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NONCONTINUOUS MULTIFUNCTIONS

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The classes of noncontinuous multifunctions studied here are characterized by their members having certain connectedness properties. A particular example is the class of connected (C, O) multifunctions whose members take connected, open sets to connected sets. Relationships between these classes are given, and some results known for connected, single valued functions are generalized to connected (C, O) multifunctions.

Section 2 contains a continuity theorem for connected (C, O) multifunctions as well as a necessary and sufficient condition for a topological space to be locally connected in terms of a condition on the class of connected (C, O) multifunctions defined on it. In § 3, with the aid of the notion of the cluster set of a multifunction at a point, sufficient conditions are given for a multifunction to be connected. Some general properties about cluster sets are also proved. Section 4 contains characterizations of continuity for linear operators, semi-norms and convex functions.

All topological spaces considered here are at least T_1 and any multifunction F from a topological space X to a topological space Y will be such that $F(x) \neq \emptyset$ for each $x \in X$. For any $A \subset X$, $F(A) = \cup\{F(x): x \in A\}$ and for any $B \subset Y$, $F^{-1}(B) = \{x \in X: F(x) \cap B \neq \emptyset\}$. F is *upper semi-continuous* (u.s.c.) at $x \in X$ if and only if for each open subset V of Y such that $F(x) \subset V$ there is a neighborhood N of x such that $F(N) \subset V$. If F is u.s.c. at each $x \in X$, then F is u.s.c. If $F(x)$ is a single element for each $x \in X$, then F will be referred to as a function and any result which is true for multifunctions may be stated also for functions. The *graph* of a multifunction $F: X \rightarrow Y$ is $\{(x, y) \in X \times Y: y \in F(x)\}$ and is denoted by $G(F)$. The multifunction $F_G: X \rightarrow X \times Y$ defined by $F_G(x) = \{x\} \times F(x)$ is called the *graph multifunction* of F. The *boundary* of a set A will be denoted by bdry A and the *closure* of A by $\bar{A}$. F is *point connected (compact)* if $F(x)$ is connected (compact) for each point $x \in X$.

In addition to connected (C, O) multifunctions we consider the following noncontinuous (properly called non-u.s.c.) multifunctions.

DEFINITION 1.1. F is called a *connected multifunction* if $F(K)$ is connected for each connected subset K of X and is called a *connectivity multifunction* if F_G is a connected multifunction. The class of connectivity multifunctions is denoted by C^{-1} and the class of connected multifunctions is denoted by C^{-2}.

DEFINITION 1.2. F is said to be in class C^{-3} if and only if $\overline{F(K)}$ is connected for every connected set K.

For functions this class becomes the class $\mathscr{U}_0$ as defined in [1].

DEFINITION 1.3. F is said to be in class C^{-4} if and only if $F(\bar{K}) \subset \overline{F(K)}$ for every connected subset K.

The C^{-n} classification of functions for $n = 0, 1, 2$ is used in [5] with C^0 being the continuous functions. For functions it is not difficult to see that $C^{-(n-1)} \subset C^{-n}$ for $n = 1, 2, 3, 4$ and, as shown in [5], the inclusions $C^0 \subset C^{-1} \subset C^{-2}$ are proper. The inclusion $C^{-2} \subset C^{-3}$ is proper as is seen in the example of the function f from the reals R to R satisfying $f(x + y) = f(x) + f(y)$ [7]. The inclusion $C^{-3} \subset C^{-4}$ is proper as is seen in the function $f\colon R \to R$ defined by $f(x) = 1$, if x is rational and $f(x) = 0$ if x is irrational.

Even though the inclusions do not all hold in general for multifunctions, we still find it convenient to use the C^{-n} notation. In certain circumstances some inclusions do hold for multifunctions. In [2] it is shown that $C^{-1} \subset C^{-2}$ and if F is a point connected, point compact multifunction which is u.s.c., then $F \in C^{-1}$. The inclusion $C^{-2} \subset C^{-3}$ clearly holds, but the inclusion $C^{-3} \subset C^{-4}$ does not hold as the multifunction $F\colon I \to I (I = \{x \in R\colon 0 \leqq x \leqq 1\})$ defined by $F(x) = 0$ for $0 \leqq x < 1$ and $F(1) = I$ shows.

DEFINITION 1.4. Let $\{S_\alpha\colon \alpha \in D\}$ be a net of sets in X, where D is a directed set [15]. We will use (S_α) to denote this net of sets if the directed set is understood.

