PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 1, 1970

ON A SIX DIMENSIONAL PROJECTIVE REPRESENTATION OF THE HALL-JANKO GROUP

J. H. LINDSEY, II

It is shown that there is a unique group G with property I: G is a central extension of Z_2 by the Hall-Janko group of order 604,800 in which a 7-Sylow subgroup S_7 is normalized by an element of order four. Also, G has a six-dimensional complex representation X. The proof is rather round-about. First, it is shown that there are at most two six-dimensional linear groups $X(G)$ projectively representing the Hall-Janko group, and all such linear groups are algebraically conjugate. The character table and generators found by M. Hall for $G/Z(G)$ are used. It is shown that a linear group L over $GF(9)$ coming from the one candidate for a six dimensional group projectively representing the Hall-Janko group actually satisfies property I. This is done by showing that L has a permutation representation on 100 three-dimensional subspaces of $(GF(9))^6$ and the image is permutation isomorphic to Hall's permutation group. Hall later studied the geometry of these subspaces. In the course of constructing the character table of any group G with property I, G was found to have a six-dimensional representation. Once this representation is known to exist, it is possible to give two easier ways of constructing generators. The faithful characters on G are given in the appendix with only one representative of each pair of algebraically conjugate characters.

This paper fills a gap in [11] concerning a six-dimensional representation X with character χ of a central extension G by the Hall-Janko group. In § 3, G is the subgroup of $GL(6, 9)$ coming from $X(G)$ in § 2 taken modulo 3. In § 4, G is any group with property I. In all cases, G turns out to be the same group. By [11], § 9, $Z = Z(G)$ has order 2 and $Q(\chi(G)) = Q(\sqrt{5})$. We let χ' be the algebraic conjugate of χ and let $Z = \langle -1 \rangle$. The characters ψ_i and ρ_i are the characters from character tables for the Hall-Janko group and $U_3(3)$ respectively, in [9]. For p a prime, we let S_p be a p-Sylow subgroup of G and let $S_7 = \langle \pi_7 \rangle$. By [11], § 9, $(7-1)/t_7 = 6 = [N(S_7): C(S_7)]$ and $C(S_7) = S_7 Z$. We let β, ε, and w be primitive seventh, fifth, and third roots of unity, respectively. Also, i is an element of order four in $GF(9)$ and $\tilde{\alpha}$ means α taken modulo a prime ideal dividing 3.

Conway, [6, p. 86], independently discovered this projective representation from the existence of the Hall-Janko group as a section in Conway's group.

2. Construction of the matrix group $\chi(G)$. Using the notation of [9], $\chi\chi' = \psi_1$ and $\chi\chi = \psi_0 + \psi_{14} +$ either ψ_{16} or ψ_{17}. Let a, b, c, and t correspond to those elements in [9]. We may let $X(a) = \operatorname{diag}(\beta, \beta^2, \beta^4, \beta^{-1}, \beta^{-2}, \beta^{-4})$, $X(c) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, and $X(t) = \begin{pmatrix} 0 & +I_3 \\ -I_3 & 0 \end{pmatrix}$. Since $c^{-1}ac = a^4$, this corresponds to multiplying the permutations in [9] from right to left. The permutation representation of G on 100 letters corresponds to $\psi_0 + \psi_1 + \psi_7$. The permutation representation of H on 36 letters corresponds to $\rho_0 + \rho_6 + \rho_7 + \rho_9$, where $H \approx U_3(3)$, [9]. We may write $X(b) = \sum A_{e,f,g} X(t^g)X(c^f)X(a^e)$ with $\sum$ over $0 \leqq -e \leqq 6$, $0 \leqq -f \leqq 2$, $-1 \leqq g \leqq 0$. We may also insist that $\sum_{0 \leqq e \leqq 6} A_{-e,f,g} = 0$ for all f and g. For $0 \leqq F \leqq 6$, $0 \leqq F \leqq 2$, and $0 \leqq G \leqq 1$; trace $X(a^E c^F t^G b) =$ trace $[X(a^E) \sum_{0 \leqq e \leqq 6} A_{-e,-F,-G} X(a^e)] = 7A_{-E,-F,-G} - \sum_{0 \leqq e \leqq 6} A_{-e,-F,-G} = 7A_{-E,-F,-G}$.

By [11], § 11, a central extension of Z_2 by $U_3(3)$ is trivial and G contains a subgroup $H \approx U_3(3)$ in which we take a, b, and c. If $G = 0$, then $a^E c^F t^G b \in H$ and its cycle structure (using the permutation representation giving $\rho_0 + \rho_6 + \rho_7 + \rho_9$) will determine $x(a^E c^F t^G b) =$ trace $[X(a^E c^F t^G b)] = 7A_{-E,-F,-G}$. If $G = 1$, then the cycle structure (using the permutation representation giving $\psi_0 + \psi_1 + \psi_7$) determines the class of $a^E c^F t^G b$ within permutation of π_1 and π_2, and within a sign. Now if "t" is the element of order 3 in H with $|C_H(\text{“}t\text{”})| = 108$, then $27 \mid |C_G(\text{“}t\text{”})|$ and "t" corresponds to T, so $X(T)$ has eigenvalues $w, w, w, \bar{w}, \bar{w}, \bar{w}$. As $C(T)/T \approx A_6$ and T is conjugate to T^{-1}, $X \mid C(T)$ has two 3-dimensional constituents representing A_6 projectively. Allowing confusion of χ with χ', $\chi(\pi) = 1 + \varepsilon^2 + \varepsilon^{-2} + \varepsilon^2 + \varepsilon^{-2} + 1$ or $1 + \varepsilon + \varepsilon^{-1} + 1 + \varepsilon^2 + \varepsilon^{-2}$. It must be the former since $\chi(\pi)\chi(\pi)' = \psi_1(\pi) = -4$ where $\varepsilon' = \varepsilon^2$. Therefore, $\chi(\pi) = 2\theta_i$ and $\chi(\pi T) = -\theta_i = (+1 \pm \sqrt{5})/2$.

