} A_i : A_i \in \mathscr{C}_i\}$ is the family of all *product-convex* sets in E. For any set $X \subset E$, the set $E(X) = \cap \{A : X \subset A \in \mathscr{C}\}$ is called the *product-convex hull* of X. Let $\pi_i : E \to E_i$ denote the usual projection. Then we can consider E as a linear space of dimension $d = \sum_{i=1}^{n} d_i$, and $E(X) = \prod_{i=1}^{n} (\operatorname{conv} \pi_i X)$ where $\operatorname{conv} B$ denotes the usual convex hull of B in each euclidean space E_i. The cardinality of B will be denoted by $|B|$. Using the notation of Bonnice-Klee [1] and others, we say that $\operatorname{int}_r B$ is the set of all points p for which there exists an r-dimensional simplex contained in B and containing p in its relative interior.

2. Caratheodory-type theorems. By a Caratheodory-type theorem we mean a result which asserts that if a point is embedded in the (axiomatically defined) hull of a set X, then it is similarly embedded in the hull of a sufficiently small subset of X. Note that the case $n = 1$ of Theorem 1 below is the result usually called Caratheodory's theorem.

THEOREM 1. *If X is any subset of an Eckhoff space $E = \prod_{i=1}^{n} E_i$ of dimension $d = \Sigma d_i$ and if $p \in E(X)$, then $p \in E(Y)$ for some $Y \subset X$ with $|Y| \leqq d + \delta$, where $\delta = 1$ if $n = 1$ and $\delta = 0$ if $n > 1$. Fur-*

thermore, if $m = |Y|$ *is the cardinality of a smallest subset* Y *of* X *for which* $p \in E(Y)$, *then* $p \in \operatorname{int}_r E(Y)$ *where*

$$\max(0, m - n) \leqq r \leqq (m - 1)|\{d_i : d_i \geqq m\}| + \Sigma_{d_i < m} d_i .$$

Proof. If $n = 1$ the upper and lower bounds on r reduce to $r = m - 1$, that is, p lies interior to the $(m - 1)$-simplex determined by the m points of Y.

Assume $n = 2$. It suffices to show that there is a set $Y \subset X$ for which $|Y| \leqq d_1 + d_2 = d$ and $\pi_i p \in \operatorname{conv} \pi_i Y$ for $i = 1, 2$. Applying Caratheodory's theorem to E_1, there is a subset $Y_1 \subset X$ with $|Y_1| \leqq d_1 + 1$ for which $\pi_1 p \in \operatorname{conv} \pi_1 Y_1$. Now if $\pi_2 p \in \operatorname{conv} \pi_2 Y_1$ as well, then $p \in E(Y_1)$ and we are done. Otherwise choose a set $Y_2 \subset X$ of minimal cardinality such that $\pi_2 p \in \operatorname{conv} \pi_2(Y_1 \cup Y_2)$. Since we may choose one of the $d_2 + 1$ points of Caratheodory's theorem arbitrarily (see [6], Lemmas 4.1–4.4) it follows that $|Y_2| \leqq d_2$. Thus letting $Y = Y_1 \cup Y_2$ it follows that $p \in E(Y)$ and $|Y| \leqq |Y_1| + |Y_2| \leqq d_1 + d_2 + 1$. We are therefore done unless both $|Y_1| = d_1 + 1$ and $|Y_2| = d_2$ and $Y_1 \cap Y_2 = \varnothing$. In this case we will show that one correctly chosen point may be removed from $Y_1 \cup Y_2$.

Case 1. $\pi_2 p \in \operatorname{conv} \pi_2 Y_2$. In this case we reverse the roles of E_1 and E_2 in the above argument, i.e., let Y_2 be as above and redefine Y_1 to be a set of minimal cardinality so that $\pi_1 p \in \operatorname{conv}(Y_2 \cup Y_1)$. Then $|Y_1| \leqq d_1$ and $|Y_2| = d_2$ and $p \in E(Y_2 \cup Y_1)$.