Superior and inferior limits are defined as follows.

$$\overline{\lim_{\alpha \in D}}\, S_\alpha = \{x \in X\colon \text{for any open set } G \subset X \text{ about } x \text{ and for any } \alpha \in D \text{ there exists a } \beta > \alpha \text{ such that } G \cap S_\beta \neq \varnothing\}.$$

$$\underline{\lim_{\alpha \in D}}\, S_\alpha = \{x \in X\colon \text{for any open set } G \text{ about } x \text{ there exists an } \alpha \in D \text{ such that } G \cap S_\beta \neq \varnothing \text{ for all } \beta > \alpha\}.$$

A net of sets is defined to be frequently (eventually) in a set S just as a net of points is frequently (eventually) in a set S.

2. Connected (C, O) multifunctions. It is not difficult to find examples which show that the class of connected (C, O) multifunctions properly contains the class of connected multifunctions and is not in general comparable to the class C^{-4}. In some results concerning connected functions it becomes evident that the same result will also hold true for the wider class of connected (C, O) functions as well if the domain space is locally connected. Section 4 indicates that several important types of functions are always connected (C, O) and are not in general connected or in C^{-4}. In this section we prove a continuity theorem for connected (C, O) multifunctions and then characterize local connectedness of a space X in terms of a condition on the class of connected (C, O) multifunctions on X.

DEFINITION 2.1. A multifunction $F: X \to Y$ is said to be *compatible with* Y if each point image $F(x)$ is separable in Y from each closed set C in its complement by a set K (possibly void) such that $\overline{F^{-1}(K)}$ does not contain x; i.e., $Y - K = A \cup B$, $F(x) \subset A$, $C \subset B$, $\bar{A} \cap B = \varnothing = A \cap \bar{B}$ and $A \neq \varnothing$, $B \neq \varnothing$. If the range space of F is understood we speak of F being *compatible*.

In [18] the range space Y is defined to be *peripherally F-normal* provided each point image $F(x)$ is separable in Y from each closed set C in its complement by a set K having a closed inverse under F. The condition of compatibility with Y of F is implied by peripheral F-normality of Y, but the converse is not true as the following example shows.

EXAMPLE 2.2. With the usual topology on R define a function $f: R \to R$ by $f(x) = \sin(1/x)$ if $x > 0$, $f(0) = 2$ and $f(x) = 0$ if $x < 0$. R is not peripherally f-normal as can be seen by considering any point $x \in R$ such that $f(x) = 3/4$ and the closed set $C = [0, 1/4] \cup \{2\}$ in the complement of $f(x)$. C and $f(x)$ can be separated only by a set K with $0 \in \overline{f^{-1}(K)} - f^{-1}(K)$ and thus $f^{-1}(K)$ is not closed. To see that f is compatible it is a simple matter to check the two cases $x = 0$ and $x \neq 0$.

The following theorem is a generalization of Theorem B of [18].

THEOREM 2.3. *If X is locally connected and $F: X \to Y$ is a compatible, connected (C, O) multifunction, then F is* u.s.c.

Proof. Let V be any proper open set about an arbitrary point image $F(x)$. Since $Y - V$ is a closed set in the complement of $F(x)$, there exists, by hypothesis, a set K separating $F(x)$ and $Y - V$ such that $\overline{F^{-1}(K)}$ does not contain x. Let $Y - K = A \cup B$, where

$F(x) \subset A$, $Y - V \subset B$ and $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. Since X is locally connected there exists a connected, open neighborhood N of x such that $N \cap \overline{F^{-1}(K)} = \emptyset$. From this it follows that $F(N) \cap K = \emptyset$, because if $a \in N$, then $a \notin F^{-1}(K)$ and by definition $F(a) \cap K = \emptyset$. Now, $F(N)$ is a connected set and since it contains $F(x)$, which is a subset of V, it cannot intersect $Y - V$, for otherwise, $A \cup B$ would separate $F(N)$ and this is a contradiction. Thus $F(N) \subset V$ and F is u.s.c. at x. Since x is arbitrary, F is u.s.c.

COROLLARY 2.4. *If X is locally connected and f is a connected (C, O) function on X to a regular space Y, then f is continuous if and only if $f^{-1}(\text{bdry } N)$ is closed for each open set $N \subset Y$.*

By a proof similar to that of Theorem 2.3, but not involving the notion of compatibility, the above corollary may be strengthened by not requiring regularity of the range space. The result becomes a generalization of Theorem 2.1 of [6] and Theorem C of [9]. We also have the following corollaries.

