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TRANSVERSALLY PERTURBED PLANAR DYNAMICAL SYSTEMS

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This paper investigates the behavior of limit cycles of a planar dynamical system which has been perturbed transversally. In particular, it is shown that if C is a limit cycle of the unperturbed dynamical system, then there are limit cycles of the perturbed dynamical systems arbitrarily close to C. Also, if C is an exterior limit cycle of the unperturbed dynamical system, then there is an outer neighborhood of C which consists solely of cycles of the perturbed dynamical systems.

In what follows R and R^2 will denote the reals and the plane respectively.

A dynamical system is an ordered pair (X, π) consisting of a topological space X and a mapping π of $X \times R$ into X such that (where $x\pi t = \pi(x, t)$)

(i) $x\pi t = x$ for all $x \in X$

(ii) $(x\pi t)\pi s = x\pi(t+s) = x_\pi(s+t)$ for all $x \in X$ and $s, t \in R$

(iii) π is continuous in the product ropology.

A point $x \in X$ is called critical if and only if $x\pi t = x$ for every $t \in R$. A point $x \in X$ is called periodic if and only if x is noncritical and $x\pi t = x$ for some $t > 0$; if X is Hausdorff the least such t is called the fundemental pariod of x. If x is periodic, $x\pi R$ is called a cycle. A cycle is a simple closed curve. Hence, if C is a cycle of a planar dynamical system (R^2, π), then C decomposes R^2 into two components; one bounded and denoted by int C; the other unbounded and denoted by ext C. A subset A of X is called a trajectorial arc if and only if there is an $x \in X$ and a compact interval $[a, b]$, $a \neq b$, such that $A = x\pi[a, b]$.

Let (R^2, π) be a dynamical system. A subset T of R^2 is called a transversal if and only if

(i) T is homeomorphic with either [0,1] or S^1, the 1-sphere

(ii) there is an $\varepsilon > 0$ such that $T \cap (T\pi t) = \varnothing$ for $0 < |t| \leqq \varepsilon$.

Our investigation depends heavily upon the following three propositions which may be found in [2, VII, 4.4], [2, VII, 4.7], and [2, VII, 4.8] respectively.

PROPOSITION A. *Let C be a trajectory and T a transversal of a planar dynamical system. If C or T is a closed curve, they have at most one intersection point; if both are closed curves, they do not* *intersect.*

PROPOSITION B. *Let $C \cup T$ be a simple closed curve with C a trajectorial arc and T a transversal of a planar dynamical system. Then one component of $R^2-(C \cup T)$ is positively invariant, the second is negatively invariant, and neither is invariant. The result is also valid if $C = \varnothing$.*

PROPOSITION C. *In a planar dynamical system the interior of each cycle, closed transversal, or simple closed curve consisting of a transversal and a trajectorial arc, all contain a critical point.*

We are interested in studying a family of dynamical systems which is defined as follows. Let $\pi: R^2 \times R \times R \rightarrow R$ be a mapping continuous in the product topology such that

(i) for each $a \in R$ the mapping π_a: $R^2 \times R \rightarrow R^2$ defined by $\pi_a(x, t) = \pi(x, t, a)$ defines a dynamical system on R^2.

(ii) critical points of the dynamical systems are independent of the index.

(iii) the noncritical trajectories of π_a are transversal to the noncritical trajectories of π_b if $a \neq b$, i.e., if T is a trajectorial arc of π_a, then T is a transversal with respact to π_b if $a \neq b$.

$C_a(x)$, $C_a^+(x)$, $L_a^+(x)$, and $L_a^-(x)$ will denote the trajectory, positive semitraiectory, positive limit set, and negative limit set, respectively, of x with respect to π_a. The family of all trajectories of π_a, a fixed, will be called a system and the family of all trajectories will be called a complete family.

In [1] and [4] sufficient conditions are given which assure that the differential equations

$$\dot{x} = P(x, y, a), \qquad \dot{y} = Q(x, y, a),$$

where the dots stand for differentiation with respect to the independent vaiable t and a is a paremeter, define a complete family.

