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THE HYPO RESIDUUM OF THE AUTOMORPHISM GROUP OF AN ABELIAN p-GROUP

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Throughout the following G denotes an abelian p-group (for some fixed prime p) and $A(G)$ its automorphism group.

The subject of this article is the question to what extent the structure of G and the structure of $A(G)$ determine each other. Theorems of the type:

G is a P-group if and only if $A(G)$ is a Q-group,

where P and Q are group theoretical properties, have been proved by R. Baer; some others are well-known.

This paper gives a characterisation of this kind for the class P of all abelian p-groups whose homogeneous direct summands have finite rank (an abelian p-group is called homogeneous if it is a direct sum of isomorphic subgroups of rank 1).

For this purpose it is convenient to define, for every group X, a characteristic subgroup ΩX, called the hypo residuum of X. This is the product of all normal subgroups N of X, such that every finite epimorphic image of N is trivial.

The main result is the following

THEOREM A. Every homogeneous direct summand of G has finite rank if and only if $\Omega A(G) = 1$.

A consequence of this theorem is the following fact: if $A(G)$ contains a quasicyclic subgroup, then the group of all permutations on a countably infinite set (and hence every countable group) is isomorphic to a group of automorphisms of G. This happens (if and) only if G possesses a homogeneous direct summand of infinite rank.

The group $\Gamma(G)$ of all automorphisms γ of G inducing the 1-automorphism in $G/p^{\omega}G$ is of special significance. It is shown that $\Omega\Gamma(G)=1$ for every reduced abelian p-group G (Theorem 2) and furthermore

THEOREM B. If G is reduced then $\Omega A(G)=1$ if and only if $A(G)/\Gamma(G)$ is residually finite.

Closely connected with the concept of the hypo residuum of a group X is the descending chain of the so called higher residua $\mathscr{R}_{\mu}X$ of X: let $\mathscr{R}_0X = X$, define $\mathscr{R}_{\lambda}X = \bigcap_{\mu<\lambda} \mathscr{R}_{\mu}X$ if λ is a limit ordinal and $\mathscr{R}_{\mu}X$ has been defined already for all ordinals $\mu < \lambda$, and let $\mathscr{R}_{\mu+1}X$ be the intersection of all subgroups of $\mathscr{R}_{\mu}X$ of finite index. Then $\Omega X = \mathscr{R}_{\sigma}X$ for sufficiently large σ, and the following result is proved:

THEOREM C. If G is an abelian p-group of Ulm type τ such that $\Omega A(G) = 1$, then $\mathscr{R}_{\tau+1}A(G) = 1$. Moreover, if G is reduced, then $\mathscr{R}_{\tau}A(G) = 1$.

2. Notation. Our notation and terminology concerning abelian groups will be standard as it can be found in [3] and [6].

As usual, ω denotes the first infinite ordinal. $A + B$ denotes the direct sum of the abelian groups A and B. $\sum$ and $\prod^*$ are our symbols for a direct sum and a cartesian or unrestricted direct product resp. The rank of an abelian p-group H is the dimension of $H[p]$ as a vector space over the primefield of characteristic p. An abelian p-group is called homogeneous if it is a direct sum of isomorphic groups of rank 1. The maximal divisible subgroup of H will be denoted by dH. If Δ is a group of automorphisms of H, then $H(\Delta - 1)$ is the set of all $x\delta - x$ where $x \in H$ and $\delta \in \Delta$. The group of all automorphisms of H shall be denoted by $A(H)$.

Following [3], we define subgroups $p^\alpha H$ inductively in the following way: $p^0 H = H$; if $p^\mu H$ is constructed already for an ordinal μ, then $p^{\mu+1}H = p \cdot p^\mu H$; and $p^\lambda H = \bigcap_{\mu<\lambda} p^\mu H$ if λ is a limit ordinal. Clearly,

$$p^\mu H \supseteqq p^{\mu+1} H$$

for every ordinal μ. The rank of the elementary abelian p-group

$$(p^\mu H)[p]/(p^{\mu+1}H)[p]$$

is called the μ-th Ulm-Kaplansky invariant of H(cf. [6], p. 27), and is denoted by $f_H(\mu)$. If H is a direct sum of cyclic groups and $k \geqq 0$ an integer, then $f_H(k)$ is the number of cyclic direct summands of order p^{k+1} in such a decomposition (see [6], p. 27). It is well known that $f_B(k) = f_H(k)$, for every basic subgroup B of H, and every natural number k. From this it follows easily that the ranks of homogeneous direct summands of H are finite if and only if $rk(dH) < \aleph_0$ and $f_H(\alpha) < \aleph_0$, for every ordinal $\alpha < \omega$.