By [9], $C(T_1)/\langle -T_1 \rangle \approx A_4$. As $X \mid C(T_1)$ has some two dimensional constituent with kernel contained in $\langle T_1 \rangle$, by [1], $C(T_1)/\langle T_1 \rangle$ is isomorphic to $SL(2, 3)$. Therefore, J_1 is conjugate to $-J_1$ in $C(T_1)$ and $T_1 J_1$ is conjugate to $-T_1 J_1$. Similarly, $C(\pi)/\langle \pi \rangle$ is the nontrivial central extension of Z_2 by A_5 and πJ_1 is conjugate to $-\pi J_1$. Therefore, all faithful, irreducible characters of G vanish on J_1, $T_1 J_1$, πJ_1, and $\pi^2 J_1$. As J_1 with eigenvalues $i, i, i, -i, -i, -i$ is represented faithfully in each of the 2-dimensional constituents of $C(\pi) \approx \langle \pi \rangle \times SL(2, 5)$, there is an element $\pi_5 \in C(\pi)$ of order 5 with $\chi(\pi_5)$ or $\chi(\pi_5)' = -3\theta_1$ or $-2\theta_1 - \theta_2$. Clearly, π_5 is not conjugate to π, and $\pi_5 \sim \pi_1$ or π_1^2. Since $1 = \psi_1(\pi_1) = \chi(\pi_1)\chi(\pi_1)'$, $\chi(\pi_1) \neq -3\theta_i$ Now $\chi(\pi_1 J)^2 = 1 + \psi_{14}(\pi_1 J) + \psi_{16 \text{ or } 17}(\pi_1 J) = 1 + \theta_i$ and $\chi(\pi_1 J) = \pm\theta_i$. Letting $\delta_i = \pm 1$

and $\theta_i = \theta_{1 \text{ or } 2}$, and using the permutations and character tables in [9]: $b \in cl(K)$ (write $b \sim K$), $7A_{0,0,0} = 0$; $ab \sim K$, $7A_{-1,0,0} = 0$; $a^2b \sim K$, $7A_{-2,0,0} = 0$; $a^3b \sim c$, $7A_{-3,0,0} = 0$; $a^4b \sim a$, $7A_{-4,0,0} = -1$; $a^5b \sim tb^{\pm 2}$, $7A_{-5,0,0}=1$; $a^6b \sim c$, $7A_{-6,0,0}=0$; $cb \sim K$, $7A_{0,-1,0}=0$; $acb \sim a$, $7A_{-1,-1,0}= -1$; $a^2cb \sim tb^{\pm 2}$, $7A_{-2,-1,0} = 1$; $a^3cb \sim u$, $7A_{-3,-1,0} = 2$; $a^4cb \sim a$, $7A_{-4,-1,0} = -1$; $a^5cb \sim c$, $7A_{-5,-1,0} = 0$; $a^6cb \sim a$, $7A_{-6,-1,0} = -1$; $c^2b \sim K$, $7A_{0,-2,0} = 0$; $ac^2b \sim tb^4$, $7A_{-1,-2,-0} = 1$; $a^2c^2b \sim a$, $7A_{-2,-2,0} = -1$; $a^3c^2b \sim u$, $7A_{-3,-2,0} = 2$; $a^4c^2b \sim a$, $7A_{-4,-2,0} = -1$; $a^5c^2b \sim K$, $7A_{-5,-2,0} = 0$; $a^6c^2b \sim a$, $7A_{-6,-2,0} = -1$; $tb \sim \pm\pi T$, $\pm\pi^2 T$, $7A_{0,0,-1} = \delta_{-1}\theta_{-1}$; $\pm atb \sim T_1 J_1$, $7A_{-1,0,-1} = 0$; $a^2tb \sim \pm TR$, $7A_{-2,0,-1} = \delta_0$; $a^3tb \sim \pm\pi^i J_1$ (here $i = 1$ or 2 will be assumed), $7A_{-3,0,-1} = 0$; $a^4tb \sim \pm\pi_1^i J$, $7A_{-4,0,-1} = \delta_{-2}\theta_{-2}$; $a^5tb \sim \pm TR$, $7A_{-5,0,-1} = \delta_1$; $a^6tb \sim \pm a$, $7A_{-6,0,-1} = \delta_2$; $ctb \sim \pm\pi_1^i J$, $7A_{0,-1,-1} = \delta_3\theta_0$; $actb \sim \pm a$, $7A_{-1,-1,-1} = \delta_4$; $a^2ctb \sim \pm K$, $7A_{-2,-1,-1} = 0$; $a^3ctb \sim \pi J_1$, $7A_{-3,-1,-1} = 0$; $a^4ctb \sim \pm a$, $7A_{-4,-1,-1} = \delta_5$; $a^5ctb \sim \pm\pi^i T$, $7A_{-5,-1,-1} = \delta_{-3}\theta_{-3}$; $a^6ctb \sim \pm a$, $7A_{-6,-1,-1} = \delta_6$; $c^2tb \sim \pm\pi^i T$, $7A_{0,-2,-1} = \delta_7\theta_3$; $ac^2tb \sim \pm R$, $7A_{-1,-2,-1} = 2\delta_{-4}$; $a^2c^2tb \sim \pm\pi^i T$, $7A_{-2,-2,-1} = \delta_9\theta_4$; $a^3c^2tb \sim \pi^i J_1$, $7A_{-3,-2,-1} = 0$; $a^4c^2tb \sim \pi^i J_1$, $7A_{-4,-2,-1} = 0$; $a^5c^2tb \sim \pm\pi_1^i J$, $7A_{-5,-2,-1} = \delta_{10}\theta_5$; $a^6c^2tb \sim \pm\pi_1^i$, $7A_{-6,-2,-1} = \delta_{-5}(1 + \theta_6)$.

Making an arbitrary choice of ε and possibly conjugating by $-I_3 \oplus I_3$ and replacing t by $-t$, we may take $\theta_{-1} = \theta_1$ and $\delta_{-1} = -1$. Now $0 = \sum_e A_{e,0,-1}$ is rational and $-\theta_1 + \delta_{-2}\theta_{-2} = 0$ or -1. As there are 3 other terms $-\theta_1 + \delta_{-2}\theta_{-2}$ is odd, and $\delta_{-2}\theta_{-2} = -\theta_2$. Similarly, $\delta_3\theta_0 + \delta_{-3}\theta_{-3}$ is odd, and $\delta_{-3}\theta_{-3} = \delta_3\theta_0'$. In $0 = \sum_e A_{e,-2,-1}$, $2\delta_{-4}$ and δ_{-5} cannot have the same sign or the 4 other θ_i terms could not cancel the ± 3. Therefore, we may let $2\delta_{-4} = -2\delta_8$ and $\delta_{-5}(1+\theta_6) = \delta_8(1+\theta_6)$.

There exists a matrix P such that $P^{-1}X(G)P$ is unitary. For all A in $X(N(a))$: $P^{-1}\bar{A}'P = P^{-1}A^{-1}P = (P^{-1}AP)^{-1} = \overline{(P^{-1}AP)'} = \bar{P}'\bar{A}'(\bar{P}')^{-1}$ and $P\bar{P}'\bar{A}'(P\bar{P}')^{-1} = \bar{A}'$. As $X \mid N(a)$ is irreducible $P\bar{P}' = \alpha I_6$ for some scalar α. Then for $B \in X(G)$, $B^{-1} = P(P^{-1}BP)^{-1}P^{-1} = P\overline{(P^{-1}BP)'}P^{-1} = P\bar{P}'\bar{B}'(\bar{P}')^{-1}P^{-1} = \alpha I_6\bar{B}'\alpha^{-1}I_6 = \bar{B}'$. Therefore, $X(G)$ is forced to be unitary when $X(N(a))$ is taken in normal form.