COROLLARY 2.5. *Let X and Y be locally connected and let $f: X \to Y$ be a one-to-one, onto function such that both f and f^{-1} are connected (C, O). Then f is a homeomorphism if and only if both f and f^{-1} take closed, nowhere dense sets to closed sets.*

COROLLARY 2.6. *Let X be locally connected and let $f: X \to R$ be a connected (C, O), real valued function. Then f is continuous if and only if there exists a dense subset D of R such that for each $x \in X$ and each open set V about $f(x)$ there exists a subset K of D, consisting of at most two points, separating $f(x)$ and $R - V$ such that $\overline{f^{-1}(K)}$ does not contain x.*

The above corollary is a characterization of continuous real valued functions on a locally connected space similar to that in Theorem 1 of [10]. It is natural now to ask the following question which is a slight variant of that posed in [10]. Is X locally connected if each connected (C, O) function on X, as in Corollary 2.4 or 2.6, is continuous? If each component of X is locally connected to begin with, then a positive result can be obtained. To see this let K be a component of X and define a function $f: X \to R$ by $f(x) = 1$ if $x \in K$ and $f(x) = 0$ if $x \notin K$. The conditions of Corollary 2.6 are satisfied by f and thus f is continuous. Therefore $f^{-1}(0) = X - K$ is closed and K is open. Similarly every other component of X is open and thus X is locally connected.

The following theorem gives an affirmative answer to a question similar to that posed above using multifunctions instead of functions.

THEOREM 2.7. *If every connected multifunction* $F: X \to R$ *for which* R *is peripherally* F*-normal is* u.s.c., *then* X *is locally connected.*

Proof. If V is any open set about an arbitrary point $x \in X$, it must be shown that the component K of x in V is open in X. Define a multifunction $F: X \to R$ by $F(x) = 0$ if $x \in K$, $F(x) = [0, 1]$ if $x \in \operatorname{bdry} V$ and $F(x) = 1$ if $x \in X - (K \cup \operatorname{bdry} V)$. F is a connected multifunction because any connected set which intersects both $X - \bar{V}$ and V must intersect bdry V. Also, any connected set in V must be entirely in K or entirely in $(X - K) \cap V$.

To see that R is peripherally F-normal there are three cases to consider. If $F(x) = 0$ and C is any closed set in $R - \{0\}$, there exist points P arbitrarily close to zero such that $F^{-1}(P) = \varnothing$ if $P < 0$ and $F^{-1}(P) = \operatorname{bdry} V$ if $P > 0$. In both cases the inverse images are closed and the points P can be chosen so as to separate $F(x)$ and C. For the second case, ($F(x) = 1$), a similar argument is used. Finally, if x is such that $F(x) = [0, 1]$ and C is a closed set in the complement of $F(x)$, there is always a set K, consisting of either one or two points, separating $F(x)$ and C such that $F^{-1}(K) = \varnothing$. Thus R is peripherally F-normal and since F is connected, F becomes u.s.c. by hypothesis. Now, the inverse image under F of any closed set in R is closed and, in particular, $F^{-1}(1) = \{x \in X: 1 \in F(x)\} = (\operatorname{bdry} V) \cup (X - (K \cup \operatorname{bdry} V)) = X - K$ is closed. Thus K is an open set in X.

COROLLARY 2.8. *The following statements are equivalent.*

(i) *X is locally connected.*

(ii) *Every connected multifunction* $F: X \to R$ *for which* R *is peripherally* F*-normal is* u.s.c.

(iii) *Every connected* (C, O) *multifunction* $F: X \to R$ *compatible with* R *is* u.s.c.

Proof. (i) implies (iii) by Theorem 2.3, (iii) implies (ii) is obvious and (ii) implies (i) by Theorem 2.7.

It is well known that if Y is compact and $f: X \to Y$ is a function with a closed graph, then f is continuous. The following theorem which is stated for multifunctions shows that the restriction on Y may be reduced somewhat if f is in addition connected (C, O) and X is locally connected. Recall that a space is *rim-compact* if it is T_2

and has a base for the open sets such that the boundary of each member is compact, It is known that a rim-compact space is regular.

THEOREM 2.9. *Let X be locally connected and let Y be rim-compact. If $F\colon X \to Y$ is a connected (C, O) multifunction which is point compact and has a closed graph, then F is* u.s.c.

Proof. By Theorem 2.3 it suffices to show that F is compatible with Y. To this end let C be a closed set in the complement of $F(x)$. For each $y \in F(x)$ there is an open neighborhood N_y of y such that $\bar{N}_y \cap C = \varnothing$ and $\operatorname{bdry} N_y$ is compact. The family of sets $\{N_y\}$, $y \in F(x)$, so chosen is an open cover of a compact set $F(x)$ and thus some finite subcover $\{N_{y_i}\}_{i=1}^n$ of $F(x)$ exists. If $N = \bigcup_{i=1}^n N_{y_i}$, then $\bar{N} \cap C = \varnothing$, $\operatorname{bdry} N$ is compact and also $\operatorname{bdry} N$ separates $F(x)$ and C. Now, since F is a closed relation, F^{-1} takes compact sets to closed sets. Consequently $F^{-1}(\operatorname{bdry} N)$ is closed and F is compatible.