If X can be imbedded into a group Y, we write $X \subsetneq Y$.

3. The hypo residuum. For an arbitrary group X we define the residuum of X as the intersection of all subgroups of finite index of X and denote it by $\mathscr{R}X$. It is a consequence of a well known theorem by Poincaré, that every subgroup of X of finite index contains a normal subgroup of X of finite index. Therefore, $\mathscr{R}X$ is also the intersection of all normal subgroups of X of finite index.

If $\mathscr{R}X$ is trivial, X is called residually finite (cf. [4], p. 16).

The property of residual finiteness is inherited by subgroups but neither by extensions nor by epimorphic images. For our purposes we therefore introduce the concept of higher residua of X, obtained by interating the process of forming the residuum. As stated in the introduction, we define $\mathscr{R}_0 X = X$, $\mathscr{R}_{\mu+1}X = \mathscr{R}(\mathscr{R}_\mu X)$, and

$$\mathscr{R}_\lambda X = \bigcap_{\mu<\lambda} \mathscr{R}_\mu X$$

if λ is a limit ordinal.

Clearly, all the $\mathscr{R}_\mu X$ are characteristic subgroups of X, called the higher residua of X; $\mathscr{R}_\mu X$ is called the μ-th residuum of X. From our definitions it follows that

$$X = \mathscr{R}_0 X \supseteq \mathscr{R}_1 X \supseteq \cdots \supseteq \mathscr{R}_\omega X \supseteq \mathscr{R}_{\omega+1} \supseteq \cdots$$

is a descending chain of subgroups of X, and that $\mathscr{R}_\mu X = \mathscr{R}_{\mu+1} X$ implies $\mathscr{R}_\mu X = \mathscr{R}_\alpha X$ for every $\alpha \geqq \mu$. Hence, for all X, there exists a least ordinal τ such that $\mathscr{R}_\tau X = \mathscr{R}_\mu X$, for all $\mu \geqq \tau$. $\mathscr{R}_\tau X$ then is called the hypo residuum of X and denoted by ΩX. Note that $\mathscr{R}_\mu X = \Omega X$ for every sufficiently large ordinal μ. Hence, $\Omega X = 1$ if and only if there exists an ordinal μ such that $\mathscr{R}_\mu X = 1$.

We remark that $\Omega X = X$ for every infinite simple group X. Also $Z(p^\infty) = \mathscr{R} Z(p^\infty) = \Omega Z(p^\infty)$, where as always $Z(p^\infty)$ denotes the quasi-cyclic p-group.

We are now going to develop the tools we will need in order to prove our results.

Throughout this section X, Y, and X_i are multiplicative groups. It is convenient for us to denote the set of all normal subgroups of finite index of X by $\mathscr{F}(X)$. Hence

$$\mathscr{R} X = \bigcap_{N \in \mathscr{F}(X)} N .$$

LEMMA 3.1. *If $X \subseteqq Y$, then $\mathscr{R}_\mu X \subseteqq \mathscr{R}_\mu Y$, for every ordinal μ.*

Proof. First we want to show

$$(+) \qquad \text{if } X \subseteqq Y, \text{ then } \mathscr{R} X \subseteqq \mathscr{R} Y .$$

Let $N \in \mathscr{F}(Y)$. Then Y/N is finite and so is $XN/N \cong X/(X \cap N)$. Hence, $X \cap N$ is a normal subgroup of X of finite index, for every $N \in \mathscr{F}(Y)$. Consequently,

$$\mathscr{R} Y = \bigcap_{N \in \mathscr{F}(Y)} N \supseteqq \bigcap_{N \in \mathscr{F}(Y)} (X \cap N) \supseteqq \bigcap_{M \in \mathscr{F}(X)} M = \mathscr{R} X$$

and we have proven $(+)$.

A simple transfinite induction on μ using $(+)$ completes the proof of Lemma 3.1.

LEMMA 3.2. *For a subgroup W of X the following properties are equivalent.*

(1) $W = \Omega X$.

(2) *W is the product of all normal subgroups N of X such*

that every finite epimorphic image of N is trivial.

(3) *W is the set theoretical union of all subgroups S of X such that every finite epimorphic image of S is trivial.*

Proof. Note that every finite epimorphic image of a group S is trivial if and only if $\mathscr{R}S = S$. Hence, ΩX is a normal subgroup of X whose finite epimorphic images are trivial, and, by Lemma 3.1, every subgroup S of X satisfying $S = \mathscr{R}S$ is contained in ΩX. This proves the theorem.