Although the θ_i and δ_i can be determined uniquely by $X(b)$ being unitary, it is easier to use ρ_1 taken mod 3. This breaks up into the sum of the 3-modular representation U obtained from the definition of $U_3(3)$ and its algebraic conjugate. We only have to check this for 3-regular elements. Let U and its algebraic conjugate have modular characters θ and θ', respectively. As $\theta(a)$ is in $GF(9)$, $\theta(a) = \beta^{\pm 1} + \beta^{\pm 2} + \beta^{\pm 4}$ where β has order 7 in $GF(3^6)$, and $\theta'(a) = -1 - \theta(a)$. Let i be in $GF(9)$ with $i^2 = -1$. As diag $(-1, i, i)$ is not conjugate to its inverse, it corresponds to b^2. Then diag $(1, i, -i)$ corresponds to u. These check since $-1 + i + i - 1 - i - i = -2 = \rho_1(b^2)$ and $1 + i - i + 1 - i + i = 2 = \rho_1(u)$. As $U(b^2) = $ diag $(-1, i, i)$, $U(b)$ may be taken with eigenvalues $\pm(-i)$, $1 - i$, $\pm(1 - i)$. It must be $-$, otherwise, b and b^2 have identical centralizers. Then $\theta(b)+\theta'(b)=0=\rho_1(b)$.

In our normal form, $X|\langle a,c\rangle$ taken mod 3 splits into distinct, irreducible, 3-dimensional subspaces. Therefore, $x_1=x_2=x_3=0$ and $x_4=x_5=x_6=0$ are the unique irreducible, proper subspaces for $X|\langle a,c\rangle$ taken mod 3, and one of these subspaces is invariant for $X|H=\langle a,b,c\rangle$ taken mod 3. In taking mod 3, the top right or bottom left 3 by 3 block of $X(b)$ vanishes.

Let $i^2=-1$ in $GF(9)$, and $'$ be the automorphism $i\to -i$ of $GF(9)$. As $(\beta+\beta^2+\beta^4)+(\beta+\beta^2+\beta^4)'=-1$ and $(\beta+\beta^2+\beta^4)(\beta+\beta^2+\beta^4)'=2$, $\beta+\beta^2+\beta^4\to 1\pm i$ in taking mod 3. As $\theta_1+\theta_2=1$ and $\theta_1\theta_2=-1$, $\theta_1\to -1\pm i$. Therefore, mod 3 either $-\theta_1\equiv\beta+\beta^2+\beta^4$ and $-\theta_2\equiv\beta^3+\beta^5+\beta^6$ or $-\theta_2\equiv\beta+\beta^2+\beta^4$ and $-\theta_1\equiv\beta^3+\beta^5+\beta^6$. The upper right 3 by 3 block of $X(b)$ is obtained from the lower left block by replacing β by β^{-1} and changing signs of all terms. Therefore, if one choice of $-\theta_1$ or $-\theta_2\equiv\beta+\beta^2+\beta^4$ makes the lower right vanish, then the other choice makes the upper right vanish and we may assume that the upper right vanishes. In the following, work mod 3. Then the (1, 4) entry of $X(b)$: $\theta_1-\delta_0\beta^2+\theta_2\beta^4-\delta_1\beta^5-\delta_2\beta^6\equiv 0$. Suppose $-\theta_1\equiv\beta^3+\beta^5+\beta^6$. Then $-\theta_2\equiv\beta+\beta^2+\beta^4$ and

$$-\beta^3+\beta^5+\beta^6-\beta-\delta_0\beta^2-\delta_1\beta^5-\delta_2\beta^6\equiv 0\ ,$$

impossible, as the coefficients of β and 1 are -1 and 0. Therefore, $-\theta_1\equiv\beta+\beta^2+\beta^4$ and $-\beta-\beta^4+\beta^2-1-\beta^3-\delta_0\beta^2-\delta_1\beta^5-\delta_2\beta^6\equiv 0$. The coefficient of all β^i is -1, so $\delta_0=-1$, $\delta_1=1$, and $\delta_2=1$. By the (1, 6) entry: $-\delta_3\theta_0-\delta_5\beta^2-\delta_6\beta^3-\delta_4\beta^4-\delta_3\theta_0'\beta^6\equiv 0$. If $-\theta_0=-\theta_1\equiv\beta+\beta^2+\beta^4$, then $\delta_3(\beta+\beta^5-\beta^2-\beta^4)-\delta_5\beta^2-\delta_6\beta^3-\delta_4\beta^4\equiv 0$, and the coefficients of 1 and β are 0 and δ_3, impossible. Therefore, $-\theta_0=-\theta_2\equiv\beta^3+\beta^5+\beta^6$ and $\delta_3(1+\beta+\beta^5+\beta^6-\beta^3)-\delta_5\beta^2-\delta_6\beta^3-\delta_4\beta^4\equiv 0$. The coefficient of all β^i is δ_3 and $\delta_5=-\delta_3$, $\delta_6=\delta_3$, and $\delta_4=-\delta_3$.

Letting $X(b)=(b_{ij})$, then $\sum b_{i,3}\bar{b}_{i,4}=0$. We may perform this calculation in $Q(\sqrt{5})[\beta]$ and collect terms where β has a certain exponent mod 7. The result is (constant) $(1+\cdots+\beta^6)$. The constant can be determined to be 0 by letting $\beta=1$ since $\sum_e A_{e,f,g}=0$ and the $b_{i,j}$ become 0. The coefficient of 1 in $49\sum b_{i,3}\bar{b}_{i,4}$ is $-3\delta_0-\theta_2+2\delta_1-2\delta_2+2\delta_5+\delta_3\theta_0'-2\delta_8-\delta_9\theta_4-\delta_8(1+\theta_6)=3-\theta_2+2-2-2\delta_3+\delta_3\theta_1-3\delta_8-\delta_8\theta_6-\delta_9\theta_4=0$. If $\delta_3=1$, then $1-\theta_2+\theta_1-3\delta_8-\delta_8\theta_6-\delta_9\theta_4=0$. The terms other than $-3\delta_8$ must add to ± 3 which is impossible. Therefore, $\delta_3=-1$, and $4-3\delta_8-\delta_8\theta_6-\delta_9\theta_4=0$. Now $\delta_8=1$. Then $1-\delta_8\theta_6-\delta_9\theta_4=0$, $\delta_9=1$, and $\theta_6'=\theta_4$. From the coefficient of β in $\sum b_{i,3}\bar{b}_{i,4}=0$: $0=\theta_1+\delta_0-\delta_1+2\delta_2-2\delta_4+\delta_3\theta_0'-2\delta_7\theta_3-2\delta_8+2\delta_8(1+\theta_6)=-2-2\delta_7\theta_3+2\theta_6$. Then $\delta_7=-1$ and $\theta_3=\theta_6'$. From $0=-\sum_e A_{e,-2,-1}=\delta_7\theta_3-2\delta_8+\delta_9\theta_4+\delta_{10}\theta_5+\delta_8(1+\theta_6)=-\theta_6'-2+\theta_6'+\delta_{10}\theta_5+1+\theta_6$, so