3. Cluster sets.

The notion of the cluster set of a function at a point has been used in analysis, see [1], [4], and has recently been extended to functions on spaces more general than euclidean spaces, see [11], and [14]. In this section cluster sets are defined for multifunctions in terms of nets rather than sequences, as is done in some of the references, and this allows theorems to be proved without the restriction of first countability on the spaces. With the aid of cluster sets some of the results in the literature proved for functions are generalized here to multifunctions on more general spaces.

DEFINITION 3.1. For any multifunction $F\colon X \to Y$ and for any $x \in X$ let $C(F; x) = \{y \in Y\colon$ there exists a net (x_α) converging to x with $y \in \overline{\lim}_\alpha F(x_\alpha)\}$. This set is called the *cluster set of* F at x and if no confusion arises we speak of *cluster set*. An equivalent definition is obtained if "$\overline{\lim}$" is replaced by "$\underline{\lim}$".

If A is any subset of X, let $C_A(F; x) = \{y \in Y\colon$ there exists a net (x_α) in A converging to x with $y \in \overline{\lim}_\alpha F(x_\alpha)\}$. This set is called the *partial cluster set of F at x* with respect to A. If A is chosen only from the family of connected sets, then we speak of *partial cluster sets with respect to connecta.*

The following properties are immediate.

1. Let X be a space in which all components are open. If partial cluster sets with respect to connecta are connected, then the cluster sets are connected.
2. For any subset K of X, $C_K(F; x) \subset \overline{F(K)}$ for each $x \in X$.

LEMMA 3.2. *If $\mathscr{N}_x$ is a system of neighborhoods of $x \in X$ and $F: X \to Y$ is a multifunction, then $C(F, x) = \cap \{\overline{F(N)}: N \in \mathscr{N}_x\}$ and is consequently a closed set.*

Proof. If $y \in C(F; x)$, there exists a net (x_α) converging to x such that $y \in \overline{\lim}_\alpha F(x_\alpha)$; i.e., every neighborhood M of y intersects $F(x_\alpha)$ frequently. If N is a neighborhood of x, then (x_α) is eventually in N and, consequently, $F(x_\alpha) \subset F(N)$ eventually. Thus $M \cap F(N) \neq \varnothing$ for each neighborhood M of y and $y \in \overline{F(N)}$. Since this is true for each $N \in \mathscr{N}_x$ we have $y \in \cap \{\overline{F(N)}, N \in \mathscr{N}_x\}$, and thus $C(F, x) \subset \cap \{\overline{F(N)}: N \in \mathscr{N}_x\}$.

Now let $y \in \cap \{\overline{F(N)}: N \in \mathscr{N}_x\}$, and let $\mathscr{M}_y$ be a neighborhood system of y. Since $y \in \overline{F(N)}$ for each $N \in \mathscr{N}_x$, $M \cap F(N) \neq \varnothing$ for each $M \in \mathscr{M}_y$ and thus for each pair (M, N) with $M \in \mathscr{M}_y$ and $N \in \mathscr{N}_x$, there exists a point $x_{(M,N)} \in N$ such that $F(x_{(M,N)}) \cap M \neq \varnothing$ and there exists then a point $y_{(M,N)} \in F(x_{(M,N)}) \cap M$. The net $(x_{(M,N)})$ converges to x and the net $(y_{(M,N)})$ converges to y. Thus $y \in \overline{\lim}_{(M,N)} F(x_{(M,N)})$ and consequently $y \in C(F, x)$.

For a given multifunction F it can be shown that the graph $G(F)$ is closed if and only if whenever a net (x_α) converges to x and $y \in \overline{\lim}_\alpha F(x_\alpha)$, then $y \in F(x)$. Also "$\overline{\lim}$" may be replaced by "$\underline{\lim}$".

THEOREM 3.3. *For any multifunction $F: X \to Y$, $G(F)$ is closed if and only if $C(F; x) = F(x)$.*

Proof. It is always true that $F(x) \subset C(F; x)$. If $G(F)$ is closed and $z \in C(F; x)$, there exists a net (x_α) which converges to x and $z \in \overline{\lim}_\alpha F(x_\alpha)$. Thus $z \in F(x)$ and $F(x) = C(F; x)$. If we assume that $C(F; x) = F(x)$ and that $y \in \overline{\lim}_\alpha F(x_\alpha)$, where (x_α) is a net which converges to x, then immediately $y \in C(F; x)$ and consequently $y \in F(x)$. This is equivalent to $G(F)$ being closed.