As immediate consequences of Lemma 3.2 we state

COROLLARY 3.3. *If $X \subseteqq Y$ and $\Omega Y = 1$ then $\Omega X = 1$.*

COROLLARY 3.4. *Let Y be a normal subgroup of X. If $\Omega Y = 1$ and $\Omega(X/Y) = 1$ then $\Omega X = 1$.*

According to these corollaries the property of having a trivial hypo residuum is inherited by subgroups and extensions. It is not inherited by epimorphic images (cf. Lemma 3.7 and note that $\Omega Z(p^\infty) = Z(p^\infty)$).

LEMMA 3.5. *Let Y be a normal subgroup of X and $\mathscr{R}_\alpha(X/Y) = 1$ for some ordinal α. Then $\mathscr{R}_\alpha X \subseteqq Y$.*

Proof. First we want to show, for a subgroup S of X,

$$(+) \qquad \text{if } \mathscr{R}(X/Y) = S/Y \text{ then } \mathscr{R}X \subseteqq S .$$

We know that

$$\mathscr{R}(X/Y) = \bigcap_{M/Y \in \mathscr{F}(X/Y)} (M/Y) = \Big(\bigcap_{M/Y \in \mathscr{F}(X/Y)} M\Big)Y = S/Y$$

and consequently

$$S = \bigcap_{M/Y \in \mathscr{F}(X/Y)} M .$$

Now $M/Y \in \mathscr{F}(X/Y)$ if and only if M is a normal subgroup of X of finite index containing Y. Hence,

$$S = \bigcap_{Y \subset M \in \mathscr{F}(X)} \supseteqq \bigcap_{N \in \mathscr{F}(X)} N = \mathscr{R}X$$

and we have proven (+).

As a subgroup of X/Y every higher residuum $\mathscr{R}_\mu(X/Y)$ of X/Y is of the form

$$\mathscr{R}_\mu(X/Y) = S_\mu/Y$$

for some subgroup S_μ of X. By a simple transfinite induction on μ using $(+)$ and Lemma 3.1 one shows easily that $\mathscr{R}_\mu X \subseteqq S_\mu$ for every ordinal μ. Hence, if $\mathscr{R}_\alpha(X/Y) = S_\alpha/Y = 1$, then $\mathscr{R}_\alpha X \subseteqq S_\alpha = Y$ as stated in the lemma.

LEMMA 3.6. *$\mathscr{R}_\mu(\prod_i^* X_i) \subseteqq \prod_i^*(\mathscr{R}_\mu X_i)$ for every ordinal μ.*

Proof. Again, the proof will be by transfinite induction on μ. First we want to show that

$$(+) \qquad \mathscr{R}\Big(\prod_i{}^* X_i\Big) \subseteqq \prod_i{}^*(\mathscr{R}X_i)\ .$$

Let $x \in \mathscr{R}(\prod_i^* X_i)$. Then, by definition of the residuum, x is contained in every normal subgroup of $\prod_i^* X_i$ of finite index. But, for every j, if N is a normal subgroup of finite index of X_j, then $N\cdot\prod_{i\neq j}^* X_i$ is a normal subgroup of $\prod_i^* X_i$ of finite index. Hence,

$$x \in \bigcap_{N\in\mathscr{F}(X_j)}\Big[N\cdot\Big(\prod_{i\neq j}{}^* X_i\Big)\Big] = \Big(\bigcap_{N\in\mathscr{F}(X_j)} N\Big)\cdot\Big(\prod_{i\neq j}{}^* X_i\Big) = (\mathscr{R}X_j)\cdot\Big(\prod_{i\neq j}{}^* X_i\Big)$$

and this is true for every j. Consequently,

$$x \in \bigcap_j\Big[(\mathscr{R}X_j)\cdot\Big(\prod_{i\neq j}{}^* X_i\Big)\Big] = \prod_i{}^*(\mathscr{R}X_i)$$

for every $x \in \mathscr{R}(\prod_i^* X_i)$. This proves $(+)$ and we have shown the lemma for $\mu = 0$ and $\mu = 1$.

Let

$$(*) \qquad \mathscr{R}_\mu\Big(\prod_i{}^* X_i\Big) \subseteqq \prod_i{}^*(\mathscr{R}_\mu X_i)$$

be true for every ordinal $\mu < \alpha$. In order to establish (*) for $\mu = \alpha$ we distinguish two cases.

