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# SYMPLECTIC BORDISM, STIEFEL-WHITNEY NUMBERS, AND A NOVIKOV RESOLUTION

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**Using an Adams type spectral sequence due to Novikov, this paper presents a proof of:**

THEOREM A. **If $M$ is a manifold representing a class in the symplectic bordism group $\Omega_m^{Sp}$, $m \neq 8k$, then $M$ bounds an unoriented manifold.**

The method of proof yields some further information; a more precise statement may be found in § 4 below.

The complex Thom spectrum $MU$ defines a (generalized) cohomology theory $U^*$. The ground ring in this theory, $\Lambda_* = U^*(pt)$ is isomorphic to the complex bordism ring $\Omega_*^U$, where $\Lambda_*$ has nonpositive grading and $\Omega_*^U$ nonnegative. Novikov [8] computed the algebra $A^U$ of operations for the theory $U^*$, $A^U \cong \Lambda^* \hat{\otimes} S$. Here $\hat{\otimes}$ denotes completed tensor product over $Z$ (cf. [5]), and $S$ is a Hopf algebra over $Z$ generated by the set of operations $s_\alpha$, one for each partition $\alpha$ of an integer $|\alpha|$. Novikov also constructed a spectral sequence

$$E_2 = \operatorname{Ext} A^U(U^*(X), \Lambda^*) \Rightarrow \pi_*(X)$$

converging to the stable homotopy ring of a ring spectrum $X$ (cf. [1]). We apply this theory to derive information about $\Omega_*^{Sp}$, the homotopy of the symplectic Thom spectrum $MSp$. In section one the structure of $U^*(MSp)$ is investigated; section two describes a resolution for $U^*(MSp)$; section three computes the necessary part of the $E_2$ term of the spectral sequence; section four completes the proof of Theorem A.

1. Recall that $\Lambda^*$ is a polynomial ring over $Z$ on generators $t_i \in \Lambda_{-2i}$. Also $H^*(BSp)$ is a polynomial ring over $Z$ on the symplectic Pontrjagin classes $P_i \in H^{4i}(BSp)$. It follows from the Thom isomorphism and the Atiyah-Hirzebruch spectral sequence that there is an isomorphism of $\Lambda_*$-modules

$$F\colon \Lambda_* \hat{\otimes} H^*(BSp) \to U^*(MSp)$$

given by

$$F(1 \hat{\otimes} P_i) = (-1)^i s_{\Delta_{2i}}(u)\ .$$

Here $u$ denotes the Thom class in $U^0(MSp)$ and $\Delta_n$ is the partition of $n$ consisting entirely of ones. The proof is similar to [3, p. 49].

In order to study the action of $A^U$ on $U^*(MSp)$, let $E\colon A^U \to U^*(MSp)$ be the map which evaluates operations on the Thom class. We will determine the "top dimension" of $E(s_\alpha)$. There is a natural transformation

$$B\colon U^*(\cdot) \to H_*(MU) \hat{\otimes} H^*(\cdot)$$

defined by the commutativity of the diagram

$$\begin{array}{ccc} U^*(X) & \xrightarrow{B} & H_*(MU) \hat{\otimes} H^*(X) \\ & {\scriptstyle i}\searrow & \uparrow{\scriptstyle \cong} \\ & & \operatorname{Hom}(H^*(MU), H^*(X)) \end{array}$$

where $i$ is defined by taking induced maps in integral cohomology. Note that on $U^*(pt) = \Lambda_*$, $B$ is just the Hurewicz map. Consider the $Z$ basis for $H^*(BU)$ consisting of an element $c_\alpha$ for every partition $\alpha$, where $c_\alpha$ is the $\alpha$ symmetric function of the Chern classes $c_i = c_{\Delta_i}$ [cf. 2]. Similarly consider the $\Lambda_*$-basis for $U^*(BU)$ consisting of the Conner-Floyd characteristic classes $cf_\alpha$ [4]. Finally let $H_*(MU)$ be given as the integral polynomial ring on classes $a_i \in H_{2i}(MU)$, and for $\omega = (i_1, \cdots, i_n)$ let $a^\omega = a_{i_1} \cdot \cdots \cdot a_{i_n}$.