The following theorem gives some sufficient conditions for the cluster set of a multifunction at a point to be connected. It is a generalization of Theorem 3.7 of [14] in the sense that connected (C, O) multifunctions are used instead of connected functions, first axiom spaces are not required and the condition of compactness on the range space is weakened.

DEFINITION 3.4. A multifunction $F: X \to Y$ is said to be *subcontinuous* if whenever a net (x_α) converges to some $x \in X$, then for any net (y_α), with $y_\alpha \in F(x_\alpha)$ for each α, there is a subnet which converges to some $y \in Y$.

Subcontinuity of a function was defined by R.V. Fuller, [3].

THEOREM 3.5. *Let X be locally connected and let Y be a normal space. If $F: X \rightarrow Y$ is a multifunction such that*

(i) *F is connected (C, O) (or $F \in C^{-3}$)* and

(ii) *F is subcontinuous,*

then $C(F; x)$ is connected for each $x \in X$.

Proof. Suppose that for some $x \in X$ $C(F; x) = A \cup B$, where A and B are nonempty sets and $\bar{A} \cap B = \varnothing = A \cap \bar{B}$. Since $C(F; x)$ is closed, both A and B are closed and by normality of Y there exist two disjoint open sets O_A and O_B containing A and B, respectively, whose closures are also disjoint. If $\mathcal{N}_x$ is the neighborhood system of x consisting of connected, open sets, then $C(F; x) = \cap\{\overline{F(N)}: N \in \mathcal{N}_x\}$, and each $\overline{F(N)}$ is connected.

Now, for each $N \in \mathcal{N}_x, F(N) \cap (Y - (O_A \cup O_B)) \neq \varnothing$ for, $F(N) \subset O_A \cup O_B$ implies that $\overline{F(N)} \subset \overline{O_A \cup O_B} = \overline{O_A} \cup \overline{O_B}$. But since $\overline{F(N)}$ is connected, it must be contained in one of $\overline{O_A}$ or $\overline{O_B}$. If $\overline{F(N)} \subset \overline{O_A}$, then $\overline{F(N)} \cap O_B = \varnothing$ and this is impossible since $B \subset \overline{F(N)} \cap O_B$. Similarly $\overline{F(N)}$ cannot be contained in $\overline{O_B}$ and thus $F(N) \not\subset O_A \cup O_B$.

For each $N \in \mathcal{N}_x$ choose an element $y_N \in F(N) \cap (Y - (O_A \cup O_B))$ and an element $x_N \in N$ such that $y_N \in F(x_N))$. The net (x_N) so obtained converges to x. By hypothesis the net (y_N) has a subnet (y'_N) which converges to some point $y \in Y$ and since each y'_N is in $Y - (O_A \cup O_B)$, which is a closed set, it follows that $y \in Y - (O_A \cup O_B)$. From the definition of cluster set, $y \in C(F; x)$ and this contradicts the fact that $C(F; x) \subset O_A \cup O_B$. Thus $C(F; x)$ is connected.

COROLLARY 3.6. *Let X be locally connected and let Y be a compact, Hausdorff space. If $F: X \rightarrow Y$ is a connected (C, O) multifunction (or is in class C^{-3}), then $C(F; x)$ is connected for each $x \in X$.*

THEOREM 3.7. *Let X be a locally connected, locally compact space and let Y be a Hausdorff space. If $F: X \rightarrow Y$ is a connected (C, O) multifunction (or is in class C^{-3}) and takes compact sets to compact sets, then $C(F; x)$ is connected for each $x \in X$.*

The proof is analogous to that of Theorem 3.5.

The following theorem is a generalization of Theorem 3.8 of [14] to connected (C, O) multifunctions on spaces which are not necessarily first countable.

THEOREM 3.8. *Let X be locally connected and let Y be compact*

Hausdorff. A connected (C, O), *point closed multifunction* $F: X \to Y$ *is* u.s.c. *if and only if* $C(F; x)$ *is the union of a countable number of disjoint closed sets one of which is* $F(x)$.

Proof. We prove the "if" part first. By Corollary 3.6 $C(F; x)$ is connected for each $x \in X$. In [19], p. 16, it is shown that a compact, connected set is never the union of a countable number (greater than 1) of disjoint closed sets. Thus $C(F; x)$, which is a compact, connected set cannot be the union of a countable number of disjoint closed sets unless $C(F; x) = F(x)$. So, for each $x \in X$ $C(F; x) = F(x)$. Thus the graph of F is closed and by Theorem 2.9 F is u.s.c.