(i) α is a limit ordinal. In this case

$$(1) \qquad \mathscr{R}_\alpha\Big(\prod_i{}^* X_i\Big) = \bigcap_{\mu<\alpha}\mathscr{R}_\mu\Big(\prod_i{}^* X_i\Big)$$

and

$$(2) \qquad \prod_i{}^*(\mathscr{R}_\alpha X_i) = \prod_i{}^*\Big(\bigcap_{\mu<\alpha}\mathscr{R}_\mu X_i\Big) = \bigcap_{\mu<\alpha}\Big(\prod_i{}^*\mathscr{R}_\mu X_i\Big)$$

as one checks easily. By comparing (1) and (2) and using the induction hypothesis it follows that

$$\mathscr{R}_\alpha\Big(\prod_i{}^* X_i\Big) = \bigcap_{\mu<\alpha}\mathscr{R}_\mu\Big(\prod_i{}^* X_i\Big) \subseteqq \bigcap_{\mu<\alpha}\prod_i{}^*(\mathscr{R}_\mu X_i) = \prod_i{}^*(\mathscr{R}_\alpha X_i)\ .$$

We have derived (*) for $\mu = \alpha$.

(ii) $\alpha = \mu + 1$. In this case we have

$$\mathscr{R}_\alpha\Big(\prod_i{}^* X_i\Big) = \mathscr{R}\mathscr{R}_\mu\Big(\prod_i{}^* X_i\Big) \tag{3}$$

and

$$\mathscr{R}\mathscr{R}_\mu\Big(\prod_i{}^* X_i\Big) \subseteqq \mathscr{R}\Big(\prod_i{}^* \mathscr{R}_\mu X_i\Big) \tag{4}$$

by induction hypothesis and Lemma 3.1. Since (+) is established for arbitrary cartesian products already, we know that

$$\mathscr{R}\Big(\prod_i{}^* \mathscr{R}_\mu X_i\Big) \subseteqq \prod_i{}^* (\mathscr{R}\mathscr{R}_\mu X_i) = \prod_i{}^*(\mathscr{R}_\alpha X_i) . \tag{5}$$

Comparing (3), (4), and (5), the statement (*) follows for $\mu = \alpha$. This proves Lemma 3.6.

LEMMA 3.7. *If A is an (additively written) abelian group, then* $\mathscr{R}A = \bigcap_{n\geqq 1} nA$.

Proof. For every subgroup S of finite index of A, there exists an integer n such that $nA \subseteqq S$. Hence, if $x \in \bigcap_{n\geqq 1} nA$, then x is contained in every subgroup of finite index and therefore

$$\bigcap_{n\geqq 1} nA \subseteqq \bigcap_{S\in\mathscr{F}(A)} S = \mathscr{R}A . \tag{1}$$

In order to show that equality holds in (1), let $a \in A$, and $a \notin \bigcap_{n\geqq 1} nA$. Then there exists an integer m such that $a \notin mA$. Since A/mA is bounded, it is a direct sum of (finite) cyclic groups (see [3], p. 44, Th. 11.2). It follows that A possesses subgroups F and H such that

$$A/mA = F/mA + H/mA , \qquad a \in F , \tag{2}$$

and F/mA is finite. Therefore

$$A/H \cong (A/mA)/(H/mA) \cong F/mA$$

is finite, i.e., H is a subgroup of finite index of A. But $a \notin H$, for otherwise, $a \in F \cap H = mA$ because of (2), which is contradiction to our choice of $a \notin mA$. Hence, $a \notin \mathscr{R}A$, and we have proven equality in (1).

4. Higher residua of automorphism groups. From now on we are concerned with automorphism groups of abelian groups. The word "p-group" always is used in the sense of *abelian p-group*.

If A is an abelian group and S a subgroup of A, the set of all automorphisms of A which fix S elementwise and induce the identity automorphism in A/S is called the *stabilizer of* S *in* A and shall be denoted by $\sum(A:S)$. By a well known theorem of Kaloujnine $\sum(A:S)$ is abelian (cf. [7], p. 88, Satz 19).