$\delta_{10}=1$ and $\theta_5=\theta_6'$. The $b_{1,5}$ entry is $\theta_6'+2\beta^2-\theta_6'\beta^3-\theta_6'\beta^4-(1+\theta_6)\beta^5\equiv 0$. If $\theta_6=\theta_1\equiv -\beta-\beta^2-\beta^4$, then the coefficient of 1 is -1 and the coefficient of β is 1, a contradiction. Therefore, $\theta_6=\theta_2$ and $X(b)$ is uniquely determined. G, a central extension of Z_2 by the Hall-Janko group with a representation of degree 6 is unique in the strong representation group sense: If G_1 and G_2 are 2 such groups and ϕ is an isomorphism: $G_1/Z(G_1)\to G_2/Z(G_2)$, then ϕ lifts to an isomorphism Φ of $G_1\to G_2$. In particular, the outer automorphism of $G/Z(G)$ lifts to G.

3. The existence of G satisfying property I. We shall now show that there exists a central extension of Z_2 by the Hall-Janko group with an element of order 4 in $N(S_7)$. First replace $X(G)$ by $(A\oplus\bar{A})X(G)(A\oplus\bar{A})^{-1}$ where $A=\begin{pmatrix}\beta & \beta^2 & \beta^4\\ \beta^2 & \beta^4 & \beta\\ \beta^4 & \beta & \beta^2\end{pmatrix}$. (Then the representation is written over $Q(\sqrt{-7},\sqrt{5})$. In fact it may be written over any field $\supseteqq Q(\sqrt{5})$ over which $X(N(\pi_7))$ can be written. It cannot be written over the reals since the 1-dimensional representation is not a constituent of the symmetric tensors of $X\otimes X$.) Now take $X(G)$ mod 3. (We shall now use G for the image of $\langle X(t), X(b)\rangle$ taken mod 3 and no longer make assumptions about G, such as G/Z is the Hall-Janko group or the representation of degree 6.) We set $\tilde{\theta}_1=-1+i$ and $\widetilde{-\beta-\beta^2-\beta^4}=-1+i$. Then identifying elements with their matrices, $t=\widetilde{X(t)}=\begin{pmatrix}0 & E\\ F & 0\end{pmatrix}$ in 3 by 3 blocks with $E=\begin{pmatrix}-1+i & 1-i & -i\\ -i & -1+i & 1-i\\ 1-i & -i & -1+i\end{pmatrix}$ and $F=\begin{pmatrix}1+i & -1-i & -i\\ -i & 1+i & -1-i\\ -1-i & -i & 1+i\end{pmatrix}$. Also $b=\begin{pmatrix}N & O\\ Q & \bar{N}\end{pmatrix}$ with $N=\begin{pmatrix}0 & -i & -1-i\\ -1 & -1+i & 1\\ -1-i & -1-i & 1\end{pmatrix}$ and $Q=\begin{pmatrix}i & i & 0\\ 1+i & i & i\\ 0 & 1-i & -1-i\end{pmatrix}$, where bar is the nonidentity automorphism of $GF(9)$. Replacing G by $\begin{pmatrix}A & 0\\ 0 & \bar{A}\end{pmatrix}G\begin{pmatrix}A^{-1} & 0\\ 0 & \bar{A}^{-1}\end{pmatrix}$ with $A=\begin{pmatrix}-i & 1 & -1+i\\ 1 & 1 & 1+i\\ 1 & 1 & 1\end{pmatrix}$, $N\to\operatorname{diag}(i,-1+i,1-i)$, $Q\to\begin{pmatrix}0 & -1 & 1+i\\ i & i & i\\ 1+i & -i & i\end{pmatrix}$, E becomes $\begin{pmatrix}1 & 0 & 1+i\\ -1+i & i & -1\\ 0 & 0 & -i\end{pmatrix}$, and F becomes $\begin{pmatrix}-1 & 0 & -1+i\\ 1+i & i & 1\\ 0 & 0 & -i\end{pmatrix}$. We may change coordinates again and replace this last G by CGC^{-1} with $C=\begin{pmatrix}1 & 0 & 1-i\\ -1-i & -i & -1\\ 0 & 0 & i\end{pmatrix}\oplus I_3$. Then E becomes I_3, F becomes $-I_3$, N becomes $\begin{pmatrix}i & 0 & i\\ -1-i & -1+i & -1\\ 0 & 0 & 1-i\end{pmatrix}$, $\bar{N}$

becomes $\mathrm{diag}(-i, -1-i, 1+i)$, and Q becomes $\begin{pmatrix} 1-i & -i & -1-i \\ -1 & -1 & 0 \\ -1-i & 1 & 1 \end{pmatrix}$. We consider permutations in [9] as acting with letters on the right and matrices as acting with vectors on the right. From [9], we may define the letters in the following way: $00 = 00$, $01 = t00$, $02 = bt00$, $03 = b^6tb^6t00$, $04 = btb^7tb^4t00$, $05 = btb^7tb^2t00$, $06 = tbtb^5t00$, $07 = tb^7tb^4t00$, $08 = btb^2t00$, $09 = b^2t00$, $10 = tb^7tb^2t00$, $11 = b^5tb^6t00$, $12 = b^6tb^2t00$, $13 = tb^6t00$, $14 = btb^3tb^7t00$, $15 = b^5tb^2t00$, $16 = b^3t00$, $17 = b^7t00$, $18 = b^2tb^2t00$, $19 = btbtb^5t00$, $20 = b^5t00$, $21 = tb^3tb^7t00$, $22 = b^4tb^2t00$, $23 = b^4t00$, $24 = b^3tb^6t00$, $25 = b^4tb^6t00$, $26 = tb^5tb^4t00$, $27 = btb^5ab^4t00$, $28 = b^3tb^2t00$, $29 = btb^6t00$, $30 = b^6t00$, $31 = b^2tb^6t00$, $32 = tb^5tb^6t00$, $33 = b^7tb^2t00$, $34 = tb^2t00$, $35 = b^7tb^6t00$, $36 = btb^5tb^6t00$, $37 = tbtb^3t00$, $38 = b^6tb^5t00$, $39 = tb^3tb^5t00$, $40 = b^3tb^7tb^7t00$, $41 = btb^4tb^6t00$, $42 = b^6tb^4t00$, $43 = b^5tb^7t00$, $44 = tb^2tb^5t00$, $45 = btb^2tb^5tb^5t00$, $46 = b^2tb^5tb^5t00$, $47 = tbtb^7t00$, $48 = b^7tb^7t00$, $49 = b^4tb^5t00$, $50 = tbtb^4t00$, $51 = tb^4tb^6t00$, $52 = tb^3tb^6t00$, $53 = tb^4t00$, $54 = tbtb^6t00$, $55 = tb^3tb^2t00$, $56 = tbtb^5tb^4t00$, $57 = b^5tb^4t00$, $58 = tb^6tb^7t00$, $59 = tb^5tb^5t00$, $60 = btbt00$, $61 = btb^7tb^7t00$, $62 = tbtbtb^7t00$, $63 = tb^3tb^4t00$, $64 = b^5tb^3t00$, $65 = tb^4tb^2t00$, $66 = b^3tb^7t00$, $67 = tb^5t00$, $68 = btb^3tb^6t00$, $69 = b^2tb^3t00$, $70 = tb^7t00$, $71 = tb^3t00$, $72 = btb^6tb^7t00$, $73 = b^4tb^3t00$, $74 = tbt00$, $75 = tbtb^2t00$, $76 = b^7tb^4t00$, $77 = btb^5t00$, $78 = b^2tb^7tb^7t00$, $79 = b^4tb^4t00$, $80 = b^3tb^4t00$, $81 = btbtb^7t00$, $82 = btb^5tb^5t00$, $83 = b^2tb^7t00$, $84 = b^5tb^5t00$, $85 = b^6tb^7t00$, $86 = b^2tb^4t00$, $87 = tbtb^4tb^6t00$, $88 = b^3tb^5tb^5t00$, $89 = b^3tb^5t00$, $90 = b^3tb^3t00$, $91 = btb^3t00$, $92 = b^7tb^5t00$, $93 = btb^4t00$, $94 = b^4tb^7t00$, $95 = tb^7tb^7t00$, $96 = btb^7t00$, $97 = tb^2tb^5tb^5t00$, $98 = btbtbtb^7t00$, $99 = b^2tb^5t00$.