Now, if we suppose that F is u.s.c., we have that graph of F is closed (see [17], Proposition 13) and thus $C(F; x) = F(x)$. The condition of the theorem is satisfied.

THEOREM 3.9. *Suppose that a point connected multifunction* $F: X \to Y$ *has the following three properties.*

(i) *If* F *is subcontinuous.*

(ii) $C(F; x)$ *is connected for each* $x \in X$.

(iii) *For each nondegenerate connected subset* C *of* X *and for each* $x \in C$, $C(F; x) \subset F(C)$.

Then F *is a connected multifunction.*

Proof. Suppose that for some nondegenerate connected subset K of X, $F(K)$ is not connected and $F(K) = A \cup B$, where $A \neq \emptyset \neq B$ and $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. If $A_1 = F^{-1}(A) \cap K$ and $B_1 = F^{-1}(B) \cap K$, then, because F is point connected, $K = A_1 \cup B_1$, A_1, B_1 are nonempty and $A_1 \cap B_1 = \emptyset$. Since K is connected, without loss of generality we may pick a point $x \in \bar{A}_1 \cap B_1$. Then $F(x) \subset B$ and there is a net (x_α) in A_1 which converges to x. The net of sets $(F(x_\alpha))$ is such that $F(x_\alpha) \cap A \neq \emptyset$ for each α and by (i) there is a net (y_α) in Y, $y_\alpha \in F(x_\alpha) \cap A$, with a subnet which converges to some point $y \in Y$, in fact $y \in \bar{A}$. This means that $y \in C(F; x)$ and by (iii) $y \in F(K) = A \cup B$. In particular, $y \in A$ since $y \in \bar{A}$ and $\bar{A} \cap B = \emptyset$. In summary $C(F; x) \subset A \cup B$ and has a nonempty intersection with both A and B. But since A and B are separated, this contradicts (ii) and thus $F(K)$ is connected.

The example on p. 488 of [14] shows that the conditions of the above theorem do not imply that F is u.s.c. and a suitable modification of this example shows that connectivity of the multifunction is not implied either. Examples also exist showing that any two of the conditions of the theorem are not adequate to imply that $E \in C^{-2}$.

The following result is proved in [3] for a function $f: X \to Y$.

"A sufficient condition that f be continuous is that f have a closed graph and whenever a net (x_α) converges to some $x \in X$, then some subnet of $(f(x_\alpha))$ converges to some point $y \in Y$. If Y is Hausdorff the condition is also necessary". Now, if the graph of a function is closed, then conditions (ii) and (iii) of Theorem 3.9 are satisfied since for each x, $C(f; x) = f(x)$. The converse is not true; see [14], p. 488. Thus, weakening the condition that f have a closed graph in [3], to conditions (ii) and (iii) does not give continuity but does insure connectedness of f.

THEOREM 3.10. *Let $F: X \to Y$ be a point connected multifunction with the following properties.*

(i) *F is subcontinuous.*

(ii) *$C_K(F; x)$ is connected for each connected set K.*

Then $\overline{F(K)}$ is connected for each connected set K; i.e., $F \in C^{-3}$.

Proof. Suppose that for some connected subset K of X, $\overline{F(K)} = A \cup B$, where A and B are not empty and $\bar{A} \cap B = \varnothing = A \cap \bar{B}$. Since this is a separation there exist two open sets O_A and O_B containing A and B, respectively, such that $\overline{F(K)} \cap O_A \cap O_B = \varnothing$. Now, O_A and O_B must each intersect $F(K)$ also, because if $F(K) \cap O_A = \varnothing$, then $\overline{F(K)} \cap O_A = \varnothing$. Similarly $F(K) \cap O_B \neq \varnothing$. Consequently A and B must each have a nonempty intersection with $F(K)$, for if $F(K) \cap A = \varnothing$, then $F(K) \cap O_A = \varnothing$. Similarly $F(K) \cap B \neq \varnothing$. Let $A_1 = K \cap F^{-1}(A)$ and $B_1 = K \cap F^{-1}(B)$. Then $K = A_1 \cup B_1$ and $A_1 \cap B_1 = \varnothing$, but K is connected so we may, without loss of generality, pick an element $x \in \bar{A}_1 \cap B_1$. There is a net (x_α) in A_1 which converges to x and by (i) there exists a $y \in \bar{A}$ such that $y \in C_K(F; x)$. Thus $C_K(F; x) \cap A \neq \varnothing \neq C_K(F; x) \cap B$ and since $C_K(F; x) \subset A \cup B$ it follows that A and B separate $C_K(F; x)$. This contradicts (ii) and so $\overline{F(K)}$ is connected.