Immediate consequences of Propositions A and C are the following two propositions.

PROPOSITION 1. *Cycles of distinct systems of a complete family do not interset.*

PROPOSITION 2. *Let x be a noncritical point of a complete family, $a \neq b$, and suppose that $C_a(x)$ and $C_b(x)$ have a point $y, y \neq x$, in common. If the trajectorial arcs of $C_a(x)$ and $C_b(x)$ connecting the points x and y have only their endpoints in common, then the region* *bounded by these trajectorial arcs contains a critical point.*

PROPOSITION 3. *Let C be a cycle of π_a. Then* $\operatorname{int} C$ *is positively invariant with respect to π_b for all $b > a$ or* $\operatorname{int} C$ *is negatively invariant with respect to π_b for all $b > a$, but in neither case is* $\operatorname{int} C$ *invariant with respect to π_b for any $b > a$. A similar result holds for $b < a$.*

Proof. Consider the sets

$A = \{b \in (a, +\infty):$ int C is positively invariant with respect to $\pi_b\}$

$B = \{b \in (a, +\infty):$ int C is negatively invariant with respect to $\pi_b\}$.

By Proposition B, int C is positively invariant or negatively invariant, but not both, with respect to each π_b, $b > a$. Thus $A \cup B = (a, +\infty)$ and $A \cap B = \emptyset$. We now show that both A and B are open. If $c \in (a, +\infty) - A = B$, then there exist $x \in \operatorname{int} C$ and $t > 0$ such that $x\pi_c t \in \operatorname{ext} C$. Since π is continuous $x\pi_b t \in \operatorname{ext} C$ for all b sufficiently close to c. Hence B is open. Similarly A is open. The connectivity of $(a, +\infty)$ implies either A or B must be empty. This completes the proof.

PROPOSITION 4. *Let C be a cycle of π_a. If* $\operatorname{int} C$ *is positively invariant with respect to every π_b, $b > a$, then* $\operatorname{ext} C$ *is positively invariant with respect to every π_b, $b < a$. A similar result holds if $b > a$ and $b < a$ are interchanged.*

Proof. Let $x \in C$ and T be a trajectorial arc of $C_c(x)$, $c > a$, which contains x as a nonend point. Then T is a transversal with respect to π_b, $b \neq c$, Moreover, if τ is the fundamental period of C, then $T\pi_a[-\tau, \tau]$ is a connected neighborhood of C which contains no critical points. Choose a neighborhood U of x, $0 < \sigma < |c\text{-}a|$, and $0 < \varepsilon < \tau$ so small that $U\pi_b[-\varepsilon, \varepsilon] \subset T\pi_a[\tau, \tau]$ for all $b \in [a\text{-}\sigma, a + \sigma]$. This is possible because π is continuous. We can now define a mapping h of $[a, a+\sigma]$ into $S = \{x\pi_b\varepsilon\colon b \in [a, a+\sigma]\}$ by $h(b) = x\pi_b\varepsilon$. h is continuous since π is continuous. For $b \neq d$, $x\pi_b\varepsilon$ and $x\pi_d\varepsilon$ cannot be equal; for if they were Proposition 2 would imply that $T\pi_a[-\tau, \tau]$ contains a critical point. Hence h is one-to-one. Obviously, h is an onto mapping. A one-to-one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism. Thus S is an arc. Since int C is, by assumption, positively invariant with respect to π_b, $b > a$, we have $S \subset \overline{\operatorname{int} C}$. Moreover, $(x\pi_a[0, \varepsilon]) \cup S \cup (x\pi_{a+\sigma}[0, \varepsilon])$ forms a simple closed curve J such that $\operatorname{int} J \subset T\pi[-\tau, \tau]$ and $\overline{\operatorname{int} J}$ is a neighborhood of $x\pi_a\varepsilon/2$ relative to $\overline{\operatorname{int} C}$. Let $y \in \operatorname{int} J$ and set

$$J_t = (x\pi_a[0, t]) \cup (x\pi_{a+\sigma}[0, t]) \cup \{x\pi_b t\colon\ b \in [a, a+\sigma]\} .$$