We let 00 be the space $x_1 = x_2 = x_3 = 0$. We have just defined the spaces i for $00 \leqq i \leqq 99$. Checking that G permutes these spaces in the same way that [9] permutes letters involves the following typical calculation: $t(b^6tb^6t)00 = t03 = 73 = b^4tb^3t00$ to equivalent to (using $b^8 = 1$) $b^5tb^4tb^6tb^6(t00) = t00$. Fixing $t00$ is equivalent to having all 0's in the bottom left 3 by 3 block of the matrix over $GF(9)$. It is sufficient to check, as George Shapiro has done by computer, that the following matrices have all 0's in the bottom left 3 by 3 block: $b^5tb^4tb^6tb^6$, $b^4tb^3tb^2tb^7tb^4$, $btb^2tb^7tbtb^7tb^4$, $btb^5tb^2tb^7tb^2$, $btbtb^6tbtb^7tb^2$, $b^2tb^6tb^6tb^2$, $b^6tbtb^2tb^3tb^7$, $b^2tb^3tb^7tbtb^3tb^7$, $b^2tbtb^5tb^2$, $b^5tb^6tb^2tb^2$, $b^2tb^3tb^2tbtb^5$, $b^2tb^5tb^7tbtbtb^5$, $b^4tbtb^2tb^5tb^4$, $b^3tb^7tb^2tb^5tb^6$, $b^5tb^7tbtbtb^3$, $b^5tb^5tb^6tb^5$, $b^3tb^3tbtb^3tb^5$, $b^6tb^5tb^4tb^7tb^7$, $b^3tb^6tb^2tb^4tb^6$, $b^4tb^6tb^6tb^4$, $b^3tbtb^5tb^7$, $b^4tb^5tbtb^2tb^5$, $btb^7tb^7tb^7tbtb^2tb^5tb^5$, $b^3tb^3tb^6tb^2tb^2tb^5tb^5$, $btb^4tb^4tb^5$, $b^2tb^4tbtbtb^4$, $b^2tb^5tbtbtb^6$, $btb^7tbtb^3tb^2$, $b^2tb^4tb^7tbtbtb^5tb^4$, b^5tb^2tb, $b^3tb^3tb^7tbtb^7tb^7$, $btb^2tbtb^3tb^4$, $b^7tb^6tb^3$, $b^4tb^7tbtb^4tb^2$, $b^3tb^5tb^2tb^3tb^6$, $b^6tb^4tb^2tb^6tb^7$, $btbtbtbtb^2$, $b^5tb^3tb^4tb^4$, $b^6tb^7tb^2tbtb^7$, $b^7tb^7tb^2tb^7$, $b^4tb^3tb^7tbtbtb^4tb^6$, $b^2tb^7tb^4tb^5tb^5$, $btbtb^5tb^3tb^5tb^5$, $btb^7tb^7tb^2tbtbtb^7$.

This permutation representation of these matrices gives a nontrivial (t interchanges $x_1 = x_2 = x_3 = 0$ and $x_4 = x_5 = x_6 = 0$) homomorphism of the matrix group $\langle t, b \rangle$ onto the permutation group

$\langle t, b\rangle$. The latter is transitive, b has order 8 and fixes 00, and tb has order 15. Therefore, (3)(8)(100) divides the order of the permutation group $\langle t, b\rangle$, which, by the classification in [9] of the large subgroups of the Hall-Janko group, must be the Hall-Janko group. Suppose that M is in the kernel K of this permutation representation. As M fixes 00 and 01, $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ in 3 by 3 blocks. We shall use the coordinates in which N, the top left block of b, is diagonal. Now $b^{-1}Mb \in K$ has diagonal block form and $b^{-1}Mb = \begin{pmatrix} N^{-1} & 0 \\ 0 & \bar{N}^{-1} \end{pmatrix} M \begin{pmatrix} N & 0 \\ 0 & \bar{N} \end{pmatrix}$. Then $\begin{pmatrix} I & 0 \\ W & I \end{pmatrix}$ commutes with M where $W = QN^{-1} = \begin{pmatrix} 0 & -1-i & i \\ 1 & -1+i & 1-i \\ 1-i & 1-i & 1-i \end{pmatrix}$. We may use b^2 instead of b and then W is replaced by $V = \begin{pmatrix} 0 & -i & -1 \\ i & 1 & i \\ 1 & i & -1 \end{pmatrix}$. For $U = W$, V, or Y where $Y = W + iV = \begin{pmatrix} 0 & -i & 0 \\ 0 & -1-i & -i \\ 1 & -i & 1+i \end{pmatrix}$ we have $UA = BU$. Also, as Y is nonsingular, A and B are similar and VY^{-1} commutes with B. Now, $VY^{-1} = \begin{pmatrix} -1+i & -i & 0 \\ -1-i & i & i \\ -1 & 1+i & 1 \end{pmatrix}$ has characteristic polynomial $-x^3 - ix^2 + ix - 1$ which has distinct roots: $-i$, $1 - i$, and $-1 + i$. Therefore, the top 3 by 3 constituent of K, and, similarly, the bottom, has a matrix with distinct eigenvalues in $GF(9)$ in its commuting algebra and can be diagonalized over $GF(9)$.

