PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 1, 1970

RANK PRESERVERS OF SKEW-SYMMETRIC MATRICES

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It is possible to study the structure of rank preservers on n-square skew-symmetric matrices over an algebraically closed field F by considering instead the linear transformations on the second Grassmann Product Space $\wedge^2\mathscr{U}$ ($\mathscr{U}$ an n-dimensional vector space) over F into itself, which preserve the irreducible lengths of the products. In this paper, it is shown that preservers of irreducible length 2 are also preservers of all irreducible lengths of the products. Correspondingly, rank 4 preservers are rank $2k$ preservers for all positive integer values of k. The structure of the preservers in each case is deduced from the fact that these preservers are in particular irreducible length 1 and rank 2 preservers respectively, whose structures are known.

A nonzero vector in $\wedge^2\mathscr{U}$ is said to have *irreducible length* k if it can be written as a sum of k and *not less than* k pure (decomposable) nonzero products in $\wedge^2\mathscr{U}$. The set of such products is denoted by $\mathscr{L}_k$ and $z \in \mathscr{L}_k$ if and only if $\mathscr{L}(z) = k$. A linear transformation $\mathscr{T}$ of $\wedge^2\mathscr{U}$ into itself is an *$\mathscr{L}$-k preserver* if and only if $\mathscr{T}(\mathscr{L}_k) \subseteqq \mathscr{L}_k$.

A linear transformation $\mathscr{S}$ which takes the set of *rank* $2k$ n-square skew-symmetric matrices into itself is a *ρ-$2k$ preserver.*

In [7], it is shown that $\mathscr{L}_k$ is isomorphic to the set of all rank $2k$ n-square skew-symmetric matrices. If this isomorphism is denoted by φ, then $\mathscr{S} = \varphi\mathscr{T}\varphi^{-1}$ is a ρ-$2k$ preserver if and only if $\mathscr{T}$ is a $\mathscr{L}$-k preserver.

To obtain the results of this paper, much use is made of $\mathscr{L}$-2 subspaces of $\wedge^2\mathscr{U}$. An *$\mathscr{L}$-k subspace* of $\wedge^2\mathscr{U}$ is a vector subspace whose nonzero members are in $\mathscr{L}_k$. An $\mathscr{L}$-2 subspace H is called a *(1, 1)-type subspace* if there exist fixed nonzero vectors $x \neq y$ such that each nonzero $f \in H$ can be written

$$f = x \wedge x_f + y \wedge y_f \,.$$

1. Intersection of (1, 1)-type subspaces.

LEMMA 1. *If V_1, V_2 are distinct (1, 1)-type subspaces of dimension $\geqq 2$ and $\dim V_1 \cap V_2 \geqq 2$, then the 2-dimensional subspaces of $\mathscr{U}$ determined by V_1, V_2 are equal.*

Proof. Let f_1, f_2 be independent in $V_1 \cap V_2$. Then $f_1 = x \wedge x_1 + y \wedge y_1$,

$f_2 = x \wedge x_2 + y \wedge y_2$ in V_1; and $f_1 = u \wedge u_1 + v \wedge v_1$, $f_2 = u \wedge u_2 + v \wedge v_2$ *in* V_2. Now $\langle x, y\rangle \subset \langle u, u_1, v, v_1\rangle \cap \langle u, u_2, v, v_2\rangle$ which has dimension 2 or 3 (Theorem 5 of [2], and Lemma 5 of [3]), and hence dim $\langle x, y\rangle \cap \langle u, v\rangle \leqq 1$. Without loss of generality, let x be in this intersection; in fact, we can take $x = u$; and $\langle u_1, v, v_1\rangle = \langle x_1 y, y_1\rangle$ and $\langle u_2 v, v_2\rangle = \langle x_2 y, y_2\rangle$ (Lemma 9 of [2]). Since $x \wedge y \wedge f_i = 0$, $i = 1, 2$, then $y \in \langle v, v_1\rangle$ and $y \in \langle v, v_2\rangle$ (proof of Lemma 7 in [3]). If $\langle v, v_2\rangle = \langle v, v_1\rangle$, then some linear combination of f_1 and f_2 has irreducible length at most one, which is impossible since f_1, f_2 are independent in $\mathscr{L}$-2 subspaces. Hence $\langle y\rangle = \langle v, v_1\rangle \cap \langle v, v_2\rangle$, and $\langle y\rangle = \langle v\rangle$, which implies $\langle x, y\rangle = \langle u, v\rangle$.

