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ON THE STRUCTURE OF COMMUTATIVE PERIODIC SEMIGROUPS

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It is well known that a commutative periodic semigroup is a semilattice of one-idempotent (or unipotent) semigroups. Thus the characterization of commutative periodic semigroups reduces to two subproblems: (1) the structure of commutative periodic unipotent semigroups, and (2) the means for putting these together in the semilattice. In this paper a complete solution is given for problem (1), while problem (2) is solved for the special case where each unipotent subsemigroup is cyclic.

If S is a semigroup with zero ($S = S^0$), then the concepts of nilpotence and the kernel of a homomorphism may be defined in the usual ring-theoretic sense. Thus $x \in S$ is said to be *nilpotent* if $x^n = 0$ for some n, and the *kernel* of a homomorphism is the complete inverse image of zero. S is said to be *nil* if every element of S is nilpotent. Let T be a semigroup with zero and S be any semigroup. Denote by T^* the nonzero elements of T. A mapping α from T^* into S is said to be a *partial homomorphism* if $a, b \in T^*$, $ab \neq 0$ implies $(ab)\alpha = (a\alpha)(b\alpha)$.

It is easily seen that a commutative semigroup S is periodic and unipotent if and only if S is the ideal extension of a periodic abelian group G by a commutative nil semigroup T. Furthermore, every such extension is determined by the partial homomorphisms of T^* into G [2, Th. 4.19]. Thus our solution to (1) is obtained by determining the structure of commutative nil semigroups and using the characterization of partial homomorphisms found in [1].

1. Commutative nil semigroups.

An element x of a semigroup S is said to be *prime* if x does not belong to S^2. S is said to have *unique factorization* if every nonzero element of S can be written uniquely as a product of powers of primes. Of course, if S is not commutative, we must take the order of the factors into account. The following result is a corollary to Theorem 1 of [1].

LEMMA 1. *$S = S^0$ is commutative nil if and only if there exists a commutative nil semigroup U with unique factorization and a homomorphism from U onto S with trivial kernel.*

If $S = S^0$ is commutative and $x \in S$ we define the annihilator of

x as in ring theory by $\mathrm{Ann}(x) = \{y: xy = 0\}$. Define a relation R on S by $0R0$, and for $x \neq 0 \neq y$, xRy if and only if $\mathrm{Ann}(x) = \mathrm{Ann}(y)$. It is straightforward to show that R is a congruence on S with trivial kernel (the class containing zero). Moreover, if τ is any congruence on S with trivial kernel, then $\tau \leqq R$.

THEOREM 2. *Let S be a commutative nil semigroup and let $\{A_i: i \in I\}$ be the set of R-classes of S. Then:*

(i) *Any partition $B_{ij}, j \in J_i$, of some R-class A_i induces a congruence τ_i with trivial kernel on S such that the B_{ij} are congruence classes of τ_i.*

(ii) *Every congruence on S with trivial kernel is the intersection of a collection of such congruences τ_i.*

Proof. Let $\{B_{ij}: j \in J_i\}$ be a partition of some R-class A_i. Define τ_i as follows:

(a) $0\tau_i 0$;

(b) for $x, y \in A_i$, $x\tau_i y \Leftrightarrow x, y \in B_{ij}$, for some $j \in J_i$;

(c) for $x \neq 0 \neq y$ and $x, y \notin A_i$, $x\tau_i y \Leftrightarrow$ for all $z \in S$ and $j \in J_i$, $xz \in B_{ij} \Leftrightarrow yz \in B_{ij}$.

τ_i is an equivalence relation on S. To show τ_i is a congruence we consider separately the last two cases of the definition.

First, suppose $x, y \notin A_i$ and $x\tau_i y$. If $xz \in A_i$ then $xz \in B_{ij}$ for some j and so $yz \in B_{ij}$ and $(xz)\tau_i(yz)$. If $xz \notin A_i$, then $yz \notin A_i$, and $x(zw) \in B_{ij}$ if and only if $y(zw) \in B_{ij}$ for all $w \in S$, so that $(xz)\tau_i(yz)$ for all $z \in S$. Secondly, let $x, y \in A_i$ and $x\tau_i y$. Then $x, y \in B_{ij}$ for some j. Now if $A_i = \{0\}$, then $(xz)\tau_i(yz)$ follows immediately, so assume $A_i \neq \{0\}$. If $xz \in A_i$ for some $z \in S$, then $xR(xz)$ which implies $xR(xz^n)$ for all n, but S is nil, so $xR0$ and $x = 0$, a contradiction. Thus for $x, y \in A_i \neq \{0\}$ we have $xz, yz \notin A_i$ for all z in S, so trivially $(xz)\tau_i(yz)$, and we have proved (i).