For each t, $0 < t < \varepsilon$, J_t is a simple closed curve. Since π is continuous, $y \in \operatorname{ext} J_t$ for t sufficiently small. But for $t = \varepsilon$, $y \in \operatorname{int} J_\varepsilon = \operatorname{int} J$. The continuity of π implies there is an $s \in (0, \varepsilon)$ such that $y \in J_s$. By the construction of J_s and since $y \in \operatorname{int} J$, y must be an element of $\{x\pi_b s\colon\ b \in [a, a+\sigma]\}$. This shows that $\overline{\operatorname{int} J}$ consists solely of trajectorial arcs from the systems π_b, $b \in [a, a+\sigma]$.

Now let V be a neighborhood of $x\pi_a\varepsilon/2$ such that $V \cap \operatorname{int} C \subset \operatorname{int} J$. Then there is an α, $0 < \alpha < \sigma$, such that $x\pi_b\varepsilon/2 \in V$ for all $b \in [a\text{-}\alpha, 0]$. For $b \in [a\text{-}\alpha, 0)$, $x\pi_b\varepsilon/2$ cannot be an element of $\overline{\operatorname{int} C}$ for then

$$x\pi_b\varepsilon/2 \subset V \cap \operatorname{int} C \subset \operatorname{int} J \subset \bigcup \{x\pi_c[0, \varepsilon]\colon\ c \in [a, a+\sigma]\} .$$

This, by Proposition 2, implies that $T\pi_a[-\tau, \tau]$ contains a critical point. Hence for $b \in [a\text{-}\alpha, 0)$ we have $x\pi_b\varepsilon/2 \in \operatorname{ext} C$ and therefore, by Proposition B, $C_b^+(x) \subset \operatorname{ext} C$. Proposition 3 now implies the desired result.

Proposition 4 allows us to assume throughout the remainder of the paper that if C is a given cycle of π_a, then $\operatorname{int} C$ is positively invariant with respect to every π_b, $b < a$, and negatively invariant with respect to every π_b, $b > a$. If the opposite invariance properties hold, the following propositions remain valid after the obvious modifications are made.

DEFINITION 5. Let C be a cycle of π_a. If there is an $x \in \operatorname{ext} C$ such that $L_a^+(x) = C$ or $L_a^-(x) = C$, then C is called an external limit cycle or a external negative limit cycle, respectively. Similarily, if there is an $x \in \operatorname{int} C$ such that $L_a^+(x) = C$ or $L_a^-(x) = C$, then C is called an internal limit cycle or a internal negative limit cycle, respectively.

DEFINITION 6. Let U be a neighborhood of a simple closed curve C. Then U-$\operatorname{int} C$ and U-$\operatorname{ext} C$ are called an outer neighborhood and an inner neighborhood, respectively, of C.

PROPOSITION 7. *Let C be an external limit cycle of π_a. Then, given any outer neighborhood U of C, there exists an $\varepsilon > 0$ such that, for each $b \in [a, a+\varepsilon]$, U contains both an external limit cycle and an internal limit cycle of π_b (the two cycles may coincide). A similar result holds for C an internal limit cycle and $b \in [a\text{-}\varepsilon, a]$.*

Proof. Let $V \subset U$ be an outer neighborhood of C containing no critical points and such that $\operatorname{int} C \cup V$ is simply connected. Let $x \in C$, $y \in \operatorname{ext} C$ be such that $L_a^+(y) = C$, and $T \subset V$ be a trajectorial arc of