Case 2. $\pi_2 p \notin \operatorname{conv} \pi_2 Y_2$. In this case $|Y_2| = d_2$ so for each point $y \in Y_1$ it is true that $\pi_2 p \in \operatorname{conv} \pi_2(\{y\} \cup Y_2)$. Thus some point of $\pi_1 Y_2$ in the space E_1 may be used to replace a particular point of Y_1, say y_1. Then $\pi_1 p \in \operatorname{conv} \pi_1(Y_2 \cup (Y_1 - \{y_1\}))$ and it is still true that $\pi_2 p \in \operatorname{conv} \pi_2((Y_1 - \{y_1\}) \cup Y_2)$. This establishes Case 2, and hence proves the first statement of the theorem if $n = 2$.

The case when $n \geqq 3$ now follows easily. As in the case $n = 2$ there exists a set $Y_1 \cup Y_2 \subset X$ for which $\pi_i p \in \operatorname{conv} \pi_i(Y_1 \cup Y_2)$ for $i = 1, 2$ and $|Y_1 \cup Y_2| \leqq d_1 + d_2$. For each $i \geqq 3$ there is a set $Y_i \subset X$, by Caratheodory's theorem in E_i, such that $|Y_i| \leqq d_i$ and $\pi_i p \in \operatorname{conv} \pi_i(Y_1 \cup Y_2 \cup Y_i)$. The set $Y = \bigcup_{i=1}^{n} Y_i$ then has the desired properties; $p \in E(Y)$ and $|Y| \leqq \sum_{i=1}^{n} d_i = d$. This establishes the first half of the theorem.

To prove the last statement, let $Y \subset X$ be a smallest subset for which $p \in E(Y)$ and suppose $|Y| = m$. Then for each i, $\pi_i p \in \operatorname{int}_{r(i)} \operatorname{conv} \pi_i Y$ for some largest nonnegative integer $r(i) \leqq d_i$, and $p \in \operatorname{int}_r E(Y)$ where $r = \sum_{i=1}^{n} r(i)$. It follows that r assumes a minimal value whenever each $r(i)$ is as small as possible, within the constraint $|Y| = m$.

This is achieved when the points of Y are used as "inefficiently as possible", specifically, when for some partition $Y = Y_1 \cup \cdots \cup Y_n$ we have $\pi_i p \in \operatorname{conv} \pi_i Y_i$ and the points of $\pi_i(Y \sim Y_i)$ are not used in E_i. For example, if $\pi_i(Y \sim Y_i)$ is a single point in E_i then the points of $\pi_i Y$ are the vertices of a simplex in E_i of dimension $|Y_i|$, and $\pi_i p$ is interior to the subsimplex $\operatorname{conv} \pi_i Y_i$. In any case, Caratheodory's theorem (case $n = 1$) implies that $r(i) = |Y_i| - 1$. Thus $r = \Sigma_i r(i) = (\Sigma_i |Y_i|) - n = m - n$, and in general $r \geqq m - n$. The other inequality on r follows from the fact that the m points of Y projected onto each space E_i can have a convex hull of dimension at most $\min(m-1, d_i)$ in E_i. This proves Theorem 1.

EXAMPLES. The following examples show that the bounds in Theorem 1 cannot, in general, be improved.

(1) For each $i = 1, 2, \cdots, n$, let X_i be a subset of E_i for which $X_i \cup \{0\}$ form the vertices of a nondegenerate d_i-simplex, and let p_i be in the relative interior of the simplex $\operatorname{conv} X_i$. Define $p \in E$ by the relations $\pi_i p = p_i$. For each point x_i in each set X_i define the point $\bar{x}_i \in E$ by the relations $\pi_i \bar{x}_i = x_i$ and $\pi_j \bar{x}_i = 0$ if $j \neq i$. Let $X \subset E$ be the set of all such points $\bar{x}_i$. Then clearly $p \in E(X)$, but $p \notin E(Y)$ for any proper subset of Y of X, and $|X| = \Sigma |X_i| = \Sigma d_i = d$. Furthermore $p \in \operatorname{int}_r E(X)$ where