2. The $\mathscr{L}$-2 preservers. The structure of $\mathscr{L}$-1 preservers is known. In fact, in [8], it is shown that if $\mathscr{T}$ is an $\mathscr{L}$-1 preserver, then $\mathscr{T}$ is a compound (i. e., if $x \wedge y \in \mathscr{L}_1$, then there exists a nonsingular matrix A such that $\mathscr{S}$ $(x \wedge y) = Ax \wedge Ay)$, *except* when dim $\mathscr{U} = 4$, in which case it may possibly be the composite of a compound and a linear transformation induced by a correlation of the 2-dimensional subspaces of $\mathscr{U}$. Thus if $\mathscr{T}$ is an $\mathscr{L}$-1 preserver, it is also an $\mathscr{L}$-k preserver for all k.

We shall show that if $\mathscr{T}$ is an $\mathscr{L}$-2 preserver, then it is also an $\mathscr{L}$-1 preserver. Since we shall make use of $\mathscr{L}$-2 subspaces and these are varied (see [3]), it will be necessary to consider several cases.

2a. dim $\mathscr{U} \geqq 7$. In [3], it is shown that if dim $\mathscr{U} = n \geqq 7$, then the *maximal* $\mathscr{L}$-2 subspaces have dimension (n-3) and are all (1, 1)-type subspaces.

LEMMA 2. *Let* $\mathscr{T}$ *be an* $\mathscr{L}$-2 *preserver*, dim $\mathscr{U} \geqq 7$. *Then* $\mathscr{T}(\mathscr{L}_1) \subset \mathscr{L}_1 \cup \mathscr{L}_2 \cup \{0\}$.

Proof. Let $u \wedge v \in \mathscr{L}_1$. Then $u \wedge v$ is expressible as $u \wedge (\alpha x_1 - x_2)$ where $\{u, x_1 x_2\}$ is independent in $\mathscr{U}$ and $0 \neq \alpha \in F$, $\alpha \neq 1$. Now $\{u, x_1, x_2\}$ can be extended to a set $\{u, x_1, \cdots, x_6\}$ of seven independent vectors in $\mathscr{U}$. Then the following 2 subspaces:

$$V_1 = \langle u \wedge x_1 + v \wedge x_4, u \wedge x_5 + v \wedge x_6, u \wedge x_3 + v \wedge x_4\rangle ,$$

$$V_2 = \langle u \wedge x_2 + v \wedge \alpha x_4, u \wedge x_5 + v \wedge x_6, u \wedge x_3 + v \wedge x_4\rangle$$

are both $\mathscr{L}$-2 subspaces and dim $V_1 \cap V_2 = 2$. Moreover

$$\begin{aligned} \mathscr{T}(u \wedge v) &= \mathscr{T}(u \wedge \alpha x_1 - x_2) \\ &= \mathscr{T}(u \wedge \alpha x_1 + \alpha v \wedge x_4 - u \wedge x_2 - \alpha v \wedge x_4) \\ &= \mathscr{T}(u \wedge \alpha x_1 + \alpha v \wedge x_4) - \mathscr{T}(u \wedge x_2 + \alpha v \wedge x_4) . \end{aligned}$$