LEMMA 4.1. *If H is a p-group and $S \subseteqq H$, then the stabilizer $\sum(H:S)$ of S in H is residually finite.*

Proof. A closer examination of the proof of [7, p. 88, Satz 19] shows that actually $\sum(H:S) \cong \mathrm{Hom}(H/S, S)$ (cf. [5], p. 153, Hilfssatz 1.4). Hence,

$$(+) \qquad \mathscr{R} \sum (H:S) \cong \mathscr{R}\, \mathrm{Hom}(H/S, S)\,.$$

By Lemma 3.7

$$\mathscr{R}\, \mathrm{Hom}(H/S, S) = \bigcap_{n \geqq 1} n \cdot \mathrm{Hom}(H/S, S)\,,$$

and since H/S is a p-group, $\mathrm{Hom}(H/S, S)$ contains no element $\neq 0$ which is divisible by every power of p (see [3], p. 206, D)). Consequently, $\bigcap_{k \geqq 1} p^k\, \mathrm{Hom}(H/S, S) = 0$, and in particular

$$\mathscr{R}\, \mathrm{Hom}(H/S, S) = \bigcap_{n \geqq 1} n \cdot \mathrm{Hom}(H/S, S) = 0\,.$$

This together with (+) proves our proposition.

LEMMA 4.2. *If H is a p-group of finite rank, $A(H)$ is residually finite.*

Proof. Let θ_k denote the set of all automorphisms of H which fix every element of $H[p^k]$. Then θ_k is a normal subgroup of $A(H)$ and $A(H)/\theta_k$ is essentially the group Φ_k of automorphisms of $H[p^k]$ which is induced by $A(H)$. Clearly, $\bigcap_{k \geqq 1} \theta_k = 1$.

We recall that for a p-group H of finite rank $H[p^k]$ is finite for every $k \geqq 1$. Hence the group Φ_k of automorphisms of $H[p^k]$ induced by $A(H)$ is finite, and every θ_k has finite index in $A(H)$. It follows that

$$\mathscr{R} A(H) \subseteqq \bigcap_{k \geqq 1} \theta_k = 1$$

and the lemma is proven.

Following the notation of L. Fuchs (see [3], p. 117f) we define subgroups G^μ of G in the following way.

Let $G^0 = G$. If G^μ is constructed already for an ordinal μ, then $G^{\mu+1} = p^\omega G_\mu$, and $G^\lambda = \bigcap_{\mu < \lambda} G^\mu$ if λ is a limit ordinal.

The Ulm type of G is the least ordinal τ such that $G^\tau = G^{\tau+1}$. Clearly, all the G^μ are characteristic subgroups of G and consequently every automorphism of G induces an automorphism in G^μ and in G/G^μ.

LEMMA 4.3. *If Δ is a group of automorphisms of G inducing the identity automorphism in G/G^μ for some ordinal μ, then Δ induces the identity automorphism in $G^1/G^{\mu+1}$.*

Proof. By hypothesis we have $y(\delta - 1) \in G^\mu$ for every $y \in G$ and every $\delta \in \Delta$. If $x \in G^1 = p^\omega G$, then there are elements $y_n \in G$ such that

$$x = p^n y_n \qquad \text{for } n = 1, 2, \cdots$$

Hence, for every $\delta \in \Delta$,

$$x(\delta - 1) = p^n y_n(\delta - 1) \in p^n G^\mu \qquad \text{for } n = 1, 2, \cdots$$

and

$$x(\delta - 1) \in \bigcap_{n \geqq 1} p^n G^\mu = p^\omega G^\mu = G^{\mu+1} ,$$

for every $x \in G^1$. Therefore, as stated in the lemma, every $\delta \in \Delta$ induces the identity automorphism in $G^1/G^{\mu+1}$.

For a p-group H, the set of all automorphisms of H which induce the identity automorphism in $H/p^\omega H$ is a normal subgroup of $A(H)$ which shall be denoted by $\Gamma(H)$.

THEOREM 1. *$\mathscr{R}_\mu\Gamma(G)$ induces the identity automorphism in G/G^μ for every ordinal μ. Moreover, if μ is finite, $\mathscr{R}_\mu\Gamma(G)$ induces the identity automorphism in $G/G^{\mu+1}$.*

Proof. The proof of both statements will be by induction on μ. In order to simplify our notation let $\Gamma = \Gamma(G)$. Since $\Gamma = \mathscr{R}_0\Gamma$ induces the identity automorphism in G/G^1, the first step of each induction has been established.