Let L be the subgroup of G fixing the spaces 00 and 01. Then by [9], L is in diagonal 3 by 3 block form and $L/K \approx PSL(2, 7)$. Either the top left of bottom right 3 by 3 constituent of L has $PSL(2, 7)$ as a constituent, say the former. Let U be the top left component of L and V the top left component of K. Then U permutes the homogeneous spaces of V. As U is not solvable, these homogeneous spaces are not 1-dimensional. Suppose that V has homogeneous spaces of dimensions 1 and 2. As V was diagonalizable over $GF(9)$, U may have its elements taken in the form: $\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}$. This is impossible as then $7 \mid |PSL(2, 7)| \mid \left|\left\langle \begin{pmatrix} b & c \\ d & e \end{pmatrix} \right\rangle\right| \mid (80)(72)$, a contradiction. Therefore, there is only one homogeneous space and V is scalar. As the bottom component of V is similar to the top, K is scalar. By unimodularity, K has order 2.

4. Existence of the six-dimensional representation of G. We now prove the following theorem:

THEOREM. *There is a unique central extension G of Z_2 by the Hall-Janko group with $N(S_7)$ having an element of order 4. Further-*

more, G has a representation of degree 6. *Uniqueness is in the strong sense that if G_1 and G_2 are two such groups and ϕ is an isomorphism: $G_1/Z(G_1) \to G_2/Z(G_2)$, then ϕ lifts to an isomorphism Φ: $G_1 \to G_2$*

We shall use the following lemma:

LEMMA. *Let the p-block B have P as a defect group where P is nonabelian of order p^3. Let $Z(P) \subseteqq Z(G)$ and $\zeta \in B$ be a character of G faithful on $Z(P)$. The $p^2 \,\|\, |G|/\zeta(1)$.*

Proof of the lemma. By [7], (87.18) there exists a p-regular class C_n with element c such that P is a Sylow-p-subgroup of $C(c)$, and $|G|\zeta(c)/|C(c)|\zeta(1)$ equals some p-local unit. Now P has p^2 linear characters with $Z(P)$ in the kernel and $(p-1)$ characters of degree p distinguished by their action on $Z(P)$. As $Z(P) \subseteqq Z(G)$, the representation U corresponding to ζ is scalar on $Z(P)$ and $U \mid P = I_m \otimes V$ where V has degree p. By [7], (51.2), $U \mid C(P) = W \otimes I_p$, where W is some representation of $C(P)$. Then as $c \in C(P)$, $p \mid \zeta(c)$. Let $p^a \,\|\, |G|$. Now p divides $\zeta(1)$ to the same power as it divides $p^a\zeta(c)/p^3$, and it divides at least to the p^{a-2} power. It cannot divide to a higher power or ζ would be in a p-block of defect 1.

Now, let G be the G in the theorem. As in [9], ψ_3 and ψ_5 are in a block $B_1(2)$ with defect group equal to a defect group of the class of π. Then the defect group is Q, the quaternions. Let χ_i, $i = 1, \cdots, n$, with degrees $x_i \equiv \delta_i \pmod 7$, $\delta_i = \pm 1$, be characters in $B_1(2) \cap B_1(7)$, where $B_1(7)$ is the nonprincipal 7-block for G. By $(\pi_7, -1)$ block orthogonality: $\sum x_i\delta_i = 288 - 160$. By $(\pi_7, -\pi_7)$ block orthogonality: $\sum \delta_i^2 = 1 + 1$. Then $B_1(7)$ has exactly 2 characters, χ_1 and χ_2 in $B_1(2) \cap B_1(7)$ and $\sum x_i\delta_i = 128$. By the lemma $64 \,\|\, x_i$. For some x_i, $(x_i, 3) = 1$ and $x_i = 2^6 5^a$. As $x_i \equiv \pm 1 \pmod 7$, $x_i = 2^6$ and $x_i = 64 =$ the other x_j. Let $\nu = \chi_1$ and $\mu = \chi_2$. Field automorphisms take ψ_3 to ψ_3, $B_1(2)$ to $B_1(2)$, $B_1(7)$ to $B_1(7)$, and $B_1(2) \cap B_1(7)$ to $B_1(2) \cap B_1(7)$. Therefore, ν and μ are either a complete set of algebraic conjugates, or are rational.

The characters ν and ψ_3 lie in $B_1(2)$ and give the same modular linear character.

Therefore, $|G|\nu(T)/(2160)(64) \equiv |G|(16)/(2160)(160) \equiv 0 \pmod 2$, $8 \mid \nu(T)$ and $\nu(T) \equiv -8 \pmod{24}$. By $(T, -\pi_7)$ block orthogonality in $B_1(2)$: $\mu(T) + \nu(T) = \psi_5(T) - \psi_3(T) = -16$. As $32^2 + 16^2 > 1080$, $\psi(T) = -8$. Similarly, $\nu(T_1) \equiv \mu(T_1) \equiv -2 \pmod 6$, $\nu(T_1) + \mu(T_1) = \psi_5(T_1) - \psi_3(T_1) = -4$. and $\nu(T_1) = -2$. Similarly, $\mu(\pi T) + \nu(\pi T) = \psi_5(\pi T) - \psi_3(\pi T) = -1$. Now $\nu(\pi T) \equiv \nu(T) \equiv 2 \pmod{\sqrt{5}}$. If $\nu(\pi T)$ is rational, then for μ or ν, say ν, $|\nu(\pi T)| \geqq 3$ and $|\nu(\pi T)|^2 + |\nu(\pi^2 T)|^2 \geqq 18 > 15$, a contradiction. Therefore, ν and μ are algebraic conjugates contained in $Q(\varepsilon + \bar{\varepsilon})$.

Since J_1 has an inverse image in G of order 4, by an earlier argument, characters faithful on G are 0 on J_1, T_1J_1, πJ_1, and $\pi^2 J_1$, or any other class C conjugate to $-C$. By [8], ν in $B_1(2)$ vanishes on elements whose 2-singular part is not conjugate to an element in the defect group of $B_1(2)$, that is not conjugate to 1, -1, or J_1. Therefore, ν vanishes on J, R, K, TJ, TR, $\pi_1 J$, and $\pi_1^2 J$.

As in [11], § 9, the automorphism of $Q(\chi_i$: χ_i a character of $G)$ which comes from lifting the automorphism of the 7-modular field: $x \to x^7$, gives an automorphism of the tree of $B_1(7)$ interchanging μ and ν, and, therefore, flipping the stem. Therefore, $B_1(7)$ has 3 pairs of algebraically conjugate characters of degrees 64, a, and b; and a rational character in the middle of the stem of degree z. Also, $B_1(7)$ has another character other than μ or ν with a degree not divisible by 3. This degree must be 8, 400, 64, or 50, for $5 \nmid$ any degree as shown later.