The first vector is in $\mathscr{T}(V_1)$, the second in $\mathscr{T}(V_2)$. Now V_1, V_2 can be extended to (n-3)-dimensional $\mathscr{L}$-2 subspaces (necessarily of (1, 1)-type). Hence $\mathscr{T}(V_1)$, $\mathscr{T}(V_2)$ are (1, 1)-type subspaces of dimension (n-3) since $\mathscr{T}$ is an $\mathscr{L}$-2 preserver, and their intersection has dimension at least two. Hence the 2-dimensional subspaces (of $\mathscr{U}$) determined by $\mathscr{T}(V_1)$ and $\mathscr{T}(V_2)$ are equal, implying that $\mathscr{T}(u \wedge v)$ has irreducible length $\leqq 2$.

THEOREM 1. *Let* dim $\mathscr{U} = n \geqq 7$. *Then* $\mathscr{T}$ *is an* $\mathscr{L}$-2 *preserver if and only if* $\mathscr{T}$ *is an* $\mathscr{L}$-1 *preserver, and* $\mathscr{T}$ *is a compound. Moreover,* $\mathscr{T}(\mathscr{L}_k) \subseteq \mathscr{L}_k$ *for all* k.

Proof. Suppose $\mathscr{T}$ is an $\mathscr{L}$-2 preserver. If $f \in \mathscr{L}_1$ and $\mathscr{T}(f)$ 0, then there exists $g \in \mathscr{L}_1$ such that $\mathscr{L}(f+g) = 2$ (use Theorem 7 of [2]). Then $\mathscr{T}(f+g) = \mathscr{T}(g) \in \mathscr{L}_2$. Hence it is sufficient to show $\mathscr{T}(\mathscr{L}_1)$ does not intersect $\mathscr{L}_2$.

Suppose $x_1 \wedge x_n \in \mathscr{L}_1$ and $\mathscr{T}(x_1 \wedge x_n) \in \mathscr{L}_2$. Consider the subspace V generated by $\{z_1 = x_1 \wedge x_n,\ z_i = x_1 \wedge x_{i+1} + x_2 \wedge x_{i+2}\}$, $2 \leqq i \leqq n\text{-}2$, where $\mathscr{U} = \langle x_1, \cdots, x_n \rangle$. Any linear combination $z = \sum_{i=1}^{n-2} \alpha_i z_i$ has irreducible length 2 except when $\alpha_2 = \cdots = \alpha_{n-2} = 0$, in which case $z = \alpha_1 z_1$ and $\mathscr{T}(\alpha_1 z_1)$ has irreducible length 2. Hence $\mathscr{T}(V)$ is an $\mathscr{L}$-2 subspace of dimension (n-2), which contradicts the fact that the maximal $\mathscr{L}$-2 subspaces have dimension (n-3). Hence $\mathscr{T}(\mathscr{L}_1) \subseteq \mathscr{L}_1$. The converse is easy to see (cf. beginning of § 2).

2b. dim $\mathscr{U} = 4, 5$. By Theorem 7 of [2], it is clear that $\mathscr{L}_k$, $k \geqq 3$, is trivial when dim $\mathscr{U} \leqq 5$. The following lemma is immediate.

LEMMA 3. *Let* dim $\mathscr{U} \leqq 5$, $\mathscr{T}$ *an* $\mathscr{L}$-2 *preserver. Then* $\mathscr{T}(\mathscr{L}_1) \subset \mathscr{L}_1 \cup \mathscr{L}_2 \cup \{0\}$.