PROPOSITION 1. *If $B\colon U^*(BU) \to H_*(MU) \hat{\otimes} H^*(BU)$ is the map defined above, then*

$$B(cf_{\Delta_k}) = \sum a^\omega \hat{\otimes} c_{\Delta_k} \cdot c_\omega \, ,$$

*where the sum is over all partitions $\omega$ of length at most $k$.*

*Proof*. Suppose $g\colon CP(\infty) \to MU(1)$ is a homotopy equivalence representing a class $y \in U^2(CP(\infty))$ which generates $U^*(CP(\infty))$ as a polynomial ring over $\Lambda_*$. Similarly let $c \in H^2(CP(\infty))$ be a generator for $H^*(CP(\infty))$. Now if $b_i \in H^{2i}(MU)$ is dual to $a_i \in H_{2i}(MU)$, we have $g^*(b_i) = c^{i+1}$. So $B\colon U^*(CP(\infty)) \to H_*(MU) \hat{\otimes} H^*(CP(\infty))$ is given by

$$B(y) = \sum_{i \geqq 0} a_i \hat{\otimes} c^{i+1} \, .$$

In the limit $CP(\infty) = BU(1) \to BU$, this is the statement of the proposition for $k = 1$, since $c_{\Delta_1} \cdot c_{(n)} = c_{(n+1)} \equiv (c_{\Delta_1})^{n+1}$ modulo the ideal generated by $c_2, c_3, \cdots$. This ideal restricts to zero in $BU(1)$, so $B(cf_{\Delta_1})$ is as claimed. The proposition now follows by an application of the splitting principle.

Let $f\colon BSp \to BU$ classify the universal symplectic bundle $\gamma$ over $BSp$. Then we have immediately:

PROPOSITION 2. *The map* $B: U^*(BSp) \to H_*(MU) \hat{\otimes} H^*(BSp)$ *is given by*

$$B(cf_{\Delta_k}(\gamma)) = \sum a^\omega \hat{\otimes} f^*(c_{\Delta_k} \cdot c_\omega) ,$$

*where the sum is over all partitions* $\omega$ *of length at most* $k$.

Note that $f^*(c_\alpha)$ is given by replacing the odd elementary symmetric fuctions in the $\alpha$ symmetric function with zero, and the $2i$th elementary symmetric function with $(-1)^i P_i$. In particular,

$$f^*(c_{\Delta_{2k+1}}) = 0$$
$$f^*(c_{\Delta_{2k}}) = (-1)^k P_k .$$

Next we consider the following commutative diagram:

$$\begin{array}{ccccc} U^*(MU) & \xrightarrow{E} & U^*(MSp) & \xleftarrow{F} & \Lambda^* \hat{\otimes} H^*(BSp) \\ \Phi \uparrow & & \Phi \uparrow & & \\ U^*(BU) & \xrightarrow{U^*(f)} & U^*(BSp) & \xrightarrow{B} & H_*(MU) \hat{\otimes} H^*(BSp) \end{array}$$

where $\Phi$ is the Thom isomorphism. By definition, $s_\alpha = \Phi(cf_\alpha)$, so we have $E(s_\alpha) = (cf_\alpha(\gamma))$. Let $K$ be the subring of $U^*(BU)$ generated by $\{cf_{\Delta_{2i}}\}$, so that $U^*(f)|_K$ is an isomorphism of $K$ with $U^*(BSp)$. Now since $B$ is a monomorphism, it will determine the Hurewicz image of coefficients in $\Lambda_*$ expressing $cf_\alpha(\gamma)$ in terms of $cf_{\Delta_{2i}}(\gamma)$. But $F$ was chosen so that $\Phi(cf_{\Delta_{2i}}(\gamma)) = s_{\Delta_{2i}}(u) = F(1 \hat{\otimes} (-1)^i P_i)$, thus we have the coefficients in $F^{-1}(E(s_\alpha))$ determined recursively. The first step is given by

PROPOSITION 3. *Let* $\rho: \Lambda_* \hat{\otimes} H^*(BSp) \to \Lambda_0 \otimes H^*(BSp)$ *be projection on the top dimension in* $\Lambda_*$. *Then*

$$\rho \circ F^{-1} \circ E(s_\alpha) = 1 \otimes f^*(c_\alpha) .$$

*Proof.* Let $\rho': H_*(MU) \hat{\otimes} H^*(BSp) \to H_0(MU) \otimes H^*(BSp)$ be projection, then by Proposition 2

$$\rho' \circ B(cf_{\Delta_k}(\gamma)) = 1 \otimes f^*(c_{\Delta_k}) .$$

Thus $\rho' \circ B(cf_\alpha(\gamma)) = 1 \otimes f^*(c_\alpha)$. Now the Hurewicz map $\Lambda_0 \to H_0(MU)$ is the identity, so $\rho' \circ B = \rho \circ F^{-1} \circ \Phi$, and the proposition follows. This formula is an explicit expression for the top dimension of $E(s_\alpha)$.