4. Applications to functions on topological vector spaces.

Because linear operators, semi-norms and convex functions each have certain connectedness properties Theorem 2.3 can be applied to some of these functions to give characterizations of continuity. All topological vector spaces (t.v.s) considered here, denoted by E or E', will be Hausdorff and the scalar field of each will be the complex numbers with the usual topology. The terminology is standard and may be found in [8].

A set A in a t.v.s. E is called *balanced* if $tA \subset A$ for all scalars t such that $|t| \leqq 1$. Denote by $\mathscr{N}$ the neighborhood system of $0 \in E$ (the zero of the t.v.s.) such that every $V \in \mathscr{N}$ is balanced. For

$x \in E$ sets of the form $V + x$, as V runs through $\mathscr{N}$, form a system $\mathscr{N}_x$ of neighborhoods for the point x. Since each balanced set in E is a connected set and since the operation of translation is a homeomorphism in E, it follows that E is a locally connected space.

LEMMA 4.1. *Any linear operator $T: E \to E'$ is a connected (C, O) function.*

Proof. Since linear operators preserve balanced sets, $T(N + x) = T(N) + T(x)$ is a connected set in E' for each $N \in \mathscr{N}$ and $x \in E$. Now, if K is a connected, open set in E, it is a simple matter to show that $T(K)$ is connected in E'.

THEOREM 4.2. *A linear operator $T: E \to E'$ is continuous if and only if $T^{-1}(\text{bdry } M)$ is closed in E for each M of some neighborhood system $\mathscr{M}$ of $0 \in E'$.*

Proof. If T is continuous, it is immediate that T^{-1} (bdry M) is closed since bdry M is closed for each $M \in \mathscr{M}$. Conversely, if T^{-1} (bdry M) is closed for each $M \in \mathscr{M}$, then by Corollary 2.4 T is continuous at 0. Therefore T is continuous.

In [6] it is shown that a linear functional f on E is continuous if and only if it is a connected function. In fact f is continuous if and only if $f \in C^{-4}$. We have not answered the question as to whether a connected linear operator is continuous in general, however, the following result is true.

THEOREM 4.3. *A linear operator $T: E \to E'$ is continuous if $T \in C^{-4}$ and for each neighborhood M of 0 in E' the closure of $T^{-1}(M)$ is a neighborhood of 0 in E.*

Proof. Let $\mathscr{M}$ be a neighborhood system of 0 in E' consisting of closed, balanced sets. $T^{-1}(M)$ is a balanced set in E for each $M \in \mathscr{M}$ and thus is connected. Also, by hypothesis, $\overline{T^{-1}(M)}$ is a neighborhood of 0 in E. Since $T \in C^{-4}$ we have $T(\overline{T^{-1}(M)}) \subset \overline{TT^{-1}(M)} = \bar{M} = M$. This means that T is continuous at 0 and thus continuous.

COROLLARY 4.4. *Let E be a second category* t.v.s. *A linear operator $T: E \to E'$ which belongs to C^{-4} is continuous.*

Proof. Since E is of second category T satisfies the conditions of Theorem 4.3, see [8], p. 97.

A finite, real valued function P on a t.v.s. E is called a *seminorm* if for all x and y in E and every scalar t

(1) $P(tx) = |t| P(x)$ and

(2) $P(x + y) \leqq P(x) + P(y)$.

From (1) and (2) it follows that $P(0) = 0$, $P(x) \geqq 0$ and $|P(x) - P(y)| \leqq P(x - y)$. This last inequality implies that P is continuous if and only if P is continuous at $0 \in E$. Denote $\{t \in R: t \geqq 0\}$ by R^+. With every semi-norm P on E there is associated a semi-norm topology $\mathscr{P}$ on E with respect to which the semi-norm is continuous and with respect to which E is a t.v.s. (not necessarily Hausdorff).

LEMMA 4.5. *Any semi-norm* $P: E \to R^+$ *is a connected* (C, O) *function.*

Proof. Let $\mathscr{N}$ denote the neighborhood system of $0 \in E$ consisting of balanced neighborhoods. Since the concept of balancedness is independent of any topology, each N is connected in $\mathscr{P}$. Since $P: (E, \mathscr{P}) \to R^+$ is continuous $P(N)$ is connected in R^+. Similarly, $P(x + N)$ is connected for each $x \in E$ and each $N \in \mathscr{N}$. As in Lemma 4.1 it can be shown now that P takes connected, open sets to connected sets.