Suppose G has a character ζ of degree 8, Then $\zeta | H \approx U_3(3) = \rho_1 + 2\rho_0$, $\rho_5 + \rho_0$, $\rho_6 + \rho_0$, or $\rho_7 + \rho_0$. By reciprocity and $\rho_0^G = \psi_0 + \psi_1 + \psi_7$, $(\psi_{i|H}, \rho_0) = 1$ if $i = 0$, 1, or 7, and 0 otherwise. In the case $\zeta | H = \rho_1 + 2\rho_0$ or $\rho_5 + \rho_0$, $(\zeta^2 | H, \rho_0) \geqq 2$. The possibly reducible characters α and β of the skew-symmetric and symmetric tensors, respectively, corresponding to ζ^2 have $(\alpha | H, \rho_0) \leqq 1$, $(\beta | \mathrm{H}, \rho_0) \leqq 1$, and we must have equalities. This is a contradiction as $B_0(7)$ of G has no character of degree $\equiv -1 \pmod 7$ and < 28. In the case $\zeta | H = \rho_5 + \rho_0$ or $\rho_7 + \rho_0$, ζ is not real and $(\zeta^2, \psi_0) = 0$. As $(\zeta^2 | H, \rho_0) = 1$, β (defined as before) $= \psi_1$. Then $4 = \psi_1(J) = \beta(J) = ((3 + 1)^2 + 8)/2 = 12$, a contradiction.

Suppose G has a rational character ζ of degree 400. Then ζ is 0 on 5-regular elements and $0 = (\nu, \zeta) = 8/189 + 1/7 - \zeta(T)/135 - \zeta(T_1)/18 = 5/27 - \zeta(T)/135 - \zeta(T_1)/18$. Now, $2 | \zeta(T_1)$ and $\zeta(T_1) \equiv 1 \pmod 3$, so $\zeta(T_1) = 4$ or -2. If $\zeta(T_1) = -2$, then $\zeta(T) = 40 > \sqrt{1080}$ and impossible. Therefore, $\zeta(T_1) = 4$, $\zeta(T) = -5$. The contribution to $1 = (\zeta, \zeta)$ from e, a, T, and T_1 is $50/189 + 1/7 + 5/216 + 4/9 = 7/8$ and 1/8 remains. Now, $\zeta(TJ) \equiv \zeta(T) \equiv \zeta(TR) \equiv -5 \pmod{2.}$ Then, $| \zeta(TJ) | = | \zeta(TR) | = 1$ and 0 is left in (ζ, ζ). Then $0 = \zeta(J) \equiv \zeta(TJ) \equiv \pm 1 \pmod 3$, a contradiction.

If the degree 400 occurs, it is in a pair and we already have degrees 64, 64, 400, 400 all $\equiv 1 \pmod 7$. There can only be one more 64, 50, or 400, otherwise, the last degree is greater than $64 + 64 + 400 + 400$. As $3 \nmid 64 + 64 + 400 + 400$, $z = 50$. Then the final pair consists of odd degrees, impossible, as then the unimodular subgroup of the linear group in the final pair complements Z_2.

If $z = 64$, then let η be the rational character of this degree. Then $0 = \eta(J) \equiv \eta(\pi_1 J) \pmod 5$ and $\eta(\pi_1 J) = 0$. Furthermore, $\eta(\pi_1) \equiv \eta(\pi_1 J) = 0 \pmod 2$, $\eta(\pi_1) \equiv 4 \pmod{10}$, and $\eta(\pi_1) = 4$, otherwise, $\eta(\pi_1) + \eta(\pi_1^2) \geqq 6^2 + 6^2 > 50$. Also, $\eta(T_1) \equiv \eta(T_1 J_1) \equiv 0 \pmod 2$, $\eta(T_1) \equiv 64 \pmod 3$, and $|\eta(T_1)| \geqq 2$. Now, $\eta(\pi T) = 0$, as otherwise,

$$\begin{aligned} 1 = (\eta, \eta) &\geqq |\eta(a)|^2/7 + |\eta(T_1)|^2/36 + |\eta(\pi_1)|^2/50 + |\eta(\pi_1^2)|^2/50 \\ &\quad + |\eta(\pi T)|^2/15 + |\eta(\pi^2 T)|^2/15 \\ &\geqq 1/7 + 1/9 + 8/25 + 8/25 + 1/15 + 1/15 > 1. \end{aligned}$$

Then $\eta(T) \equiv \eta(\pi T) = 0 \pmod 5$, $4 \mid \eta(T)$, $\eta(T) \equiv 1 \pmod 3$, $\eta(T) \equiv -20 \pmod{60}$, and $\eta(T) = -20$. Then

$$1 = (\eta, \eta) \geqq 1/7 + 1/9 + 400/1080 + 8/25 + 8/25 > 1 ,$$

a contradiction.

If both a, $b = 64$, then $z = (6)(64)$ corresponds to a character in a 2-block of G of defect 1. By $(1, -1)$ block orthogonality, this block has a character with Z in the kernel a contradiction. If $a = 64$, $b \neq 64$, then $B_1(7)$ has 1 (mod 3) characters with degree 50, and $z = 50$. This is impossible as b would then be odd.

Therefore, a, b, and z are all distinct from 8, 64, and 400. The number of degrees equal to 50 is 2 (mod 3). Therefore, we may take $a = 50$, and $b \equiv z \equiv 0 \pmod 3$. As $3 \| 228 = 2(50 + 64)$, b or z is divisible exactly by 3. Such a degree must be 6; 48; 384 has 2-defect 1, impossible; or 300. The possibilities are $228 - 12 = \underline{216}$, $228 - 96 = 132$, $600 - 228 = 372$, $114 - 3 = 111$, $114 - 24 = 90$ divisible exactly by 5, and $150 - 114 = \underline{36}$. The last case is impossible by $3 - 7$ block separation as 36 is the only degree in $B_1(7)$ corresponding to a 3-block of defect 1. The degree equation must be $50 + 50 + 64 + 64 = 6 + 6 + 216$ and G has a representation of degree 6. Some G has been given by 6 by 6 matrices over $GF(9)$ and uniqueness of G follows from the uniqueness of $X(b)$. The character table of G is completed in the appendix. As $t_7 = 1$ for G, if $G_1 \triangleright G$ and G_1 has a unimodular representation of degree 6, then by [10], $7 \| |G|$. As in the proof of [3], $3F$, $C_{G_1}(S_7) = Z(G_1)S_7$. As $[N_{G_1}(S_7) : C_{G_1}(S_7)] = [N_G(S_7) : C_G(S_7)] = 6$, $G_1 = GZ(G_1)$.