2. From this information on the $A^U$-module structure of $U^*(MSp)$, we will construct a resolution for $U^*(MSp)$. Let $\kappa_\alpha$ be the unique

element of the subring $K$ of $U^*(BU)$ such that $U^*(f)(\kappa_\alpha) = U^*(f)(cf_\alpha)$. Let $\mathscr{R}_\alpha = \Phi(\kappa_\alpha)$, so $(s_\alpha - \mathscr{R}_\alpha)$ is an element of the kernel of $E$. Let $\Theta_n$ be the set of those partitions $\omega$ of $n$ which cannot be written $\omega = (\alpha, \alpha)$, and let $\Theta = \bigcup_{n>0} \Theta_n$.

THEOREM 1. *The set* $\{(s_\beta - \mathscr{R}_\beta): \beta \in \Theta\}$ *generates the kernel of* $E$ *as a free* $\Lambda_*$*-module.*

For the proof of this theorem, we require some data on symmetric functions. Recall the classes $c_\omega \in H^*(BU)$, and define $c^\alpha = c_{\Delta_{i_1}} \cdot \cdots \cdot c_{\Delta_{i_n}}$, if $\alpha = (i_1, \cdots, i_n)$. Introduce a linear ordering, $>$, on the set of partitions of $k$ by taking the longest first and ordering lexicographically among partitions of the same length. For every partition $\omega$ of $k$, we define another partition $T(\omega)$ of $k$ as follows: $T(\omega) = (r_1 + \cdots + r_q, r_2 + \cdots + r_q, \cdots, r_q)$, where $q$ is the largest integer in $\omega$, and $r_j$ is the number of $j$'s in $\omega$. Note that $\beta \notin \Theta$ if and only if $T(\beta) = 2\alpha$. Then the following lemmas are elementary.

LEMMA 1. *There are integers* $m(\alpha, \beta)$ *for every pair of partitions* $\alpha, \beta$ *of* $k$ *such that* $c^\alpha = \sum m(\alpha, \beta) c_\beta$. *Moreover,* $m(\beta, T(\beta)) = 1$ *and* $m(\alpha, \beta) = 0$ *for* $\beta > T(\alpha)$.

LEMMA 2. *There are integers* $\bar{m}(\beta, \alpha)$ *for every pair of partitions* $\alpha, \beta$ *of* $k$ *such that* $c_\beta = \sum \bar{m}(\beta, \alpha) c^\alpha$. *Moreover,* $\bar{m}(\beta, T(\beta)) = 1$ *and* $\bar{m}(\beta, T(\gamma)) = 0$ *for* $\gamma > \beta$.

Now suppose for every partition $\alpha$ of $|\alpha|$ there is given an element $x_\alpha \in \Lambda_{2|\alpha|-d}$, so that $\sum x_\alpha s_\alpha$ is an operation of degree $d$ in $A^U$, written in Novikov's notation [8]. Suppose that $E(\sum x_\alpha s_\alpha) = 0$, and that $x_\alpha = 0$ for $|\alpha| < k$. We write $\rho_k$ for the projection $S \hat{\otimes} \Lambda_* \to S_k \otimes \Lambda_*$ onto elements of degree $k$ in $S$. Now proceeding by induction on $k$, for the proof of Theorem 1 it will suffice to show

$$\rho_k(\sum x_\alpha s_\alpha) = \rho_k\Big(\sum_{\beta \in \Theta} y_\beta (s_\beta - \mathscr{R}_\beta)\Big)$$

for some unique coefficients $y_\beta \in \Lambda$.

First consider the case of odd $k$. For $|\alpha| = k$ odd, we have $\alpha \in \Theta$. From Proposition 2 we have that $\rho' \circ B(cf_{\Delta_k}(\gamma))$ is zero for odd $k$. Thus $\kappa_\alpha = \sum_{|\gamma|>k} y_\alpha c f_\gamma$, and $\rho_k(\mathscr{R}_\alpha) = 0$, and $\rho_k(\sum x_\alpha s_\alpha) = \rho_k(\sum_{|\alpha|=k} x_\alpha(s_\alpha - \mathscr{R}_\alpha))$. By Proposition 3, $k \geqq 1$, so this also provides the initial case for the induction, $k = 1$.