First we want to prove the second part of Theorem 1. Assume, that $\mathscr{R}_n\Gamma$ fixes every coset of G/G^{n+1} for some finite $n \geqq 0$. Let Δ be the set of all elements in $\mathscr{R}_n\Gamma$ which induce the 1-automorphism in G/G^{n+2}. Then Δ is a normal subgroup of $\mathscr{R}_n\Gamma$ and $(\mathscr{R}_n\Gamma)/\Delta$ is essentially a group $\sum$ of automorphisms of G/G^{n+2}. By Lemma 4.3, $\mathscr{R}_n\Gamma$ and therefore $\sum$ induces the 1-automorphism in

$$G^1/G^{n+2} \supseteqq G^{n+1}/G^{n+2} ,$$

and $\sum$ likewise fixes $(G/G^{n+2})/(G^{n+1}/G^{n+2})$ elementwise. Hence, $\sum \cong (\mathscr{R}_n\Gamma)/\Delta$ is a subgroup of the stabilizer of G^{n+1}/G^{n+2} in G/G^{n+2}, which according to Lemma 4.1 is residually finite. Lemma 3.1 then implies $\mathscr{R}\sum = 1$ and, by using Lemma 3.5, we obtain

$$\mathscr{R}_{n+1}\Gamma = \mathscr{R}\mathscr{R}_n\Gamma \subseteqq \Delta .$$

Consequently $\mathscr{R}_{n+1}\Gamma$ induces the identity automorphism in G/G^{n+2}. Finite induction proves the second part of the theorem.

To show the first result we make the induction hypothesis

$$(+) \qquad G(\mathscr{R}_\mu\Gamma - 1) \subseteqq G^\mu \qquad \text{for all } \mu < \lambda .$$

If λ is a limit ordinal then $\mathscr{R}_\lambda\Gamma = \bigcap_{\mu<\lambda} \mathscr{R}_\mu\Gamma$ and $(+)$ implies

$$G(\mathscr{R}_\lambda\Gamma - 1) \subseteqq \bigcap_{\mu<\lambda} G(\mathscr{R}_\mu\Gamma - 1) \subseteqq \bigcap_{\mu<\lambda} G^\mu = G^\lambda .$$

So, in this case, $(+)$ holds for $\mu = \lambda$.

If $\lambda = \mu + 1$, then $G^\lambda = p^\omega G^\mu$ and, by what we have shown already, we may assume $\mu > 0$. Let Δ denote the set of all elements of $\mathscr{R}_\mu\Gamma$ which induce the 1-automorphism in G/G^λ. As before, $\mathscr{R}_\mu\Gamma/\Delta$ is essentially a group of automorphisms of G/G^λ which, by $(+)$ and Lemma 4.3, fixes $(G/G^\lambda)/(G^\mu/G^\lambda)$ and $G^1/G^{\mu+1} \supseteqq G^\mu/G^{\mu+1} = G^\mu/G^\lambda$ elementwise. As a subgroup of a stabilizer this group is residually finite (Lemmas 4.1 and 3.1). Hence, Lemma 3.5 implies

$$\mathscr{R}_\lambda\Gamma = \mathscr{R}_{\mu+1}\Gamma = \mathscr{R}\mathscr{R}_\mu\Gamma \subseteqq \triangle ,$$

and $\mathscr{R}_\lambda\Gamma$ induces the identity automorphism in G/G^λ.

Transfinite induction proves $(+)$ for all ordinals μ.

THEOREM 2. *$\Omega\Gamma(G) = 1$ for every reduced p-group G.*

Proof. Let τ be the Ulm type of G. Then $G^\tau = 0$ and $G/G^\tau = G$. By Theorem 1, $\mathscr{R}_\tau\Gamma(G)$ induces the identity in G/G^τ. Hence, $\mathscr{R}_\tau\Gamma(G) = 1$ and therefore $\Omega\Gamma(G) = 1$.

THEOREM 3. *If G is a p-group without elements of infinite height such that $f_G(\alpha)$ is finite for every ordinal $\alpha < \omega$, then its automorphism group $A(G)$ is residually finite.*

Proof. It is easy to verify that the finiteness of $f_G(\alpha)$ for every $\alpha < \omega$ is equivalent to the condition that

$$(+) \qquad G[p^k]/(p^nG)[p^k] \text{ is finite for all } n,\ k \geqq 1 .$$

In order to prove our theorem we consider the set $\Delta(n, k)$ of all automorphisms of G which induce the identity automorphism in

$$G[p^k]/(p^nG)[p^k] .$$

All the $G[p^k]$ and p^nG are characteristic subgroups of G; therefore every $\Delta(n, k)$ is a normal subgroup of $A(G)$ and $A(G)/\Delta(n, k)$ is essentially a group of automorphisms of $G[p^k]/(p^nG)[p^k]$, which according to $(+)$ is finite. Hence, every $\Delta(n, k)$ has finite index in $A(G)$ and

$$(++) \qquad \mathscr{R}A(G) \subseteqq \bigcap_{n,k\geqq 1} \Delta(n, k) = \Delta .$$

