INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONALS ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

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S. Johansen has given a characterization of infinitely divisible characteristic functions on the real line analogous to the Bochner's theorem characterizing the characteristic functions through their nonnegative definiteness. Recently the present author was able to extend this result to infinitely divisible characteristic functionals on a Hilbert space and on locally compact abelian groups. We shall now obtain a similar theorem for infinitely divisible characteristic functionals on locally convex topological vector spaces whose dual spaces are nuclear.

Let E be a locally convex separable topological vector space. Let U be an open circled convex neighborhood of zero. For every $x \in E$, let

$$\|x\|_U = \text{Inf}\{\lambda: \lambda > 0, x \in \lambda U\} .$$

$\|x\|_U$ is a semi-norm on E. Let E_U denote the associated norm space (quotient space of E by the subspace such that $\|x\|_U = 0$). Its completion is the Banach space $\hat{E}_U$. If $\hat{E}_U$ is a Hilbert space, then U is called a *prehilbertian* neighborhood. Let π_{UV} denote the canonical mapping from E_U to E_V extended by continuity from $\hat{E}_U$ to $\hat{E}_V$. A *nuclear space* is a separated locally convex space such that there exists a fundamental system $\mathscr{T}$ of neighborhoods of zero which are prehilbertian neighborhoods and such that for every $U \in \mathscr{T}$, there exists a $V \in \mathscr{T}$, $V \subset U$ such that the mapping from $\hat{E}_U$ to $\hat{E}_V$ is Hilbert-Schmidt. Let $\mathscr{B}$ denote the σ-algebra of Borel subsets of E i.e., the σ-algebra generated by the open sets in E. A probability measure μ on the measurable space $(E, \mathscr{B})$ is a measure μ such that $\mu(E) = 1$. A probability measure μ is said to be *tight* if given any $\varepsilon > 0$, there exists a compact set K_ε such that $\mu(K_\varepsilon) > 1 - \varepsilon$. Let F denote the dual space of E. The characteristic functional $\varphi(y)$ for $y \in F$ is defined by the relation

$$\varphi(y) = \int_E e^{i<x, y>} \mu(dx) .$$

It can be shown that this functional φ possess the following properties. (1) $\varphi(0) = 1$. (2) φ is positive-definite, i.e., for any N, any $y_1, y_2, \cdots, y_N \in F$ and any complex numbers $\alpha_1, \cdots, \alpha_N$,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \varphi(y_i - y_j)\alpha_i \bar{\alpha}_j \geq 0 \ .$$

(3) For every fixed $y \in F$, the function $\varphi(ty)$ of a real argument t is continuous. In general, these three properties *do not* characterize the characteristic functionals of countably additive probability measures μ on $(E, \mathscr{B})$. Any functional φ on F which satisfies the properties (1), (2), and (3) is the characteristic functional of cylinder set measure or a weak distribution on E. For the definitions of cylinder set measures or weak distributions and other results, the reader is referred to Prohorov [9], Mourier [6] or Gelfand and Vilenkin [3].

The following theorem can be proved from the results of Minlos [5] and Schwartz [10] as pointed out by Mourier [6]. Recently Badrikian [1] was able to extend this results to more general spaces.

MINLOS-SCHWARTZ THEOREM. *Let $(E, \mathscr{T})$ be a locally convex separable topological vector space such that the dual space F is nuclear when provided with the topology of uniform convergence on compact convex subsets of E. Then every positive definite continuous functional φ on F with $\varphi(0) = 1$ is the characteristic functional of a tight probability measure on $(E, \mathscr{B})$.*

Our aim in this paper is to give a similar characterization for infinitely divisible characteristic functionals on locally convex separable topological vector spaces E whose dual spaces F are nuclear when provided with the topology of uniform convergence on compact convex subsets of E. Similar results were obtained by Johansen [4] in the case of real line and by Prakasa Rao [7, 8] for Hilbert spaces and locally compact abelian groups.

2. **Infinitely divisible distributions.** Unless otherwise stated, we shall suppose here after that E is a locally convex separable topological vector space, $\mathscr{B}$ is the σ-field of Borel subsets of E and the dual space F of E is nuclear when provided with the topology of uniform convergence on compact convex subsets of E. A tight probability measure μ on $(E, \mathscr{B})$ is said to be *infinitely divisible* if there exists another probability measure λ_n on $(E, \mathscr{B})$ such that $\mu = \lambda_n^n$ for each n where λ_n^n denotes the measure obtained by convoluting λ_n for n times. Structures of tight infinitely divisible laws and Levy-Khintchin type representation of their characteristic functionals have been discussed by Tortrat [11, 12] and Fernique [2]. It is clear from the theorem and the corollary in Tortrat [12] (see page 325) that the characteristic functional $\varphi(.)$ of a tight infinitely divisible law on $(E, \mathscr{B})$ does not vanish and $\varphi(y) = \exp \psi(y)$ where

$\psi(.)$ is a complex-valued function on F with $\psi(0) = 0$. We shall call ψ the logarithm of the characteristic functional φ.