To prove (ii), let τ be any congruence on S with trivial kernel. Then $\tau \leqq R$, and τ induces a partition B_{ij} on each R-class A_i. If we define τ_i to be the congruence induced on S by each such partition (as in the proof of (i)), then it follows directly that $\tau = \cap\{\tau_i: i \in I\}$ since τ and τ_i agree on A_i for all i.

All commutative nil semigroups with unique factorization are easily determined [1]. Let F be the free commutative semigroup on X with ideal K. Then F/K is nil if and only if K contains some positive power of each x in X. Combining Theorem 2 with Lemma 1 we obtain all commutative nil semigroups.

We remark that there are sufficient congruences in Theorem 2 to separate distinct elements of S.

2. Semilattices of cyclic semigroups. A cyclic semigroup S has an idempotent if and only if it is finite, and for these semigroups the concepts of *index* and *period* are defined as in [2, p. 19]. If S is infinite cyclic we may say it has infinite index and zero period, so that these terms are defined for all cyclic semigroups. With that convention in mind the following theorem characterizes semilattices of all cyclic semigroups, not just those with idempotents, and the solution of problem (2) mentioned in the introduction is obtained by assuming each cyclic semigroup is finite.

By Z, Z^+ and Z_0 we mean the integers, positive integers and nonnegative integers respectively.

THEOREM 3. *Let Y be any semilattice and $\{S_\alpha: \alpha \in Y\}$ a collection of disjoint cyclic semigroups where $S_\alpha = \langle a_\alpha \rangle$ with index n_α, and period p_α. For all $\alpha \geqq \beta \in Y$, choose $f(\alpha, \beta) \in Z_0$ such that*

(i) $f(\alpha, \alpha) = 1$;

(ii) $n_{\delta\lambda} \neq 1 \Rightarrow f(\lambda, \delta\lambda) + f(\delta, \delta\lambda) \neq 0$.

Define $g: Y \times Y \rightarrow Z$ such that:

(iii) $g(\alpha, \beta) = g(\beta, \alpha)$ *for all* $\alpha, \beta \in Y$;

(iv) $\alpha \neq \alpha\beta \neq \beta$ *and* $f(\alpha, \alpha\beta) + f(\beta, \alpha\beta) = 0 \Rightarrow g(\alpha, \beta) = 1$;

(v) $\alpha \neq \alpha\beta \neq \beta$ *and* $f(\alpha, \alpha\beta) + f(\beta, \alpha\beta) = n_{\alpha\beta} - 1 \Rightarrow g(\alpha, \beta) = 1$ *or* 0;

(vi) $\alpha \neq \alpha\beta \neq \beta$ *and* $f(\alpha, \alpha\beta) + f(\beta, \alpha\beta) = n_{\alpha\beta} - 1 + kp_{\alpha\beta} + S$ *for some* $k \in Z^+$ *and* $0 \leqq S \leqq p - 1 \Rightarrow g(\alpha, \beta) = -k$ *or* $1 - k$;

(vii) $g(\alpha, \beta) = 0$ *otherwise.*

Let $S = \cup\{S_\alpha \mid \alpha \in Y\}$ and define multiplication in S by

(1) $a_\alpha^i a_\beta^j = a_{\alpha\beta}^{if(\alpha,\alpha\beta)+jf(\beta,\alpha\beta)+g(\alpha,\beta)p_{\alpha\beta}}$, *for all* $\alpha, \beta \in Y$.

Further assume that f and g are defined such that

(viii) $a_\alpha^i(a_\beta^j a_\gamma^k) = a_\beta^j(a_\alpha^i a_\gamma^k) = a_\gamma^k(a_\alpha^i a_\beta^j)$ *for all* $\alpha, \beta, \gamma \in Y$, *and* $i, j, k \in \{1, 2\}$ *such that* $i + j + k \leqq 4$.

Then S is a commutative semigroup. Conversely, every commutative semigroup which is a semilattice of cyclic semigroups may be constructed in this manner.

Proof. Suppose $S = \cup\{S_\alpha: \alpha \in Y\}$ is a commutative semigroup which is a Y-semilattice of the S_α. Denote by G_α the maximal subgroup of S_α, if it exists. For $\alpha > \beta$ in Y, define $f(\alpha, \beta) = \exp(a_\alpha a_\beta) - 1$ where $\exp(a_\alpha a_\beta)$ is the least positive integer t such that $a_\alpha a_\beta = a_\beta^t$. Define $f(\alpha, \alpha) = 1$ for all $\alpha \in Y$.