$C_c(x)$, $c < a$, containing x as an endpoint. Then T is a transversal with respect to π_b, $b \neq c$. Since $L_a^+(y) = C$, $y \in \text{ext}\, C$, and V is an outerneighborhood of C, there is a $\tau > 0$ such that $y\pi_a[\tau, +\infty) \subset V$. Let $y_1, y_2 \in y\pi_a[\tau, +\infty)$ be consecutive points of intersection between $C_a^+(y)$ and T with $y_2 \in C_a^+(y_1)$. Then the trajectial arc of $C_a^+(y)$ and the subarc of T connecting y_1 and y_2 form a simple closed curve $J \subset V$ such that $\text{int}\, J\text{-int}\, C \subset V$. Now $L_a^+(y_1) = L_a^+(y) = C \subset \text{int}\, J$ and Proposition B imply $y_2\pi_a(0, +\infty) \subset \text{int}\, J$. Since $y_2 \in C_a^+(y_1)$ and π is continuous there is an $\varepsilon > 0$ such that $C_b^+(y_1)$ intersects $\text{int}\, J$ for $|b\text{-}a| < \varepsilon$. If $y_1\pi_b t \in \text{int}\, J$ for some $t > 0$, then $y_1\pi_b[t, \infty)$ must be a subset of $\text{int}\, J$; for if it were not $y_1\pi_b[t, \infty)$ would intersect J and Proposition 2 would imply $\text{int}\, J - \text{int}\, C$, and hence V, contains a critical point. Moreover, by the continuity of π, and the fact $L_a^+(y_1) = C$, we may assume that ε was chosen so small that $C_b^+(y_1)$, $|b\text{-}a| < \varepsilon$, intersects T at least twice between y_2 and x. This is true because $C_a^+(y_1)$ intersects T infinitely many times and the only limit point of the intersections is x, [2, VIII, 1.2] and [2, VIII, 1.5]. The trajectorial arc connecting two such consecutive points of intersection and the corresponding subarc of T form a simple closed curve J_b such that $\text{int}\, J_b \subset \text{int}\, J$ and $\text{int}\, J_b\text{-int}\, C \subset V$. Moreover, $\text{int}\, J_b$ is positively invariant with respect to π_b by Propoisition B. Thus $\text{int}\, J_b$ and $\text{ext}\, C$ are both positively invariant with respect to π_b. Hence $\text{int}\, J_b\text{-int}\, C$ is positively invariant, so that $C_b^+(x) \subset \overline{\text{int}\, J_b - \text{int}\, C}$ which is compact and contains no critical points. By the Poincaré-Bendixson Theorem, [2, VII, 1.14], $L_b^+(x)$ is a cycle C_b. Since $\text{int}\, J_b$ is positively invariant, but not invariant by Proposition B, and $C_b \cap C = \varnothing$ by Proposition 1, we have $C_b \cap \partial(\text{int}\, J_b\text{-int}\, C) = \varnothing$. Thus C_b is an internal limit cycle of π_b contained in $\text{int}\, J_b \subset U$. For c sufficiently large $y_1\pi_b[c, \infty) \subset \text{int}\, J_b$ and therefore $y_1\pi_b[c, \infty) \subset \text{int}\, J_b - \overline{\text{int}\, C_b}$. The Poincaré-Bendixson Theorem now implies the existence of an external limit cycle. This completes the proof.

In a similar manner it can be shown that

PROPOSITION 8. *Let C be an external negative limit cycle of π_a. Then, given any outer neighborhood U of C, there exists an $\varepsilon > 0$ such that, for each $b \in [a\text{-}\varepsilon, a]$, U contains both an external negative limit cycle and an internal negative limit cycle of π_b (the two cycles may coincide). A similar result holds for C an internal negative limit cycle and $b \in [a, a+\varepsilon]$.*

LEMMA 9. *Let D_1 and D_2 be cycles of a complete family such that $D_1 \subset \text{int}\, D_2$ and that $\text{int}\, D_2 - \text{int}\, D_1$ contains no critical points.*

If C_1 *and* C_2 *are distinct cycles in* $\operatorname{int} D_2 - \operatorname{int} D_1$, *then* $C_1 \subset \operatorname{int} C_2$ or $C_2 \subset \operatorname{int} C_1$.