THEOREM 4.6. *A semi-norm* $P: E \to R^+$ *is continuous if and only if there exists a* $t > 0$ *such that* $\overline{P^{-1}(t)}$ *does not contain* $0 \in E$.

Proof. If P is continuous, then $P^{-1}(t)$ is closed for each $t \in R^+$ and if $t > 0$, then $0 \notin P^{-1}(t)$.

To prove the converse we make use of the known result that P is continuous if and only if it is bounded on some open subset of E. Because of the hypothesis for some $t \in R^+, t > 0$, there exists a balanced neighborhood N of $0 \in E$ such that $N \cap \overline{P^{-1}(t)} = \varnothing$. Since $P(N)$ is connected, does not contain t and does contain 0 it must be that $P(N)$ is bounded above by t. Thus P is continuous.

THEOREM 4.7. *A semi-norm* $P: E \to R^+$ *in class* C^{-4} *is continuous if for each neighborhood* M *of* $0 \in R^+$, $\overline{P^{-1}(M)}$ *is a neighborhood of* $0 \in E$.

Proof. It is sufficient to show continuity of P at $0 \in E$. For any neighborhood $[0, t)$ of $0 \in R^+$ consider the neighborhood $[0, 1/2t)$. By hypothesis $\overline{P^{-1}([0, 1/2t))}$ is a neighborhood of 0 and is furthermore connected since $P^{-1}(0, 1/2t))$ is convex. Since $P \in C^{-4}$ we have $P(\overline{P^{-1}([0, 1/2t))}) \subset \overline{PP^{-1}([0, 1/2t))} \subset [0, 1/2t] \subset [0, t)$. Thus P is continuous at $0 \in E$.

In [13] it is shown that if E is a second category t.v.s. and

$P: E \to R^+$ is a semi-norm such that $\{x: P(x) \leqq 1\}$ is a Baire set then P is continuous. (A set S is a Baire set if $S = (G - P) \cup R$, where G is open and P and R are first category sets. G may be replaced by a closed set.) From this we have the following.

THEOREM 4.8. *A semi-norm in class C^{-4} on a second category* t.v.s. *is continuous.*

Proof. In [15] it is shown that for functions in C^{-4} the inverse image of a closed, connected set is closed. Thus $\{x \in E: P(x) \leqq 1\}$ is closed and thus a Baire set. By the above remarks P is continuous.

A real valued function $f: E \to R$ is called *convex* if $f(x + y/2) \leqq f(x) + f(y)/2$ for every x and y in E. Every semi-norm as well as every additive functional $f(x + y) = f(x) + f(y)$ is a convex function and in [12] it is shown that, just as for semi-norms, a convex function which is bounded above on a nonempty open subset is continuous. Convex functions have the following connectivity property.

THEOREM 4.9. *If $f: E \to R$ is a convex function and N is a balanced neighborhood of $0 \in E$, then $\overline{f(N)}$ is connected.*

Proof. We make use of Theorem 4.5 of [1] which states that a convex function defined on the real line is in C^{-3} and hence in C^{-4}. The restriction of f to any line $L = \{tx + (1 - t)y: t \in R\}$ in E is a convex function. Now, a balanced neighborhood N of $0 \in E$ is a union of line segments, namely $N = \bigcup_{x \in N}[0, x]$. By the above mentioned result $\overline{f([0, x])}$ is connected for each $x \in N$ and since each such connected set contains the common point $f(0)$, the set $\overline{\bigcup_{x \in N}([0, x])}$ is connected in R. It now follows readily that $\overline{f(N)}$ is connected.

Similarly it can be shown that $\overline{f(N + x)}$ is connected for each $x \in X$ and each balanced neighborhood N of $0 \in E$.

THEOREM 4.10. *A convex function $f: E \to R$ is continuous if and only if there is an interval I with $f(0) < \inf_{t \in I}\{t\}$ and $0 \notin \overline{f^{-1}(I)}$.*

Proof. If f is continuous, then $f^{-1}([a, b])$ is closed and does not contain 0 if $a > f(0)$. Such an interval $[a, b] = I$ can always be chosen.

Conversely, if $0 \notin \overline{f^{-1}(I)}$ for some interval I as in the hypothesis, there exists a neighborhood N of 0 such that $N \cap \overline{f^{-1}(I)} = \varnothing$ and $\overline{f(N)}$ is connected. Now, $0 \in f(N), f(0) < \inf_{t \in I}\{t\}$ and $f(N) \cap I = \varnothing$ together imply that $\overline{f(N)}$ is bounded above by I. Thus $f(N)$ is bounded above and f is continuous.

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