Now let $\alpha, \beta \in Y$. Then $a_\alpha a_\beta = a_{\alpha\beta}^t$ for some least positive integer t. We have $a_{\alpha\beta}^{t+1} = a_\alpha(a_\beta a_{\alpha\beta}) = a_{\alpha\beta}^{f(\alpha,\alpha\beta)+f(\beta,\alpha\beta)+1}$, so

$$t = f(\alpha, \alpha\beta) + f(\beta, \alpha\beta) + g(\alpha, \beta)p_{\alpha\beta}$$

for some integer $g(\alpha, \beta)$. By induction

$$a_\alpha^i a_\beta^j = a_{\alpha\beta}^{if(\alpha,\alpha\beta)+jf(\beta,\alpha\beta)+g(\alpha,\beta)p_{\alpha\beta}}, \qquad \text{for all } i, j \in Z^+.$$

Suppose $\alpha \neq \alpha\beta \neq \beta$ in Y in the following three cases.

Case 1. $f(\alpha, \alpha\beta) + f(\beta, \alpha\beta) = 0$: Then $f(\alpha, \alpha\beta) = f(\beta, \alpha\beta) = 0$ so $n_{\alpha\beta} = 1$ and $p_{\alpha\beta} = t$. Thus (ii) and (iv) are satisfied because $g(\alpha, \beta) = 1$.

Case 2. $f(\alpha, \alpha\beta) + f(\beta, \alpha\beta) = n_{\alpha\beta} - 1$: Then $(a_{\alpha\beta})^{n_{\alpha\beta}} = a_{\alpha\beta}^{t+1}$. If $a_\alpha a_\beta \notin G_{\alpha\beta}$, then $t = n_{\alpha\beta} - 1$ and $g(\alpha, \beta) = 0$. On the other hand if $a_\alpha a_\beta \in G_{\alpha\beta}$, then $t = p_{\alpha\beta} + n_{\alpha\beta} - 1$ so that $g(\alpha, \beta) = 1$. Thus (v) holds.

Case 3. $f(\alpha, \alpha\beta) + f(\beta, \alpha\beta) = n_{\alpha\beta} - 1 + kp_{\alpha\beta} + S$ where $k \in Z^+$, $0 \leqq S \leqq p - 1$: If $a_\alpha a_\beta \notin G_{\alpha\beta}$, then $t = n_{\alpha\beta} - 1$, $S = 0$, and $g(\alpha, \beta) = -k$. If $a_\alpha a_\beta \in G_{\alpha\beta}$ and $S \neq 0$, then $t = n_{\alpha\beta} - 1 + S$ and $g(\alpha, \beta) = -k$. If $a_\alpha a_\beta \in G_{\alpha\beta}$ and $S = 0$, then $t = n_{\alpha\beta} - 1 + p$ and $g(\alpha, \beta) = 1 - k$.

In every other situation we see $a_{\alpha\beta}^t = a_{\alpha\beta}^{f(\alpha,\alpha\beta)+f(\beta,\alpha\beta)}$ where the exponents are in fact equal, so that $g(\alpha, \beta) = 0$, giving (vii).

Now that $g(\alpha, \beta)$ is defined for all $\alpha, \beta \in Y$ it is clear that (iii) is satisfied, and (viii) is obvious by the associativity of S.

Conversely, suppose $S = \cup\{S_\alpha : \alpha \in Y\}$ and the functions f and g are defined satisfying (i)-(viii) with multiplication given by (1). Multiplication is commutative by (1) and (iii). For $\alpha > \beta > \gamma$ in Y, condition (viii) implies

$$(2) \qquad \begin{aligned} &f(\alpha, \gamma) \equiv f(\alpha, \beta)f(\beta, \gamma) \bmod p_\gamma \text{ or} \\ &f(\alpha, \gamma) = f(\alpha, \beta)f(\beta, \gamma) . \end{aligned}$$

Let $a_\alpha^i, a_\beta^j, a_\gamma^k$ be any three elements of S. Then $a_\alpha^i(a_\beta^j a_\gamma^k)$, $a_\beta^j(a_\alpha^i a_\gamma^k)$ and $a_\gamma^k(a_\alpha^i a_\beta^j)$ are powers of $a_{\alpha\beta\gamma}$ (using (1)) with exponents which we denote e_1, e_2 and e_3 respectively. If $\min\{e_1, e_2, e_3\} \geqq n_{\alpha\beta\gamma}$, then the associativity follows by applying (2) to corresponding parts of the exponents e_1, e_2 and e_3. So we may assume that, say, $e_1 < n_{\alpha\beta\gamma}$. Then from (viii) it follows that the exponents of $a_\alpha(a_\beta a_\gamma)$, $a_\beta(a_\alpha a_\gamma)$ and $a_\gamma(a_\alpha a_\beta)$, which we denote by r_1, r_2 and r_3 respectively, are equal. If $i \geqq 2$, then $a_\alpha^2(a_\beta a_\gamma) = a_\beta(a_\alpha^2 a_\gamma) = a_\gamma(a_\alpha^2 a_\beta) \notin G_{\alpha\beta\gamma}$ so that the exponents of these expressions are equal, and in conjunction with the previous statement we get

$$(3) \qquad f(\alpha, \alpha\beta\gamma) = f(\alpha, \alpha\beta)f(\alpha\beta, \alpha\beta\gamma) = f(\alpha, \alpha\gamma)f(\alpha\gamma, \alpha\beta\gamma) .$$