Proof. Since $\operatorname{int} D_2 - \operatorname{int} D_1$ contains no critical points, we must have $D_1 \subset \operatorname{int} C_i$, $i = 1, 2$. Thus $\operatorname{int} C_1 \cap \operatorname{int} C_2 \neq \emptyset$. Then $\operatorname{int} C_1 \subset \operatorname{int} C_2$ or $\operatorname{int} C_1 \cap \operatorname{ext} C_2 \neq \emptyset$. In the first case $\overline{\operatorname{int} C_1} \subset \overline{\operatorname{int} C_2}$. Therefore $C_1 \subset \operatorname{int} C_2$ or $C_1 \cap C_2 \neq \emptyset$. The latter is impossible by Proposition 1. In the second case, $\partial(\operatorname{int} C_2) \cap \operatorname{int} C_1 \neq \emptyset$. Therefore $C_2 \cap \operatorname{int} C_1 \neq \emptyset$ and $C_2 \subset \operatorname{int} C_1$ since $\operatorname{int} C_1$ is either positively invariant or negatively invariant for the system containing C_2 (Proposition 3).

Let D_1 and D_2 be as in the statement of Lemma 9. Then

LEMMA 10. *If* C_1 *and* C_2 *are distinct cycles in* $\operatorname{int} D_2 - \operatorname{int} D_1$ *such that* $C_1 \subset \operatorname{ext} C_2$, *then* $C_2 \subset \operatorname{int} C_1$

Proof. By Lemma 9, $C_2 \subset \operatorname{int} C_1$ or $C_1 \subset \operatorname{int} C_2$. C_1 cannot be contained in both $\operatorname{int} C_2$ and $\operatorname{ext} C_2$. Therefore $C_2 \subset \operatorname{int} C_1$.

In a topological space X, it is possible to define limits of nets of subsets $X_i \subset X$ as follows. Let $\lim\inf X_i$ consist of all limits of nets of points $x_i \in X_i$; let $\lim\sup X_i$ consist of all limits of subnets of points $x_i \in X_i$. Obviously $\lim\inf X_i \subset \lim\sup X_i$. If equality holds, the net X_i is said to converge to its limit and we write

$$\lim X_i = \lim\inf X_i = \lim\sup X_i .$$

DEFINITION 11. A net (R^2, π_i), i contained in a directed set containing 0, of dynamical system is called regular if

(i) $\pi_i \to \pi_0$ in the sense that if $x_i \to x$ and $t_i \to t$ then $x_i \pi_i t_i \to x \pi_0 t$.

(ii) critical points are independent of the index i.

(iii) to each noncritical point x there corresponds a subset T of R^2 which is a transversal with respect to each π_i and contains x as a nonend point.

In [3] the following theorem is proved.

THEOREM D. *Let* (R^2, π_i) *be a regular net of dynamical systems. Let* $C_i(x_i)$ *be a cycle of* (R^2, π_i) *with fundamental period* $\tau_i(x_i)$. *If* $\lim\inf C_i(x_i) \neq \emptyset$, *then*

(1) *If* $\tau_i(x_i) \to 0$, *then* $\lim C_i(x_i)$ *exists and is a single critical point.*

(2) *If* $\lim\inf C_i(x_i)$ *intersects a cycle* $C_0(x)$, *then* $\tau_i(x_i) \to \tau_0(x)$ *and* $\lim C_i(x_i) = C_0(x)$.

(3) *If* $\lim \inf C_i(x_i)$ *intersects a noncyclic trajectory, then* $\tau_i(x_i) \to +\infty$.

DEFINITION 12. Let $C_a(x)$ be a cycle of π_a. Then $\tau_a(x)$ will denote the fundamental period of x with respect to π_a.