As $|C(\pi_1)/\langle \pi_1 \rangle| = 20 < 5^2$, $C(\pi_1)/\langle \pi_1 \rangle$ has no 5-block of defect 0, and by [7], 88.8, is not a defect group. Since J_1 has order 4 in $C(\pi)$, $C(\pi)/\langle \pi \rangle \approx SL(2, 5)$ and has no character of 5-defect 0 faithful on $\langle -1 \rangle$. Therefore, G has only 1 5-block of defect 1 and representations in this block have Z in the kernel. Then 5 does not exactly divide any degree of a faithful irreducible character of G.

5. An alternative construction. There is a simple way to construct matrix generators of G, but it would be hard to show directly that these generators generate a finite group. Let $Q \subset G$ be a common 2-Sylow subgroup of $C(\pi)$ and $C(T_1)$ isomorphic to the quaternions, by choosing T_1 appropriately. If U is the unique irreducible nonlinear representation of Q, then $X \mid Q \simeq I_3 \otimes U$. By [7], (51.7), $X(N(Q)) = R \otimes S$ for some linear groups R and S. From $C(T_1)/\langle T_1 \rangle \simeq SL(2, 3)$, we see that $3 \mid [N(Q): C(Q)]$ and $3 \mid [S: Z(S)]$. By [7], (51.2), π and T_1 lie in $R \otimes I_2$ so $2^2 \| [S: Z(S)]$ and $S \simeq SL(2, 3)$. As no conjugate of T_1 commutes with π, R does not have a normal 5-Sylow subgroup. By [1]'s classification of two and three-dimensional groups, $R/Z(R) \simeq A_5$. If R has a two dimensional constituent, then an element $\operatorname{diag}(1, -1, -1) \otimes I_2$ is centralized by $SL(2, 5) \times SL(2, 3)$, a contradiction. This determines R since the 3-dimensional representations of A_5 are related by automorphism from S_5. An element u of order three in $I_3 \otimes S$ is centralized by π in $R \otimes I_2$, so u has eigenvalues $w, w, w, \bar{w}, \bar{w}, \bar{w}$. This determines S.

We may take $v = \operatorname{diag}(\varepsilon, \bar{\varepsilon}, 1) \in R$ and $\pi = v \otimes I_2 \in X(G)$. Then $X((C(\pi))')$ is block diagonal in 2 by 2 blocks and each diagonal block represents $SL(2, 5)$. The diagonal blocks differ by conjugation by matrices and possible algebraic conjugation: $\varepsilon \rightarrow \varepsilon^2$. As the matrix conjugation fixes S elementwise, the conjugating matrices are scalar. We shall add to $R \otimes S$ an element $y = A \oplus B \oplus C$ in 2 by 2 blocks with $\langle S, A \rangle \simeq SL(2, 5)$. As before, S uniquely determines the matrix group $\langle S, A \rangle$ and we may take A with eigenvalues ε, $\bar{\varepsilon}$. Then πy has an eigenvalue one and is conjugate to π or π^2. As πy has another eigenvalue 1, B has eigenvalues ε, $\bar{\varepsilon}$. Then C has eigenvalues ε^2 and ε^{-2}. These eigenvalues determine the representations of $SL(2, 5)$ by the second and third diagonal blocks and determine B and C. Then $\langle y, R \otimes S \rangle$ is a subgroup of order at least $5[(2)(720)]$ of G and, by [9]'s classification of large subgroups of $G/Z(G)$, is G.

Alternatively, we could have replaced v by $C(u)$ of order $2(1080)$. This is facilitated by replacing $R \otimes S$ by $S \otimes R$ and taking $S(u)$ to be diagonal. Then $C(u)$ is block diagonal in 3 by 3 blocks. The two diagonal blocks elementwise are related by interchanging w and $\bar{w}$, and are identical elementwise on $N(Q) \cap C(u) \simeq Z_6 \times A_5$. In either of these constructions generators of $X(G)$ may be gotten from the generators of the two and three-dimensional groups in [1].

Appendix

$G/Z \simeq$ Hall-Janko group $|Z| = 2$, the notation follows [9],
$\theta_1 = (1 + \sqrt{5})/2$, $\theta_2 = (1 - \sqrt{5})/2$

e	a	J	R	K	TJ	TR	$\underline{\pi_1 J}$	T	T_1	$\underline{\pi}$	$\underline{\pi_1}$	$\underline{\pi T}$
$\underline{6}$	-1	-2	2	0	1	-1	θ_2	-3	0	$2\theta_2$	$-1-\theta_1$	$-\theta_1$
$\underline{50}$	1	10	2	$2i$	1	-1	0	5	2	0	0	0
216	-1	24	0	0	0	0	-1	0	0	6	1	0
$\underline{64}$	1	0	0	0	0	0	0	-8	-2	$2+4\theta_2$	$-2\theta_1$	$-\theta_2$
14	0	6	2	0	0	2	1	-4	2	4	-1	1
84	0	4	4	0	1	1	-1	-15	0	-6	-1	0
$\underline{126}$	0	-10	2	0	-1	-1	0	-9	0	$4-6\theta_1$	$2\theta_1$	1
252	0	-20	4	0	1	1	0	9	0	2	2	-1
$\underline{56}$	0	-8	0	0	-2	0	$-\theta_2$	2	2	$2\theta_2$	$-1-\theta_1$	$-\theta_1$
448	0	0	0	0	0	0	0	16	-2	-2	-2	1
350	0	-10	-6	0	2	0	0	-10	2	0	0	0
336	0	16	0	0	-2	0	1	-6	0	-4	1	-1

BIBLIOGRAPHY

1. H. F. Blichfeldt, *Finite Collineation Groups*, University of Chicago Press, Chicago, 1917.
2. R. Brauer, *Investigations on group characters*, Ann. of Math. **42** (1941), 936-958.
3. ———, *On groups whose order contains a prime to the first power*, I, II, Amer. J. Math. **64** (1942), 401-440.
4. ———, *Uber endlichen lineare Gruppen von Primzahlgrad*, Math. Ann. **169** (1967), 73-96.
5. R. Brauer and H. F. Tuan, *On simple groups of finite order*, I, Bull. Amer. Math. Soc. **51** (1945), 756-766.
6. J. Conway, Bull. London Math. Soc. **1** (1969), 79-88.
7. C. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York, 1962.
8. J. A. Green, *Blocks of modular representations*, Math. Zeit. **79** (1962) 100-115.
9. M. Hall and D. Wales, *The simple group of order* 604,800, J. of Algebra. **9** (1968), 417-450.
10. J. H. Lindsey, II, *A generalization of Feit's theorem*, (to appear in Trans. Amer. Math. Soc.)
11. ———, *Finite linear groups of degree six*, (to appear)

Received February 4, 1970.

NORTHERN ILLINOIS UNIVERSITY