We want to show $\Delta = 1$. Let $\delta \in \Delta$ and $x \in G$ arbitrary. Then $p^k x = 0$ for some integer $k \geqq 1$, and $\delta \in \Delta \subseteqq \bigcap_{n\geqq 1} \Delta(n, k)$ implies

$$x(\delta - 1) \in (p^nG)[p^k] \subseteqq p^nG \text{ for all } n \geqq 1 .$$

Hence

$$x(\delta - 1) \in \bigcap_{n\geqq 1} p^nG = p^\omega G = 0$$

and

$$x\delta = x .$$

This being true for all $x \in G$ and every $\delta \in \Delta$ gives us $\Delta = 1$ and $(++)$ proves the theorem.

THEOREM 4. *Let G be a p-group whose homogeneous direct summands have finite rank. If τ is the Ulm type of G, then $\mathscr{R}_{\tau+1}A(G) = 1$. Moroever, if G is reduced, $\mathscr{R}_\tau A(G) = 1$.*

Proof. Let $B = \sum_{i\geqq 1} B_i$ be a basic subgroup of G where B_i is a direct sum of cyclic groups of order p^i. Then every B_i is a bounded pure subgroup of G and therefore a direct summand (cf. [3], p. 80, Theorem 24.5). Hence all the B_i and, by a structure theorem for divisible groups (cf. [3], p. 64, Th. 19.1), also dG are homogeneous direct summands of G. Our hypothesis implies

$$(0) \qquad B_i \text{ is finite for all } i \geqq 1$$

$$(\mathrm{d}) \qquad rk(dG) < \aleph_0 .$$

Again, $f_B(\alpha) = f_G(\alpha)$ for all finite α and $f_B(i) = rk(B_{i+1})$. So, (0) implies

$$(1) \qquad f_G(\alpha) < \aleph_0 \text{ for all } \alpha < \omega .$$

If $\tau = 0$, G is divisible and $G = dG$ has finite rank according to (d). In this case Theorem 4 is a consequence of Lemma 4.2. So we may assume that

$$(2) \qquad \tau \geqq 1.$$

Again, $A(G)$ induces in $G/p^\omega G$ a group θ of automorphisms, and

$$(3) \qquad \theta \cong A(G)/\Gamma(G) \subsetneq A(G/p^{\omega}G) ,$$

where as before $\Gamma(G)$ denotes the normal subgroup consisting of all elements of $A(G)$ which induce the 1-automorphism in $G/p^{\omega}G$. $G/p^{\omega}G$ is a p-group without elements of infinite height whose basic subgroups are isomorphic to the basic subgroups of G. Hence, $f_{G/p^{\omega}G}(\alpha) = f_G(\alpha)$ for all $\alpha < \omega$, and by (1), $G/p^{\omega}G$ satisfies the hypothesis of Theorem 3. It follows that $A(G/p^{\omega}G)$ is residually finite and (3) together with the Lemmas 3.1 and 3.5 gives us $\mathscr{R}A(G) \subseteqq \Gamma(G)$. Applying Lemma 3.1 to this inclusion it follows that

$$(4) \qquad \mathscr{R}_{\mu}\cdot\mathscr{R}A(G) \subseteqq \mathscr{R}_{\mu}\Gamma(G) \qquad \text{for all } \mu .$$

Clearly, if μ is finite, we have $\mathscr{R}_{\mu}\mathscr{R}A(G) = \mathscr{R}_{\mu+1}A(G)$. If μ is infinite, then $\mathscr{R}_{\mu}\mathscr{R}A(G) = \mathscr{R}_{1+\mu}A(G) = \mathscr{R}_{\mu}A(G)$. Hence, we can rewrite (4) in the form

$$(4)^* \qquad \begin{cases} \mathscr{R}_{\mu+1}A(G) \subseteqq \mathscr{R}_{\mu}\Gamma(G) & \text{for } \mu < \omega , \\ \mathscr{R}_{\mu}A(G) \subseteqq \mathscr{R}_{\mu}\Gamma(G) & \text{for } \mu \geqq \omega . \end{cases}$$

Let Φ be the set of all automorphisms of G which induce the identity in G/dG. If τ is the Ulm type of G, then $G^{\tau} = dG$. We claim that

$$(+) \qquad \mathscr{R}_{\tau}A(G) \subseteqq \Phi .$$

To prove $(+)$ we distinguish two cases. By Theorem 1, $\mathscr{R}_{\tau}\Gamma(G)$ induces the identity automorphism in $G/G^{\tau} = G/dG$. So, in the case τ is infinite, $(+)$ follows from the second part of $(4)^*$.