Similarly, if $j \geqq 2$ or $k \geqq 2$, then we have the respective equations

$$(4) \qquad f(\beta, \alpha\beta\gamma) = f(\beta, \alpha\beta)f(\alpha\beta, \alpha\beta\gamma) = f(\beta, \beta\gamma)f(\beta\gamma, \alpha\beta\gamma)$$

or

$$(5) \qquad f(\gamma, \alpha\beta\gamma) = f(\gamma, \alpha\gamma)f(\alpha\gamma, \alpha\beta\gamma) = f(\gamma, \beta\gamma)f(\beta\gamma, \alpha\beta\gamma) .$$

Adding $(i-1)f(\alpha, \alpha\beta\gamma) + (j-1)f(\beta, \alpha\beta\gamma) + (k-1)f(\gamma, \alpha\beta\gamma)$ to each member of the equation $r_1 = r_2 = r_3$, and using (3), (4) and (5) when appropriate, we obtain $e_1 = e_2 = e_3$. Therefore S is associative.

For the special case when each S_α is infinite cyclic, Theorem 3 is greatly simplified, and we state that result in the following corollary.

COROLLARY 4. *Let Y be any semilattice and $\{S_\alpha = \langle a_\alpha \rangle : \alpha \in Y\}$ a collection of disjoint infinite cyclic semigroups indexed by Y. For all $\alpha \geqq \beta$ in Y choose $f(\alpha, \beta) \in Z_0$ such that*

(i) $f(\alpha, \alpha) = 1$

(ii) $f(\lambda, \delta\lambda) + f(\delta, \lambda\delta) \neq 0, \qquad \lambda, \delta \in Y.$

(iii) $\alpha \geqq \beta \geqq \gamma \Rightarrow f(\alpha, \gamma) = f(\alpha, \beta)f(\beta, \gamma).$

Let $S = \cup\{S_\alpha : \alpha \in Y\}$ and define multiplication in S by

$$a_\alpha^i a_\beta^j = a_{\alpha\beta}^{if(\alpha,\alpha\beta)+jf(\beta,\alpha\beta)} , \qquad \alpha, \beta \in Y.$$

Then S is a commutative semigroup. Conversely, every commutative semigroup which is a semilattice of infinite cyclic semigroups is determined in this manner.

Proof. It is easily verified that (iii) is sufficient for the equality of the three exponents that arise from the product $a_\alpha^i a_\beta^j a_\gamma^k$. Conversely, if S is associative and $\alpha \geqq \beta \geqq \gamma$ then the exponents from $a_\alpha(a_\beta a_\gamma) = a_\gamma(a_\alpha a_\beta)$ will give (iii).

We remark that condition (viii) of Theorem 3 says essentially that associativity of third degree and fourth degree terms is sufficient to guarantee all associativity. We conclude with an example to show that in this respect Theorem 3 is the best possible result.

Let Y be the semilattice consisting of $\alpha, \beta, \gamma = \alpha\gamma, \alpha\beta$, and $\alpha\beta\gamma = \beta\gamma$. Let the cyclic semigroups indexed by Y be chosen such that $n_{\beta\gamma} = 11$, $p_{\alpha\beta} = p_{\beta\gamma} = 1$ and $n_{\alpha\beta} = 2$. Define f and g by $f(\alpha, \gamma) = 1$, $f(\alpha, \alpha\beta) = f(\alpha, \beta\gamma) = f(\gamma, \beta\gamma) = 0$, $f(\beta, \alpha\beta) = 2$, $f(\beta, \beta\gamma) = f(\alpha\beta, \beta\gamma) = 11$, and $g(\alpha, \beta) = g(\beta, \gamma) = g(\gamma, \alpha\beta) = -1$. Conditions (i)-(vii) of Theorem 3 are satisfied. It can be shown that any term of the form xyz where x, y, z are first powers of the generators is associative. In fact, any term of the form $a_\alpha^i a_\beta a_\gamma^k$ is associative. However $(a_\alpha a_\beta^2)a_\gamma = a_\alpha(a_\beta^2 a_\gamma)$ so that the union of these is not a semigroup under the multiplication (1).

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