PROPOSITION 13. *Let C be an external limit cycle of π_a. There exists an outer neighborhood U of C and an $\varepsilon > 0$ such that U consists entirely of periodic points of the systems π_b, $b \in [a, a+\varepsilon]$. A similar result holds for C an internal limit cycle and $b \in [a\text{-}\varepsilon, a]$.*

Proof. Let $x \in C$ and V be an outer neighborhood of C which contains no other cycles of π_a or critical points and such that $V \cup \operatorname{int} C$ is simply connected. Moreover, by Theorem D, V may be chosen along with a $\sigma > 0$ such that if $C_b(y)$ is a cycle of π_b in V with $|b\text{-}a| < \sigma$, then $|\tau_a(x)\text{-}\tau_b(y)| < 1/2\tau_a(x)$. By Proposition 7 there is an ε, $0 < \varepsilon < \sigma$ such that, for each $b \in [a, a+\varepsilon], V$ contains a cycle of π_b. Thus the fundamental periods cycles of $\pi_{a+\varepsilon}$ which lie in V are contained in $[1/2\tau_a(x), 3/2\tau_a(x)]$. This, Theorem D with each $i = a + \varepsilon$, and the fact that cycles of distinct systems do not intersect imply that there is a cycle D of $\pi_{a+\varepsilon}$ in V such that $\operatorname{int} D - \operatorname{int} C$ contains no cycle of $\pi_{a+\varepsilon}$. Set $U = \overline{\operatorname{int} D} - \operatorname{int} C$. U is an outer neighborhood of C by Lemma 10. Let A denote the set of periodic points of π_b, $b \in [a, a+\varepsilon]$, which are contained in U. We will show that $A = U$. Assume the contrary that there exists a $w \in U - A$ and consider the sets

$$F = \{\overline{\operatorname{int} C_b(y)}: \quad y \in A, \ C_b(y) \text{ a } cycle, \ w \in \operatorname{ext} C_b(y)\}$$
$$G = \cup F.$$

Since $w \in U$, we have $w \in \operatorname{ext} C = \operatorname{ext} C_a(x)$, so that $F \neq \emptyset$. If $C_b(y) \subset G \subset U$, then $\tau_b(y) \in [1/2\tau_a(x), 3/2\tau_a(x)]$. Proposition 7 and Theorem D now imply, respectively, that $\partial G \cap \operatorname{ext} C \neq \emptyset$ and ∂G consists entirely of periodic points. Lemma 9 implies that $\partial G \cap \operatorname{ext} C$ is a cycle $C_d(z)$ where $z \in U$ and $d \in [a, a+\varepsilon]$. Moreover, since $w \in \operatorname{ext} C_b(y)$ for each $\overline{\operatorname{int} C_b(y)}$ in F and $C_b(w)$ is not a cycle for any $b \in [a, a+\varepsilon]$, we have $w \in \operatorname{ext} C_d(z)$. $d \neq a$ since $C_d(z) = \partial G \cap \operatorname{ext} C \subset V$ and the only cycle of π_a in V is C. Since $U \neq A$, $C_d(z) \neq D$. Hence $d \neq a + \varepsilon$. Also, by the construction of $C_d(z)$, there is no cycle B of π_b, $b \in [a, a+\varepsilon]$, in U such that $C_d(z) \subset \operatorname{int} B$ and $w \in \operatorname{ext} B$. Thus C_d is either an external limit cycle or an external negative limit cycle, [2, VIII, 3. 3]. Proposition 7 or 8, respectively, now implies the existence of a $c \in [a, a+\varepsilon]$ such that a cycle C_1 of π_c has the property that $C_d(z) \subset \operatorname{int} C_1$ and $w \in \operatorname{ext} C_1$. This contradiction implies $A = U$. This completes the proof.

In a similar manner it can be shown that

PROPOSITION 14. *Let C be an external negative limit cycle of π_a. There exists an outer neighborhood U of C and an $\varepsilon > 0$ such that U consists entirely of periodic points of the systems π_b, $b \in [a\text{-}\varepsilon, a]$. A similar result holds for C an internal negative limit cycle and $b \in [a, a+\varepsilon]$.*

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