THEOREM 2. *Let* dim $\mathscr{U} = 4$. *Then* $\mathscr{T}$ *is an* $\mathscr{L}$-2 *preserver if and only if* $\mathscr{T}$ *is an* $\mathscr{L}$-1 *preserver.*

Proof. Suppose $\mathscr{T}$ is an $\mathscr{L}$-2 preserver. Suppose $x_1 \wedge x_2 \in \mathscr{L}_1$ and $\mathscr{T}(x_1 \wedge x_2) = 0$. Extend $\{x_1, x_2\}$ to a basis $\{x_1, \cdots, x_4\}$ of $\mathscr{U}$. Then $x_1 \wedge x_2 + x_3 \wedge x_4$ has irreducible length 2 and hence

$$\mathscr{T}(x_1 \wedge x_2 + x_3 \wedge x_4) = \mathscr{T}(x_3 \wedge x_4)\ .$$

has irreducible length 2. Hence the above and Lemma 3 imply it is sufficient to show only that $\mathscr{T}(\mathscr{L}_1) \not\cap \mathscr{L}_2$.

Suppose $\mathscr{T}(x_1 \wedge x_3)$ has irreducible length 2 for $x_1 \wedge x_3 \in \mathscr{L}_1$. Consider the subspace V generated by the products $z_1 = x_1 \wedge x_3$;

$$z_2 = x_1 \wedge x_2 + x_3 \wedge x_4 \text{ where } \mathscr{U} = \langle x_1, \cdots, x_4 \rangle\ .$$

Then any linear combination $z = \alpha z_1 + \beta z_2$ has irreducible length 2 unless $\beta = 0$, in which case $\mathscr{T}(z) = \mathscr{T}(\alpha z_1)$ which has irreducible length 2 by assumption. Hence $\mathscr{T}(V)$ is an $\mathscr{L}$-2 subspace of dimension 2. But this contradicts the fact that the $\mathscr{L}$-2 subspaces have dimension one and no more (Theorem 10 of [2]). The result follows. The converse is easy to see.

THEOREM 3. *Let* $\dim \mathscr{U} = 5$. *Then* $\mathscr{T}$ *is an* $\mathscr{L}$-2 *preserver if and only if* $\mathscr{T}$ *is an* $\mathscr{L}$-1 *preserver.*

Proof. As in the proof of Theorem 2, it is sufficient to show $\mathscr{T}(\mathscr{L}_1) \not\subset \mathscr{L}_2$. Let $\mathscr{U} = \langle u_1, \cdots, u_5 \rangle$. Suppose $u_1 \wedge u_5 \in \mathscr{L}_1$ and $\mathscr{T}(u_1 \wedge u_5) \in \mathscr{L}_2$. Then consider the subspace V generated by the products

$$\begin{aligned} z_1 &= u_1 \wedge u_5 \,, \\ z_2 &= u_1 \wedge u_4 + u_2 \wedge u_3 \,, \\ z_3 &= u_1 \wedge u_3 + u_2 \wedge u_5 \,, \\ z_4 &= u_2 \wedge u_4 + u_3 \wedge u_5 \,. \end{aligned}$$

Then $z = \sum_{i=1}^{4} \alpha_i z_i$ has irreducible length 2 except when $\alpha_2 = 0 = \alpha_3 = \alpha_4$, in which case $z = \alpha_1 z_1$ and $\mathscr{T}(\alpha_1 z_1) \in \mathscr{L}_2$. Hence $\mathscr{T}(V)$ is an $\mathscr{L}$-2 subspace of dimension 4. But this contradicts the fact that the maximal $\mathscr{L}$-2 subspaces have dimension 3 (see Theorem 1 of [3]).

2c. $\dim \mathscr{U} = 6$. The following lemma is clear from Theorem 7 of [2].

LEMMA 4. *Let* $\dim \mathscr{U} = 6$, $\mathscr{T}$ *an* $\mathscr{L}$-2 *preserver. Then*

$$\mathscr{T}(\mathscr{L}_1) \subset \left\{ \bigcup_{i=1}^{3} \mathscr{L}_i \right\} \cup \{0\} \,.$$

It is thus necessary to consider also the $\mathscr{L}$-3 subspaces.

If $z \in \mathscr{L}_k$, then we can associate a *unique* 2k-dimensional subspace $[z]$ of $\mathscr{U}$ with z (Theorem 5 of [2]).