For $k$ even, since $E(\sum x_\alpha s_\alpha) = 0$ we have

$$\rho \circ F^{-1} \circ E(x_\alpha s_\alpha) = 0 ,$$

so

$$\sum_{|\alpha|=k} x_\alpha \otimes f^*(c_\alpha) = 0 ,$$

and

$$\sum_{|\alpha|=k} (-1)^{|\gamma|} x_\alpha \bar{m}(\alpha, 2\gamma) = 0$$

for every $\gamma$ with $2|\gamma| = k$. Now by Lemma 2, these equations may be solved uniquely for $x_\alpha$, $\alpha \notin \Theta$ in terms of $x_\alpha$, $\alpha \in \Theta$. Thus it suffices to prove that the matrix indexed by $\alpha, \beta \in \Theta_k$ whose $(\alpha, \beta)$ entry is the coefficient of $s_\alpha$ in $(s_\beta - \mathscr{R}_\beta)$ is invertible. Notice that Proposition 3 implies

$$\rho_{|\beta|}(\mathscr{R}_\beta) = \sum_{2|\gamma|=|\beta|} (-1)^{|\gamma|} \bar{m}(\beta, 2\gamma)\Big(\sum_\eta m(2\gamma, \eta) s_\eta\Big) .$$

Then by Lemmas 1 and 2, if the coefficient of $s_\eta$ is $\mathscr{R}_\beta$ is nonzero, we have $\eta < \beta$. This completes the proof of Theorem 1.

We now construct the first stage of a resolution; the remaining stages may be obtained by a simple iteration. Let $C_0 = A^U$ and let $C_1$ be the free $A^u$-module generated by $\{G_\beta : \beta \in \Theta\}$. Define $d_1 : C_1 \to C_0$ by $d_1(G_\beta) = s_\beta - \mathscr{R}_\beta$. Then the following sequence is exact:

$$0 \longleftarrow U^*(MSp) \xleftarrow{E} C_0 \xleftarrow{d_1} C_1 .$$

There is an isomorphism Hom $A^U(A^U, \Lambda_*) \cong \Omega^U_*$ defined by evaluation on the Thom class followed by the Atiyah duality isomorphism. The gradings are nonnegative here, so we take $\Omega^U_*$ rather than $\Lambda_*$. Thus if $g_\beta : C_1 \to \Lambda_*$ is the dual of $G_\beta$, we have

$$\Omega^U_* \cong \operatorname{Hom}_{A^U}(C_0, \Lambda_*) \xrightarrow{d_1} \operatorname{Hom}_{A^U}(C_1, \Lambda_*)$$

given by

$$d_1^*(y) = \sum_{\beta \in \Theta} (s_\beta - \mathscr{R}_\beta)(y) g_\beta .$$

3. At this point we may compute

$$E_2^{0,*} = \operatorname{Ext}^{0,*}_{A^U}(U^*(MSp), \Lambda_*) = \ker d_1^* .$$

LEMMA 3. *Let $X \in \Omega^U_{2n}$ be dual to $z \in \Lambda_{-2n}$. Then $d_1^*(X) = 0$ if and only if $(s_\omega - \mathscr{R}_\omega)(z) = 0$ for all $\omega \in \Theta_n$.*

*Proof.* Suppose there is a $\beta \in \Theta$, $|\beta| \neq n$, such that $(s_\beta - \mathscr{R}_\beta)(z) \neq 0$.

It will suffice to find $\gamma \in \Theta_n$ with $(s_\gamma - \mathscr{R}_\gamma)(z) \neq 0$. Let $(s_\beta - \mathscr{R}_\beta)(z) = y \in \Lambda_{-2k}$, $y \neq 0$, $k \neq 0$. Then there is an $\alpha$, $|\alpha| = k$, such that $s_\alpha(y) \neq 0 \in \Lambda_0$. By Theorem 1, we may express $s_\alpha(s_\beta - \mathscr{R}_\beta)$ in terms of $\{s_\gamma - \mathscr{R}_\gamma\colon \gamma \in \Theta\}$, so there is a $\gamma \in \Theta_n$ with $(s_\gamma - \mathscr{R}_\gamma)(z) \neq 0$.

THEOREM 2. *$E_2^{0,*}$ is a polynomial ring over $Z$ with one generator $X_i$ in every dimension $4i \geqq 0$.*

*Proof.* Since $E_2^{0,*}$ is a subring of $\Omega_*^U$ given as the kernel of a map of free abelian groups, it suffices to count dimensions. The theorem now follows from Lemma 3.

It is interesting to note that Lemma 3 together with Proposition 3 gives an explicit criterion for the elements $X_i \in \Omega_{4i}^U$. These elements $X_i$ are polynomial generators for $\Omega_*^{Sp} \otimes Q$.