MAIN THEOREM. *Let $(E, \mathscr{T})$ be a locally convex separable topological vector space such that the dual space F is nuclear when provided with the topology of uniform convergence on compact convex subsets of E. Then a necessary and sufficient condition for a complex-valued functional ψ on F to be the logarithm of the characteristic functional of a tight infinitely divisible probability measure μ on $(E, \mathscr{B})$ is that*

(i) *$\psi(0) = 0, \psi(y) = \overline{\psi(-y)}$ for $y \in F$,*

(ii) *for every choice y_i, $1 \leqq i \leqq N$ in F and complex numbers α_i, $1 \leqq i \leqq N$ such that $\sum_{i=1}^{N} \alpha_i = 0$,*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \psi(y_i - y_j)\alpha_i \bar{\alpha}_j \geqq 0, \ and$$

(iii) *ψ is a continuous functional on F.*

We shall now state a lemma which will be used in the proof of the main theorem.

LEMMA. *Let $\psi(.)$ be any complex-valued functional on F such that $\psi(0) = 0, \psi(y) = \overline{\psi(-y)}$. Then the following conditions are equivalent.*

(1) *For every choice y_i, $1 \leqq i \leqq N$ in F and complex numbers α_i, $1 \leqq i \leqq N$ and $\lambda > 0$,*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} e^{\lambda \psi(y_i - y_j)}\alpha_i \bar{\alpha}_j \geqq 0 .$$

(2) *For every choice y_i, $1 \leqq i \leqq N$ in F and complex numbers α_i, $1 \leqq i \leqq N$ such that $\sum_{i=1}^{N} \alpha_i = 0$,*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \psi(y_i - y_j)\alpha_i \bar{\alpha}_j \geqq 0 .$$

(3) *For every choice y_i, $1 \leqq i \leqq N$ in F and complex numbers α_i, $1 \leqq i \leqq N$,*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} [\psi(y_i - y_j) - \psi(y_i) - \psi(-y_j)]\alpha_i \bar{\alpha}_j \geqq 0 .$$

This lemma has been proved for complex-valued functions on the real line by Johansen [4]. We shall omit the proof of this lemma as it is similar to the proof in the case of real line.

3. **Proof of the main theorem.** Suppose φ is the characteristic functional of a tight infinitely divisible probability measure μ on $(E, \mathscr{B})$. Let ψ denote the logarithm of φ. (i) follows from the

definition of the functional φ. Since μ is infinitely divisible, $\varphi^{1/n}$ is a uniquely determined characteristic functional of probability measure for every n and hence φ^r is a characteristic functional of a probability measure for every rational number r. Hence by the property of a characteristic functional, for any choice of y_i, $1 \leq i \leq N$ in F and complex numbers α_i, $1 \leq i \leq N$, we have

$$\sum_{i=1}^{N} \sum_{j=1}^{N} [\varphi(y_i - y_j)]^r \alpha_i \bar{\alpha}_j \geq 0 .$$

Let r_n approach $\lambda > 0$ as n approaches infinity. Then it follows that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} [\varphi(y_i - y_j)]^\lambda \alpha_i \bar{\alpha}_j \geq 0$$

which in turn proves that $\psi = \log \varphi$ satisfies the condition (ii) of the theorem in view of the lemma. Since φ is a continuous functional on F and is nonvanishing, $\psi = \log \varphi$ is a continuous functional on F. This proves the condition (iii) of the theorem which completes the proof of the necessity.

Conversely suppose ψ is a complex-valued function satisfying conditions (i), (ii) and (iii) of the theorem. Let $\varphi(y) = \exp \psi(y)$ for $y \in F$. Clearly $[\varphi(0)]^\lambda = 1$ for any $\lambda > 0$ and φ^λ is a continuous functional on F for any $\lambda > 0$. Since

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \psi(y_i - y_j) \alpha_i \bar{\alpha}_j \geq 0$$

for every choice of y_i, $1 \leq i \leq N$ in F and complex numbers α_i, $1 \leq i \leq N$ such that $\sum_{i=1}^{N} \alpha_i = 0$, it follows from the lemma,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} e^{\lambda \psi(y_i - y_j)} \alpha_i \bar{\alpha}_j \geq 0$$

or equivalently,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} [\varphi(y_i - y_j)]^\lambda \alpha_i \bar{\alpha}_j \geq 0$$

for every choice of y_i, $1 \leq i \leq N$ in F and every choice of complex numbers α_i, $1 \leq i \leq N$. In other words φ^λ is a positive definite functional on F. Furthermore φ^λ is a continuous functional on F for any $\lambda > 0$ since φ is a continuous functional on F. Hence by the Minlos-Schwartz theorem, φ^λ is a characteristic functional of a tight probability measure for every $\lambda > 0$. In particular, it follows that $\varphi^{1/n}$ is a characteristic functional of a tight probability measure on $(E, \mathscr{B})$. Hence φ is the characteristic functional of an infinitely divisible tight probability measure μ on $(E, \mathscr{B})$. This completes the proof of sufficiency.

4. **Remarks.** Similar results have been obtained for infinitely

divisible characteristic functionals on locally compact abelian groups and on Hilbert spaces in Prakasa Rao [7, 8]. Recently Badrikian [1] has extended Minlos-Schwartz theorem to more general spaces and our main theorem can also be extended to such spaces.

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