If $\tau \geqq 1$ (cf. (2)) is finite, $\mathscr{R}_{\tau-1}\Gamma(G)$ already induces the identity in $G/G^{\tau} = G/dG$, again according to Theorem 1. In this case $(+)$ follows from the first part of $(4)^*$, and is therefore proven in general.

In particular, if G is reduced, $(+)$ implies $\mathscr{R}_{\tau}A(G) = 1$ as claimed in the theorem.

Let Δ be the set of all automorphisms of G fixing dG elementwise. Then, as before, Δ is a normal subgroup of $A(G)$ and $A(G)/\Delta$ is essentially a group of automorphisms of dG. But dG has finite rank according to (d), hence by Lemma 4.2 its automorphism group is residually finite. We conclude (Lemmas 3.1 and 3.5)

$$(++) \qquad \mathscr{R}A(G) \subseteqq \Delta .$$

Comparing $(+)$ and $(++)$ and recalling that $\tau \geqq 1$ (see (2)) it follows that

$$(5) \qquad \mathscr{R}_{\tau}A(G) \subseteqq \Phi \cap \Delta .$$

But $\Phi \cap \Delta$ is the stabilizer of dG in G, hence by Lemma 4.1 residually finite. Again applying Lemma 3.1 the statement (5) finally gives us

$$\mathscr{R}_{\tau+1}A(G) = \mathscr{R}\mathscr{R}_{\tau}A(G) = 1$$

and the theorem is proven.

REMARK. Unfortunately, we were not able to determine the smallest ordinal α such that $\mathscr{R}_{\alpha}A(G) = 1$ (if $\Omega A(G) = 1$). We do not even know whether $\mathscr{R}_{\alpha}A(G) = 1$ is possible for an ordinal α less than the Ulm type of G.

A consequence of Theorem 4 is the following

THEOREM 5. *For a p-group G the following statements are equivalent.*

(1) *G possesses a quasicyclic group of automorphisms.*

(2) *Every countable group is isomorphic to a group of automorphisms of G.*

(3) *The group of all permutations on a countably infinite set is isomorphic to a group of automorphisms of G.*

(4) $\Omega A(G) \neq 1$.

(5) *G possesses a homogeneous direct summand of infinite rank.*

(6) *Either* $rk(dG) \geqq \aleph_0$ *or* $f_G(\alpha) \geqq \aleph_0$ *for some ordinal* $a < \omega$.

REMARK. For an arbitrary abelian torsion group G the statements (1)-(5) still are equivalent. (6) has to be modified in an obvious way (cf. Lemma 3.6).

Proof. As stated several times before, (5) and (6) are equivalent.

Clearly, (5) implies (3). Since by a well known theorem every countable group is a group of permutations on a countably infinite set, (2) follows from (3). (1) is a trivial consequence of (2).

Let us assume the validity of (1) and recall that $\Omega Z(q^\infty) = Z(q^\infty)$ for every prime q. Therefore, by Corollary 3.3, $\Omega A(G) \neq 1$ and we have derived (4) from (1).

It remains to show that (4) implies (5). But this is just the previous Theorem 4. The proof of the theorem is completed.

Theorem A, stated in the introduction, is an obvious consequence of this result.

Theorem C above follows readily from Theorems 5 and 4.

It remains to give a

Proof of Theorem B. Let G be reduced and assume first that $A(G)/\Gamma(G)$ is residually finite. Then $\Omega(A(G)/\Gamma(G)) = 1$ and, by Theorem 2, also $\Omega\Gamma(G) = 1$. Hence Corollary 3.4 implies $\Omega A(G) = 1$.

Conversely, suppose $\Omega A(G) = 1$. Then $f_G(\alpha) < \infty$ for all finite ordinals α by the previous Theorem 5. Since $f_G(\alpha) = f_{G/p^\omega G}(\alpha)$ for all $\alpha < \omega$ and $G/p^\omega G$ has no elements of infinite height, Theorem 3 applies to $G/p^\omega G$ and we obtain

$$(+) \qquad \mathscr{R}A(G/p^\omega G) = 1 .$$

Since, by definition, $A(G)/\Gamma(G)$ is essentially a group of automorphisms of $G/p^\omega G$, Lemma 3.1 and (+) imply the residual finiteness of $A(G)/\Gamma(G)$.

All the theorems stated in the introduction are proven.

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