LEMMA 5. *Let* $z \in \mathscr{L}_k$ *and* $x_1 \in [z]$. *Then there is a representation* $z = x_1 \wedge u_2 + u_3 \wedge u_4 + \cdots + u_{2k-1} \wedge u_{2k}$ *where* $\langle u_2, \cdots, u_{2k} \rangle = [z] - \langle x_1 \rangle$.

Proof. Let x_1 be extended to a basis $\{x_1, \cdots, x_{2k}\}$ of $[z]$. Then

$$\begin{aligned} z &= \sum \alpha_{ij} x_i \wedge x_j \; (1 \leqq i < j \leqq 2k) \\ &= x_1 \wedge \left(\sum_{j=2}^{2k} \alpha_{1\,j} x_j\right) + \sum \alpha_{i\,j} x_i \wedge x_j (2 \leqq i \wedge j \leqq 2k) \,. \end{aligned}$$

By Corollary 8 of [2] and the fact that $\mathscr{L}(z) = k$, the second term in the expression of z has irreducible length (k-1). The result follows.

THEOREM 4. *Let* dim $\mathscr{U} = 6$. H *an* $\mathscr{L}$-3 *subspace. Then* dim $H = 1$.

Proof. If $u_1 \in \mathscr{U}$ and f is any nonzero member of H, then $u_1 \in [f]$. Hence f can be represented $f = u_1 \wedge u + y$, where $u \in \mathscr{U}$ and $y \in \mathscr{L}_2$, $[y] \subset \mathscr{U} - \langle u_1 \rangle$; (Lemma 5). This latter subspace has dimension 5. Thus, if f_1, f_2 are any 2 nonzero members of H, then $f_1 = u_1 \wedge u_2 + u_3 \wedge u_4 + u_5 \wedge u_6$, where $\mathscr{U} = \langle u_1, \cdots, u_6 \rangle$, and f_2 can be expressed as $f_2 = u_1 \wedge y_1 + u_3 \wedge y_2 + u_5 \wedge y_3$ where $y_i = \sum_{j=2}^{6} a_{ij} u_j$, using the fact that $\langle f_1, f_2 \rangle$ is an $\mathscr{L}$-3 subspace, Corollary 8 of [2] and Corollary 1 of [3].

Consider $f = \gamma f_1 + f_2$, $\gamma \in F$. Now $f = u_1 \wedge [(\gamma + a_{12})u_2 + a_{13}u_3 + a_{14}u_4 + a_{15}u_5 + a_{16}u_6] + u_3 \wedge [a_{22}u_2 + (\gamma + a_{24})\, u_4 + a_{25}u_5 + a_{26}u_6] + u_5 \wedge [a_{32}u_2 + a_{33}u_3 + a_{34}u_4 + (\gamma + a_{36})\, u_6] = w_1 \wedge w_2 + w_3 \wedge w_4 + w_5 \wedge w_6$, putting $w_1 = u_1$, $w_2 = [(\gamma + a_{12})u_2 + a_{13}u_3 + a_{14}u_4 + a_{15}u_5 + a_{16}u_6]$, and so on. Then $\mathscr{L}(f) = 3$ if and only if the vectors $w_1, \cdots, w_6$ are independent (Theorem 7 of [2]); i.e., if and only if the determinant of the matrix (a_{ij}), where a_{ij} is the coefficient of u_i in w_j; $i, j = 1, \cdots, 6$; is *nonzero.* However this determinant is a monic polynomial in γ of degree 3; viz., $(\gamma + a_{12})((\gamma + a_{24})(\gamma + a_{36}) - a_{34}a_{26}) - a_{22}(a_{14}(\gamma + a_{36}) - a_{34}a_{16})) + a_{32}(a_{14}a_{26} - a_{16}(\gamma + a_{24}))$, whose constant term *must* be nonzero since the vectors u_1, $u_2\, u_3$, y_1, y_2, y_3 are independent. Hence there is a *nonzero* value of γ in F for which the determinant is zero (since F is algebraically closed). For this value of γ, $\mathscr{L}(f) < 3$. Hence there is at most one basis member in H.