4. The proof of Theorem A requires two further facts.

PROPOSITION 6. *For $X \in E_2^{0,*}$, the image $[X]_2$ of $X$ in the unoriented bordism ring $\mathfrak{N}_*$ is a fourth power.*

*Proof.* It will suffice to show that the dual Stiefel—Whitney numbers $\bar{w}_\alpha(X)$ vanish for $\alpha \neq (\gamma, \gamma, \gamma, \gamma)$. Recall [10, p. 256] that the $\omega$ symmetric function, $\omega \in \Theta$, is contained in the ideal generated by 2 and the odd elementary symmetric functions. Thus $\rho_{|\omega|}(\mathscr{R}_\omega)$ is divisible by 2, and $s_\omega(z) \equiv 0 \pmod 2$ for $\omega \in \Theta_{2n}$, and $z$ the dual of $X \in \ker d_1^*$ in dimension $4n$. But for such $X$ and $\omega$, $s_\omega(z) = c_\omega(\nu X)$, the normal Chern numbers. These reduce mod 2 to the dual Stiefel—Whitney numbers.

$$c_\omega(\nu X) \equiv \bar{w}_{\omega,\omega}(X) \bmod 2 ,$$

so for $\omega \in \Theta_{2n}$, $\bar{w}_{\omega,\omega}(X) = 0$. Since $X \in \Omega_*^U$, $[X]_2$ is a square [7], so $\bar{w}_\alpha(X) = 0$ for $\alpha \neq (\omega, \omega)$. The only possible $\alpha$ for which $\bar{w}_\alpha(X) \neq 0$ is thus $\alpha = (\gamma, \gamma, \gamma, \gamma)$.

Novikov shows that $\mathrm{Ext}_{AU}^{s,*}(U^*(Y), \Lambda_*)$ is a torsion group for $s > 0$, for any $Y$ [8]. Thus integral multiples of the $X_i$ are generators for $\Omega_{4i}^{Sp}$. Moreover the $E_2$ term contains only 2-torsion, as may be seen from [6, 8], so the multipliers are all powers of two. Recall the generators $t_i \in \Omega_{2i}^U$, and let $t^\omega = t_{i_1} \cdot \cdots \cdot t_{i_n}$ for $\omega = (i_1, \cdots, i_n)$.

PROPOSITION 7. *Let $X_i$ be as in Theorem 2, with $X_i = \sum a(\omega) t^\omega$ for integer coefficients $a(\omega)$. Suppose $[X_i]_2 \neq 0$. Then there is an $\omega = (2\alpha, 2\alpha)$ with $a(\omega) \equiv 1 \pmod 2$.*

*Proof.* By Proposition 6 there are $Y, Y' \in \Omega_*^U$ such that $X_i = Y^2 + 2Y'$, since $[Y^2]_2$ is a fourth power, by [7]. Thus $a(\omega) \equiv 0 \pmod 2$ unless $\omega = (\beta, \beta)$. However if $\beta$ contains an odd number the symplectic Pontrjagin numbers of $t^\beta$ are all zero for dimensional reasons. Thus if $a(2\alpha, 2\alpha) \equiv 0 \pmod 2$ for all $\alpha$, the Stiefel—Whitney numbers of $X_i$ vanish, and $[X_i]_2 = 0$.

THEOREM 3. *Suppose $X \in \Omega_*^{Sp}$ and $[X]_2 \neq 0$. Then $X$ is in the subring of $\Omega_*^{Sp}$ generated by those $X_{2i} \in E_2^{0,8i} \subset \Omega_{8i}^U$ on which all differentials in the spectral sequence vanish.*

*Proof.* Since $|(2\alpha, 2\alpha)| = 4|\alpha|$, it follows from Proposition 7 that $[X_i]_2 \neq 0$ implies $i$ is even. The rest of the statement follows immediately from the existence of the spectral sequence.

Now Theorem A is just a simplification of Theorem 3. It should be noted that the map $\Omega_*^{Sp} \to \mathfrak{N}_*$ factors thru $\Omega_*^U$, so any torsion element of $\Omega_*^{Sp}$ bounds in $\mathfrak{N}_*$. Moreover $\Omega_*^{Sp} \otimes Q$ is a polynomial algebra on $X_i \in \Omega_{4i}^{Sp} \otimes Q$, so for $X \in \Omega_n^{Sp}$, $[X]_2 = 0$ unless $n = 4k$. Thus the content of Theorem A is that $[\Omega_{8k+4}^{Sp}]_2 = 0$.

The author has been informed of some recent work of E. E. Floyd which overlaps considerably with the above results. Using very different methods, Floyd gives a more refined upper bound for the image of $\Omega_*^{Sp}$ in $\mathfrak{N}_*$.

This work formed part of the author's doctoral thesis at Northwestern University, under the direction of Professor Mark Mahowald. A summary appeared as [9].

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