$$r = \Sigma(|X_i| - 1) = |X| - n = m - n .$$

(2) As a second example, let m be any integer for which $1 \leqq m \leqq \max\{d_i + 1 : i = 1, \cdots, n\}$. For each subspace E_i if $m \leqq d_i + 1$ let $\{x_{ij}\}_{j=1}^{m}$ be the vertices of a nondegenerate simplex in E_i. If $m > d_i + 1$ let $\{x_{ij} : j = 1, \cdots, d_i + 1\}$ be the vertices of a nondegenerate simplex, and let $x_{i1} = x_{ij}$ for $j = (d_i + 2), \cdots, m$. In either case choose a point p_i in the relative interior of this simplex. Now define $p \in E$ by the relations $\pi_i p = p_i$ and let $X = \{x_j\}_{j=1}^{m} \subset E$ where each x_j is defined by $\pi_i x_j = x_{ij} \in E_i$. Then $|X| = m$ and $p \in E(X)$ but $p \notin E(Y)$ for any proper subset Y of X. Also $p \in \operatorname{int}_r E(X)$ where

$$r = (m - 1)|\{d_i : d_i \geqq m\}| + \Sigma_{d_i < m} d_i .$$

The case where $n = 1$ and $r = d$ in Theorem 2 below is commonly called Steinitz's theorem.

THEOREM 2. *If X is any subset of an Eckhoff space $E = \prod_{i=1}^{n} E_i$ and if $r \geqq 0$ is the largest integer for which $p \in \operatorname{int}_r E(X)$, then $p \in \operatorname{int}_r E(Y)$ for some subset $Y \subset X$* with $|Y| \leqq 2r + \delta$ *where δ is the number of spaces E_i for which $\pi_i p \notin \operatorname{int}_1 \operatorname{conv} \pi_i X$.*

Proof. Let $r(i)$ be the largest integer for which $\pi_i p \in \text{int}_{r(i)} \text{conv} \pi_i X$. Thus $r = \Sigma r(i)$. By the Bonnice-Klee theorem (see [1], Th. 2.5) and the maximality of $r(i)$ there is a subset Y_i of X for which $\pi_i p \in \text{int}_{r(i)}$ $\text{conv}\, \pi_i Y_i$ and $|Y_i| \leqq 2r(i)$ if $r(i) > 0$, and $|Y_i| = 1$ if $r(i) = 0$. Thus letting $Y = \bigcup_{i=1}^n Y_i$ we have $p \in \text{int}_r E(Y)$ and $|Y| = |\cup Y_i| \leqq \Sigma |Y_i| \leqq 2\Sigma_{r(i)>0} r(i) + \Sigma_{\{i \mid r(i)=0\}} 1 = 2r + \delta$. This proves Theorem 2.

Using the techniques from the examples given above, it is easy to construct sets X in Eckhoff spaces which show that the bounds of Theorem 2 cannot, in general, be improved. A further generalization may be obtained by considering $p \in \text{int}_s E(X)$ where $0 < s < r$, and ask the cardinality of the smallest $Y \subset X$ for which $p \in \text{int}_s E(Y)$. This is the spirit of the Bonnice-Klee Theorem (see [1] and [6]). Another approach is to add further information about the set X, and ask how the bound on $|Y|$ may be improved. For example, if it is known that k_i is the dimension of the highest-dimensional simplex with vertices in $\pi_i X \subset E_i$ and having $\pi_i p$ in its relative interior, then the bound on $|Y|$ can, in general, be improved. See Bonnice-Reay [2] for a bibliography and results of this type. Also connectedness or symmetry conditions on X may lead to an improvement of the bound on $|Y|$. See [6] for a bibliography and results of this type.

These theorems and others which depend even more upon the structure of X are similar to the above theorems, but are much more complicated and are therefore omitted.

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Received October 16, 1969.

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