THEOREM 5. *Let* dim $\mathscr{U} = 6$. *Then* $\mathscr{T}$ *is an* $\mathscr{L}$-2 *preserver if and only if* $\mathscr{T}$ *is an* $\mathscr{L}$-1 *preserver.*

Proof. It is sufficient to prove that $\mathscr{T}(\mathscr{L}_1)$ does not intersect $\mathscr{L}_2 \cup \mathscr{L}_3$ (cf. proof of Theorem 2 and use Lemma 4).

Suppose $\mathscr{U} = \langle u_1, \cdots, u_6 \rangle$ and $\mathscr{T}(u_1 \wedge u_6) \in L_2$. Consider $V = \langle z_1, \cdots, z_4 \rangle$ where

$$z_1 = u_1 \wedge u_6;\ z_2 = u_1 \wedge u_3 + u_2 \wedge u_4;\ z_3 = u_1 \wedge u_4 + u_2 \wedge u_5\,;$$

$$z_4 = u_1 \wedge u_5 + u_2 \wedge u_6\,.$$

Then $\mathscr{T}(\mathscr{V})$ is an $\mathscr{L}$-2 subspace of dimension 4, contradicting the fact that the maximal $\mathscr{L}$-2 subspaces have dimension 3 (Theorem 11 of [3]).

Suppose $\mathscr{T}(u_1 \wedge u_5) \in \mathscr{L}_3$. Let $\mathscr{V} = \langle z_1, z_2 \rangle$ where $z_1 = u_1 \wedge u_5$; $z_2 = u_1 \wedge u_4 + u_2 \wedge u_3 + u_6 \wedge u_5$. Then $\mathscr{T}(\mathscr{V})$ is an $\mathscr{L}$-3 subspace of dimension 2, contradicting Theorem 4.

3. The main results. We can now assert:

THEOREM 6. *$\mathscr{T}$ is an $\mathscr{L}$-2 preserver if and only if $\mathscr{T}$ is an $\mathscr{L}$-1 preserver. If $\mathscr{T}$ is an $\mathscr{L}$-2 preserver, then $\mathscr{T}$ is an $\mathscr{L}$-k preserver, $k = 1, 2, \cdots, [n/2]$, $\dim \mathscr{U} = n$, and $\mathscr{T}$ is a compound except when $n = 4$, in which case $\mathscr{T}$ may possibly be a composite of a compound and a linear transformation induced by a correlation of the 2-dimensional subspaces of $\mathscr{U}$.*

Using the results in [7], we can also assert the following.

THEOREM 7. *$\mathscr{S}$ is a ρ-4 preserver if and only if $\mathscr{S}$ is a ρ-2 preserver. If $\mathscr{S}$ is a ρ-4 preserver, then $\mathscr{S}$ is a ρ-$2k$ preserver, $k = 1, 2, \cdots, [n/2]$. Moreover, if A is any n-square skew-symmetric matrix, then $\mathscr{S}(A) = \alpha PAP'$ or $\mathscr{S}(A) = \beta PA'P'$ for α, β nonzero in F and some nonsingular n-square matrix P except when $n = 4$, in which case $\mathscr{S}$ may possibly be of the form*

$$\mathscr{S}(A) = \alpha P \left\| \begin{matrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{matrix} \right\| P'$$

where $A = (a_{ij})$, $a_{ij} = -a_{ji}$.

REMARK. These results are not necessarily true when the underlying field F is *nonalgebraically closed* (cf. § 2b. and end of [2]).

ACKNOWLEDGEMENT. Many thanks are due to Professor R. Westwick for his invaluable help in obtaining these results.

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Received May 16, 1969.

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